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Lectures on Hall Algebras

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Claim:

$$U_q n_+ \cong \mathrm{Hall}(\mathrm{Rep}_q(Q))$$

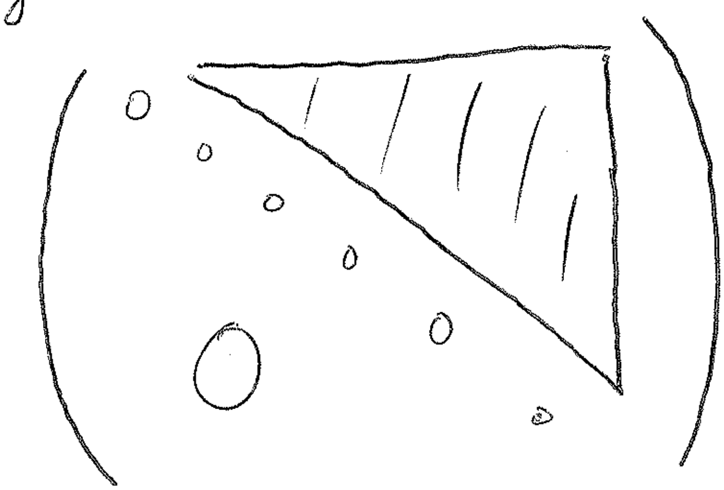
q a prime power

Consider the case

$$Q = A_n = \bullet \to \bullet \to \bullet \to \bullet \to \bullet \quad (n \text{ dots})$$

associated to $\mathfrak{g} = \mathfrak{sl}(n+1)$, with

$n_+ =$



What's Un_+ like? n_+ has a basis of elementary matrices e_{ij} $(i<j)$. Un_+ is generated by the e_{ij} with relations:

$$e_{ij}e_{kl} - e_{kl}e_{ij} = \delta_{jk}e_{il} - \delta_{li}e_{kj}$$

By Poincaré - Birkhoff - Witt, Un_+ has a basis like this:

$$e_{12}^{n_{12}} e_{13}^{n_{13}} e_{14}^{n_{14}} \cdots e_{1\,n+1}^{n_{1\,n+1}} e_{23}^{n_{23}} \cdots$$

$U_q n_+ \cong Un_+$ as a vector space, so we can use the same sort of basis, but the multiplication is different!

So: can we see how Un_+ is isomophic to $H = \mathrm{Hall}(\mathrm{Rep}_q(Q))$ as vector spaces? Recall, H has a basis of isomorphism classes of reps of Q in $\mathrm{Vect}_{\mathbb{F}_q}$. Every rep is a direct sum of indecomposable reps.

We can guess, therefore, that the indecomposable reps correspond to the elementary matrices e_{ij} $(1 \le i < j \le n+1)$; then general reps will correspond to our basis vectors for U_{n_+} & we'll get an isomorphism

$$U_{n_+} \cong H$$

as vector spaces.

Let's prove our guess right. Let

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \longrightarrow \cdots \xrightarrow{f_{n-1}} V_n$$

be an <u>indecomposable</u> rep of Q. What can V_1 be like? It could be 0-dimensional, it could be 1-dimensional, but not 2-or-more-dimensional.

Why? If it were, we could write

$$V_1 = V_1' \oplus V_1''$$

with both V_1', V_1'' nonzero. Then we have:

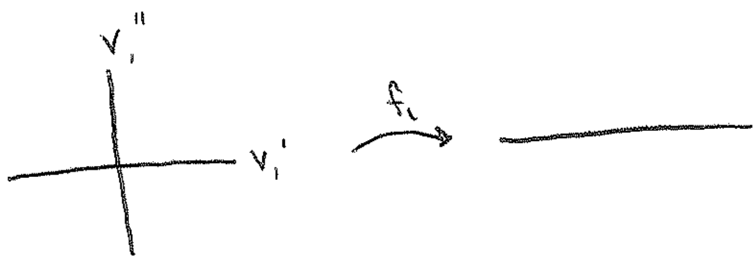
$$V_1' \oplus V_1'' \xrightarrow{f_1' \oplus f_1''} V_2' \oplus V_2'' \xrightarrow{f_2' \oplus f_2''} \cdots \xrightarrow{f_{n-1}' \oplus f_{n-1}''} V_n' \oplus V_n''$$

where

$$V_2' = \operatorname{im} V_1' \quad \& \quad f_1' = f_1|_{V_1'}$$

$$V_2'' = \operatorname{im} V_1'' \oplus W$$

Problem:



$$V_1 = F_q^2 \qquad\qquad V_2 = F_q$$

Then $\operatorname{im} V_1' = \operatorname{im} V_1''$, so $\operatorname{im} V_1' \cap \operatorname{im} V_1'' \neq 0$!

Can we chop V_1 into $V_1' \cap V_1''$ such that $\operatorname{im} V_1' \cap \operatorname{im} V_1'' = 0$? Yes - write

$$V_1 = \ker f_1 \oplus A_1$$

where $f_1|_{A_1}$ is 1-1.

Write $V_1 = V_1' \oplus V_1''$ where $\ker f_1 \subseteq V_1'$.

Then $\operatorname{im} V_1' \cap \operatorname{im} V_1'' = 0$.

Then we can march ahead:

$$V_2' = \operatorname{im} V_1'$$

$$V_2'' = \operatorname{im} V_1'' \oplus W$$

where W is chosen so $V_2' \oplus V_2'' = V_2$.

Etc... but this requires more thought.

Same kind of arguments show that none of V_i can be 2 or more dimensional.

So, an indecomposable rep of Q might look like:

$$\mathbb{F}_q \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{F}_q \xrightarrow{\alpha} \mathbb{F}_q \xrightarrow{\beta} \mathbb{F}_q \xrightarrow{0} 0 \xrightarrow{0} \mathbb{F}_q$$

But this one is decomposable!

It's

$$\mathbb{F}_q \to 0 \to 0 \to \mathbb{F}_q \xrightarrow{\alpha} \mathbb{F}_q \xrightarrow{\beta} \mathbb{F}_q \to 0 \to 0$$

$$\oplus$$

$$0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \mathbb{F}_q$$

What are some indecomposable ones?

$$0 \to 0 \to \mathbb{F}_q \xrightarrow{\alpha} \mathbb{F}_q \xrightarrow{\beta} \mathbb{F}_q \qquad \alpha, \beta \neq 0$$

is okay, but it is isomorphic to:

$$0 \to 0 \to \mathbb{F}_q \xrightarrow{1} \mathbb{F}_q \xrightarrow{1} \mathbb{F}_q$$

as follows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{F}_q \arrow[r, "\alpha"] \arrow[d, "1"'] & \mathbb{F}_q \arrow[r, "\beta"] \arrow[d, "\alpha^{-1}"'] & \mathbb{F}_q \arrow[d, "\beta^{-1}\alpha^{-1}"] \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{F}_q \arrow[r, "1"'] & \mathbb{F}_q \arrow[r, "1"'] & \mathbb{F}_q
\end{tikzcd}$$

This is indecomposable:

$$\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q$$

since it's

$$\mathbb{F}_q \longrightarrow 0$$

$$\oplus$$

$$0 \longrightarrow \mathbb{F}_q$$

Every indecomposable rep is isomorphic to one of these

$$E_{ij} = 0 \longrightarrow 0 \longrightarrow \underset{\substack{\uparrow \\ i^{th}\text{ place}}}{\mathbb{F}_q} \xrightarrow{1} \mathbb{F}_q \xrightarrow{1} \underset{\substack{\uparrow \\ (j-1)^{th}\text{ place}}}{\mathbb{F}_q} \longrightarrow 0 \longrightarrow 0$$

$(1 \leq i < j \leq n+1)$

Example: $\quad 0 \longrightarrow \mathbb{F}_q \longrightarrow 0$

$$i = 2, \quad j = 3$$

Next we'd want to work out the product in the Hall algebra.

Example: $n = 2$ Study short exact sequences:

$$0 \longrightarrow E_{12} \longrightarrow T \longrightarrow E_{23} \longrightarrow 0$$

&

$$0 \longrightarrow E_{23} \longrightarrow T \longrightarrow E_{12} \longrightarrow 0$$

They're very different, so H is noncommutative.

In fact $\mathrm{Hall}(\mathrm{Rep}_q(Q))$ is a Hopf algebra – and the Hopf algebra structure can be described in terms of $\mathrm{Rep}_q(Q)$.

We can hope to groupoidify some of this structure – working with $\mathrm{Rep}_q(Q)_0$.

Ringel studies $U_q \mathfrak{b}$ using Hall algebras, where:

$$\underbrace{n_- \oplus h \oplus n_+}_{\mathfrak{b}}$$

$\mathfrak{b}$ – upper triangular matrices when $\mathfrak{g} = \mathfrak{sl}(n+1)$