Monomials $\longrightarrow$ Polynomials

Monoid $\longrightarrow$ Ring

We have complicated freely generated expressions such as

$$(3x+1)(2x+5) + (x+3)$$

but distributivity allows us to view addition as a finishing touch.

$$\mathrm{Cat} \xrightarrow{F} \text{Cat w. colimits}$$

$$C \longmapsto F(C) = \mathrm{Set}^{C^{op}}$$

$$\mathrm{Set} \xrightarrow{\text{"Free"}} \text{Comm Mon}$$

$$S \longmapsto \mathbb{N}^S$$

Let's look at an example:

$$C = \mathrm{FinSet}_0$$

We saw this C when we were studying the free symmetric monoidal category.

$$F(C) = \mathrm{Set}^{\mathrm{FinSet}_0} = \text{"formal sums of finite sets"}$$

$$\begin{aligned} \mathrm{FinSet}_0 &\longrightarrow \mathrm{Set} \\ N &\longmapsto 2 \cdot N \end{aligned}$$

or

$$N \longmapsto \text{"2-colorings of } N\text{"} = 2^N$$

The power series for $N \longmapsto 2 \cdot N$ is

$$2x + \frac{4}{2}x^2 + \frac{6}{6}x^3 + \frac{8}{24}x^4 + \cdots$$

$$= \sum_N \frac{2N}{N!} x^N$$

For $N \longmapsto 2^N$ we have

$$1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \cdots$$

$$= \sum_N \frac{2^N}{N!} x^N$$

$$\begin{tikzcd} \mathrm{Elt}(G) \arrow[d] & \\ \mathrm{FinSet}_0 \arrow[r, "G"] & \mathrm{Set} \end{tikzcd}$$

$$\begin{tikzcd} (N, \arrow[d, "B"'] & x \text{ in } G(N)) \arrow[d, "G(B)(x) = x'"] \\ (N', & x' \text{ in } G(N')) \end{tikzcd}$$

Since we are dividing by factorials in our power series

we have a special kind of colimits:

"co-invariants of a group action"

"co-equilizer"

See

Joyal, "Especes de structure"

Species of structure