Free Symmetric Monoidal Category on one object X

A monoidal category has a functorial tensor product, so we can start taking products of our one object X.

$$X \qquad X \otimes X$$

& then

$$X \otimes (X \otimes X) \qquad \text{or} \qquad (X \otimes X) \otimes X$$

due to issues of associativity, but we will sweep these issues under the rug for now. & we have

$$X \otimes X \otimes X$$

& so on

$$X^{\otimes n}.$$

And at the 0^{th} level, we have

$$1.$$

To be symmetric we want to imitate a commutative monoid. We need a canonical isomorphism

$$Y \otimes Z \xrightarrow{sw_{Y,Z}} Z \otimes Y$$

Let $Y = X$ & $Z = X \otimes X$

$$X \otimes (X \otimes X) \xrightarrow{sw} (X \otimes X) \otimes X$$

Note: this is not associativity.

We can ask if sw^2 is the identity. By the definition of symmetric monoidal category it is.

$$\begin{tikzcd}
Y \otimes Z \arrow[d, "sw_{Y,Z}"] \arrow[dd, bend right=60, "1"'] \\
Z \otimes Y \arrow[d, "sw_{Z,Y}"] \\
Y \otimes Z
\end{tikzcd}$$

We also have

$$\begin{tikzcd}
 & A \otimes B \otimes C \arrow[dl, "sw_{A,B} \otimes C"'] \arrow[dd, "sw_{A, B \otimes C}"] \\
B \otimes A \otimes C \arrow[dr, "B \otimes sw_{A,C}"'] & \\
 & B \otimes C \otimes A
\end{tikzcd}$$

What about morphisms?

We can't duplicate or delete data so morphisms must preserve the number of copies of X & all morphisms will be invertible.

The automorphism group on $X^{\otimes n}$ is the symmetric group on n elements.

So the free symmetric monoidal category on one object is equivalent to FinSet_0, the groupoid of finite sets, with tensor product $+$.

$$(\mathrm{FinSet}_0, +)$$

What about the free monoidal category on one object X?

We have the same objects but only identity morphisms.

$$(\mathbb{N}, +)$$

or we could write

$$(\mathrm{List}(*), \underset{\text{concatenation}}{+})$$

We can also consider the free braided monoidal category on one object X.

$$(\mathrm{Planar\ Set}_0, +)$$

These have associated "dimensions"

$(\mathrm{List}(*), +)$	$(\mathrm{Planar\ Set}_0, +)$	$(\mathrm{FinSet}_0, +)$
1	2	3

We have been talking about the tensor product, but we also want some categorified version of addition.

We consider now the free symmetric monoidal category with finite colimits on one object X.
(we are assuming a nice distributivity property)

"Colimit of a diagram"

$$
\begin{tikzcd}
K \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \arrow[r] \arrow[d] \arrow[dr] & A \arrow[d] \\
& B \arrow[r] & C
\end{tikzcd}
$$

X is the limit of the diagram

$$
\begin{tikzcd}
& A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}
$$