Hall Algebras

A <u>quiver</u> is a category freely generated by a directed graph.

A <u>representation</u> of a quiver Q is a functor

$$R : Q \longrightarrow \mathrm{Vect}_F$$

For example:

$$Q = \mathbb{A}_n = \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \quad (n=4)$$

A rep of Q is just:

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_n$$

There's a category $\mathrm{Rep}(Q)$ with:

- functors $R : Q \longrightarrow \mathrm{Vect}_F$ as objects
- natural transformations

$$\begin{tikzcd} Q \arrow[r, bend left, "R"{name=U}] \arrow[r, bend right, "R'"'{name=D}] & \mathrm{Vect}_F \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

as morphisms.

For $Q = A_n$, a morphism in $\mathrm{Rep}(Q)$ is like this:

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & V_n \arrow[d] \\ V_1' \arrow[r] & V_2' \arrow[r] & \cdots \arrow[r] & V_n' \end{tikzcd}$$

with commuting squares.

In fact $\mathrm{Rep}(Q)$ is an abelian category.

A category is abelian if:

1) given any objects x, y, $\hom(x,y)$ is an abelian group & $\circ : \hom(x,y) \times \hom(y,z) \longrightarrow \hom(x,z)$ is bilinear. So we say the category is "enriched over abelian groups".

2) every morphism $f : x \longrightarrow y$ has a kernel & cokernel

$$\mathrm{Ker}(f) \longrightarrow x \xrightarrow{f} y \longrightarrow \mathrm{cok}(f)$$

3) Every pair of objects has a biproduct ("direct sum") – i.e. product & coproduct in a compatible way

$$\begin{tikzcd} & a \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & x \times y \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

product

$$\begin{tikzcd} x \arrow[dr] \arrow[ddr, bend right] & & y \arrow[dl] \arrow[ddl, bend left] \\ & x + y \arrow[d, dashed, "\exists!"] & \\ & a & \end{tikzcd}$$

coproduct

$$\begin{tikzcd} x \arrow[dr] \arrow[dd, "1"'] & & y \arrow[dl] \arrow[dd, "1"] \\ & x \oplus y \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

biproduct is a product & a coproduct making these triangles commute

4) There is a <u>zero object</u>, i.e. an object 0 s.t. $\forall x$ $\exists!\ f : x \to 0$ (0 is <u>terminal</u>) & $\exists!\ g : 0 \to x$ (0 is <u>initial</u>)

5) Every subobject x of an object y is the kernel of some map $f : y \to z$

$$0 \longrightarrow x \longrightarrow y \xrightarrow{f} z \longrightarrow 0$$

(this is exact), & every quotient object z of an object y is the cokernel of some $g : x \to y$:

$$0 \longrightarrow x \xrightarrow{g} y \longrightarrow z \longrightarrow 0 .$$

Examples of abelian categories:

1) $\mathrm{Vect}_F = R\mathrm{Mod}$ where $R = F$

2) $\mathrm{AbGrp} = R\mathrm{Mod}$ where $R = \mathbb{Z}$

3) $R\mathrm{Mod}$ where R is any ring

4) $\mathrm{Rep}(Q) \cong R\mathrm{Mod}$ where R is the category algebra of Q.

Let $\mathcal{C}$ be an abelian category (think $\mathrm{Rep}(Q)$). We say $X \in \mathcal{C}$ is <u>irreducible</u> if we don't have any exact sequence:

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

where both A & B are nonzero.

Note: we always have exact sequences

$$0 \longrightarrow X \xrightarrow{1} X \longrightarrow 0 \longrightarrow 0$$

& $$0 \longrightarrow 0 \longrightarrow X \xrightarrow{1} X \longrightarrow 0$$

but here A or B is zero.

i.e. X has no subobjects except 0 & X.

We say X is <u>indecomposable</u> if we don't have $X \cong A \oplus B$ where both A & B are nonzero.

Note: if $X \cong A \oplus B$ we would get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[dr] & A \oplus B \arrow[r] \arrow[d, phantom, "\cong"] & B \arrow[r] & 0 \\
& & X \arrow[ur] & &
\end{tikzcd}$$

so decomposable $\Longrightarrow$ reducible.

This kind of exact sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$

<u>splits</u>, i.e. we have

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & A \oplus B \arrow[r, shift right, "p"'] & B \arrow[l, shift right, "i"'] \arrow[r] & 0
\end{tikzcd}$$

s.t. $pi = 1_B$.

Theorem (Ringel) - If Q is a quiver, $\mathrm{Rep}(Q)$ has finitely many irreducible objects iff Q is one of these:

i) $Q = A_n$  $(n=5)$

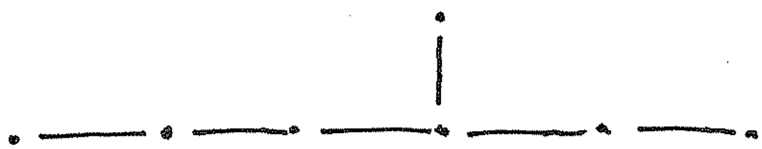
2) $Q = D_n$

$(n=6)$

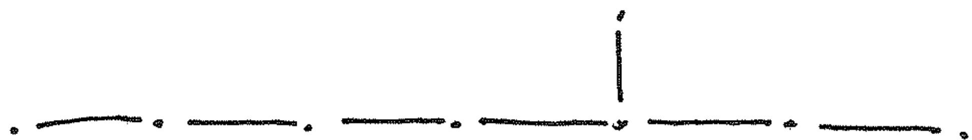
3) $Q = E_6, E_7, E_8$:

E_6

E_7



E_8



In all cases, an arrow can point in either direction.

Given an abelian category $\mathcal{C}$, we can form a vector space $\mathrm{Hall}(\mathcal{C})$ w. basis consisting of isomorphism classes of objects in $\mathcal{C}$:

$$\mathrm{Hall}(\mathcal{C}) = H_0(\mathcal{C})$$

We can try to make this into an algebra with

$$[A] \cdot [B] = \sum_{\substack{\text{iso class of} \\ \text{exact sequences} \\ 0 \to A \to X \to B \to 0}} [X]$$

Here two exact sequences

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow X' \longrightarrow B \longrightarrow 0$$

are isomorphic if we have:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "1"'] & X \arrow[r] \arrow[d, "f", "\wr"'] & B \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A \arrow[r] & X' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

making this commute. If this sum is always finite, we get an associative algebra $\mathrm{Hall}(\mathcal{C})$.

Jim sketched the proof of associativity & groupoidified the Hall algebra, get a multiplication on the groupoid $\mathcal{C}_0$

$$\begin{tikzcd}
0 \to A \to X \to B \to 0 \arrow[d, mapsto] & SES(\mathcal{C}_0) \arrow[dl] \arrow[dr] & 0 \to A \to X \to B \to 0 \arrow[d, mapsto] \\
(A,B) \in \mathcal{C}_0 \times \mathcal{C}_0 & & \mathcal{C}_0 \ni X
\end{tikzcd}$$

Any A, D, E quiver gives rise to a complex Lie algebra $\mathfrak{g}$ - namely $\mathfrak{sl}(n+1, \mathbb{C})$ for A_n. This has a maximal nilpotent sub-Lie-algebra n_+, & this gives an associative algebra Un_+. We can "q-deform" this & get an assoc. alg $U_q n_+$.

Theorem (Ringel-Hall) - If Q is an A, D, E quiver & $U_q n_+$ is the algebra obtained from it, then

$$U_q n_+ \cong \mathrm{Hall}(\mathrm{Rep}(Q))$$

where q is a prime power & we define $\mathrm{Rep}(Q)$ using vector spaces over $\mathbb{F}_q$.