Geom. Rep. Theory Seminar   John Baez   Lecture 37

Suppose $\mathfrak{g}$ is a simple (complex) Lie algebra.

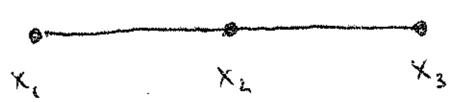
There is a simple Lie group G associated to it
— i.e. having no normal subgroups except 1 & G.

Pick a maximal abelian subgroup $H \subseteq G$ (any two are isomorphic). This has a Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$.

$\mathfrak{h}$ has a god-given inner product on it, & a lattice $L \subseteq \mathfrak{h}$ given by:

$$L = \ker\left(\exp : \mathfrak{h} \longrightarrow H \subseteq G\right)$$

Sometimes L has a basis of vectors that are all the same length & at angles $90°$ or $120°$. Then you can draw a "simply-laced Dynkin diagram":



An edge between dots corresponding to basis vectors x_i & x_j if they are at $120°$ angle; no edge otherwise.

Thm - The only diagrams you get this way are

A_n, D_n, E_6, E_7, E_8.

Example: $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

Here $\mathfrak{h} = \{$diagonal matrices w. trace zero$\}$

and

$$\exp : \mathfrak{h} \longrightarrow H$$

$$\begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{pmatrix} \longmapsto \begin{pmatrix} e^{\alpha_1} & & & 0 \\ & e^{\alpha_2} & & \\ & & \ddots & \\ 0 & & & e^{\alpha_n} \end{pmatrix}$$

So

$$L = \left\{ \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} : \begin{array}{c} \alpha_1 + \cdots + \alpha_n = 0 \\ \alpha_i \in 2\pi\mathbb{Z} \end{array} \right\}$$

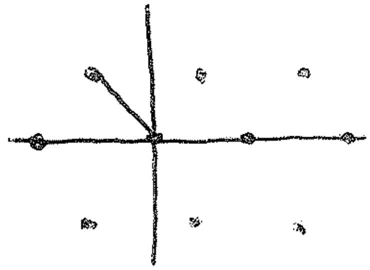
If $\mathfrak{g} = \mathfrak{sl}(3)$ then L has a basis

$$\begin{pmatrix} 2\pi & & \\ & -2\pi & \\ & & 0 \end{pmatrix} \; \& \; \begin{pmatrix} 0 & & \\ & 2\pi & \\ & & -2\pi \end{pmatrix},$$

which are at a $120°$ angle with respect to

$$\langle x, y \rangle = c \operatorname{tr}(xy)$$

$L \subseteq \mathfrak{h}$



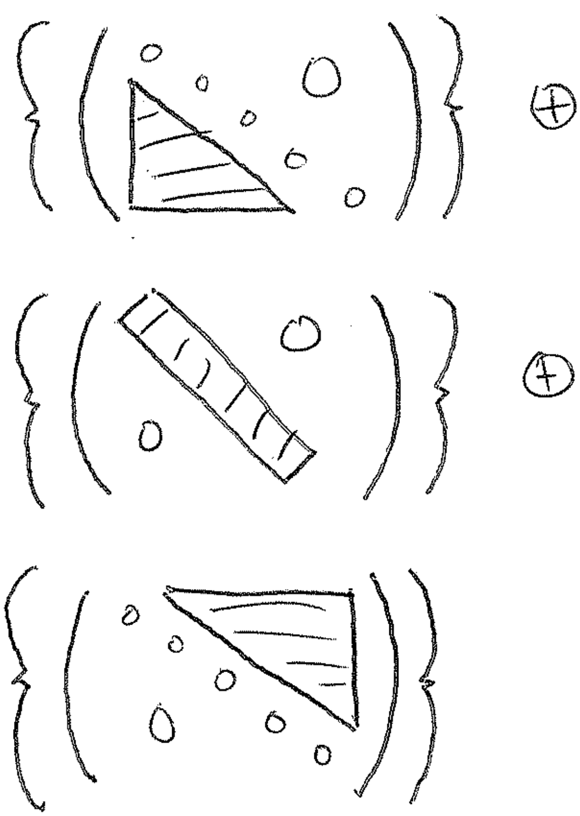
In this general situation we always have:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

as vector spaces where $\mathfrak{h}$ is maximal abelian ("Cartan") & $\mathfrak{n}_\pm$ are maximal nilpotent.

Example: If $\mathfrak{g} = \mathfrak{sl}(n)$ then

$\mathfrak{g} =$



$$\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

In this example

$$[\mathfrak{h}, \mathfrak{n}_\pm] \subseteq \mathfrak{n}_\pm$$

and this holds in general.

In this example of $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{n}_+$ has a basis of elementary matrices e_{ij} $(i < j)$

Ringel - Hall - If $Q = A_n, D_n, E_6, E_7, E_8$ then for q any prime power we have an isomorphism of associative algebras:

$$\mathrm{Hall}(\mathrm{Rep}_q(Q)) \cong U_q \mathfrak{n}_+$$

where

$$\mathrm{Rep}_q(Q) = \hom(Q, \mathrm{FinVect}_{\mathbb{F}_q})$$

and $U_q \mathfrak{n}_+$ is the "q-deformed universal enveloping algebra of $\mathfrak{n}_+$", part of the "quantum group" $U_q \mathfrak{g}$.

But what is $U_q \mathfrak{n}_+$? First, what is $U \mathfrak{n}_+$?

U means "universal enveloping algebra" – the trick for turning Lie algebras into associative algebras. There is a functor

$$\begin{tikzcd} \mathrm{Assoc\,Alg} \arrow[d, "F"] \\ \mathrm{Lie\,Alg} \end{tikzcd}$$

"forgetful functor"

where given $A \in \mathrm{AssocAlg}$, FA has the same underlying vector space with Lie bracket:

$$[a,b] = ab - ba.$$

This functor has a left adjoint:

$$\begin{tikzcd} \mathrm{Assoc\,Alg} \arrow[d, shift left, "F"] \\ \mathrm{Lie\,Alg} \arrow[u, shift left, "U"] \end{tikzcd}$$

i.e. given $A \in \text{AssocAlg}$; $L \in \text{LieAlg}$

$$\hom(L, FA) \cong \hom(UL, A)$$

set of Lie algebra homomorphisms — set of assoc. alg. homomorphisms

Concretely, UL is the assoc. alg. freely generated by elements of L modulo relations:

$$x(\alpha y + \beta z) = \alpha xy + \beta xz$$
$$(\alpha x + \beta y) z = \alpha xz + \beta yz \qquad x, y, z \in L$$
$$xy - yx = [x, y]$$

Then given Lie algebra homomorphism $f: L \to FA$ it extends uniquely to an assoc. alg. homomorphism

$$\tilde{f} : UL \to A$$

$$\begin{tikzcd} UL \arrow[r, dashed, "\exists! \tilde{f}"] & A \\ L \arrow[u, hook] \arrow[r, "f"'] & FA \arrow[u, "\cong"'] \end{tikzcd}$$

Example: $L = \mathfrak{n}_+ \subseteq \mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

$A = \mathrm{End}(\mathbb{C}^n)$

Then $FA = \mathfrak{gl}(n, \mathbb{C})$ - all $n \times n$ matrices w. $[a,b] = ab - ba$

$UL = U\mathfrak{n}_+$

We have an inclusion of Lie algebras,

$$\begin{array}{ccc} f : L & \longrightarrow & FA \\ \parallel & & \parallel \\ \mathfrak{n}_+ & & \mathfrak{gl}(n, \mathbb{C}) \end{array}$$

So we get an assoc. alg. homo:

$$\begin{array}{ccc} \tilde{f} : UL & \longrightarrow & A \\ \parallel & & \parallel \\ U\mathfrak{n}_+ & & \mathrm{End}(\mathbb{C}^n) \end{array}$$

where $\tilde{f}$ sends any $x \in \mathfrak{n}_+$ to the element of $\mathrm{End}(\mathbb{C}^n)$ that it secretly is!

We also have an assoc. alg. homomorphism

$$\mathrm{End}(\mathbb{C}^n) \longrightarrow \mathrm{End}(S\mathbb{C}^n)$$

where $S\mathbb{C}^n$ is the commutative assoc. algebra generated by $\mathbb{C}^n$ - i.e. $\mathbb{C}[x_1, \ldots, x_n]$ — polynomials in n variables. This is the "Fock space" we have been talking about all along! So we get

$$U\mathfrak{n}_+ \longrightarrow \mathrm{End}(\mathbb{C}^n) \longrightarrow \mathrm{End}(S\mathbb{C}^n)$$

which sends $e_{ij} \in \mathfrak{n}_+$ to the "transmotation operator" $a_i^* a_j$. This is the thing we would like to categorify (we have <u>done</u> that, getting groupoidified transmutation operators $A_i^* A_j$ for <u>all</u> i, j) & q-deform (but here we will be stuck with $i < j$, since we are only using $U\mathfrak{n}_+$ instead of $U\mathfrak{g}$).

Now, what is $U_q \mathfrak{n}_+$? As a vector space, it is isomorphic to $U\mathfrak{n}_+$, but it has a different product, $\cdot_q$, depending on $q \in X \subseteq \mathbb{C}$, with

$$\underset{U_q n_+}{x \cdot_q y} \longrightarrow \underset{\text{product in } U n_+}{xy} \quad \text{as } q \to 1$$

Next time we will assume

$$U_q n_+ \cong \mathrm{Hall}(\mathrm{Rep}_q(Q))$$

& show

$$U_q n_+ \cong U n_+$$

as vector spaces in some nice way

& work out the product in $U_q n_+$ & show

$$x \cdot_q y \longrightarrow xy \quad \text{as } q \to 1.$$