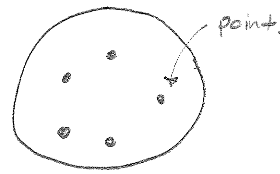
Category Theories:

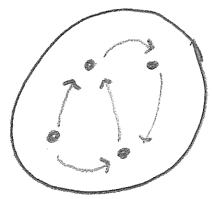
- Unifies mathematics.
- Studies the mathematics of mathematics (similar to mathematical logic)
- Moves towards higher dimensional algebra ("homotopifying" mathematics)

We could build math from set theory which builds up from basic elements



(0-dimensional)

But category theory uses one more step (arrows and points)



(1-dimensional)

$\boxed{\text{Def}}$ A category C consists of

- A class $ob(C)$ of objects (Read Zermelo Frankel for def of class)
  (if $x \in ob(C)$ we write $x \in C$
- Given $x, y \in C$, there is a set $hom(x,y)$, called a homset whose elements are called morphisms or arrows from x to y

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \end{tikzcd}$$

If $f \in hom(x,y)$ we write f maps x to y.

- Given $f: x \to y$ $g: y \to z$ there is a morphism called their composite $g \circ f: x \to z$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "g\circ f"'] & y \arrow[d, "g"] \\ & z \end{tikzcd}$$

- Composition is associative : $(h\circ g)\circ f = h\circ(g\circ f)$ if either side is well-defined.

$$\begin{tikzcd}
\cdot \arrow[r, "g"] \arrow[dr, dashed, "h\circ g"] & \cdot \arrow[d, "h"] \\
\cdot \arrow[u, "f"] \arrow[ur, "g\circ f"] & \cdot
\end{tikzcd}$$

$h\circ(g\circ f) = (h\circ g)\circ f$

- For any object $x \in C$, there is an identity morphism $1_x : x \to x$

$$\begin{tikzcd}
x \arrow[loop right, "1_x"]
\end{tikzcd}$$

s.t. $1_x \circ f = f$ for any $f: x' \to x$

$f \circ 1_x = f$ for any $f: x \to x'$

Examples of Categories

Categories of mathematical objects

For any kind of mathematical object, there is a category with objects of that kind and morphisms being the structure preserving maps between objects of that kind.

- **Set** is the category with sets as objects and functions as morphisms.
- Grp is the category with groups as objects and homomorphisms as morphisms.
- For k any field, Vect_k of vector spaces over k and linear maps.
- Ring is the category of rings and ring homomorphisms.

These are categories of "algebraic" gadgets, namely:

- A set } stuff
- with operations } structure
- obeying equations } properties

With morphisms being functions that preserve the operations.

All of this is formalized in "Universal algebra", using "algebraic theories".

There are also categories of non-algebraic gadgets:

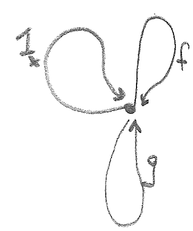
- **Top**, the category of topological spaces and continuous maps.
- **Met**, the category of metric spaces and continuous maps.
- **Meas**, the category of measurable spaces and measurable maps.

Categories as mathematical objects

There are lots of small, more manageable categories:

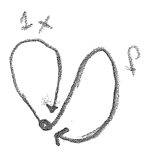
$\boxed{\text{Def}}$ A monoid is a category with one object

(then $\hom(X,X)$ for this object X is a set w. associative product and unit)



All interesting activity will be with the maps.

example



with $1_X \circ f = f$, $f \circ 1_X = f$, $f \circ f = 1_X$ — usually called $\mathbb{Z}_2$ or $\mathbb{Z}/2$ with $\circ$ = or

or

$1_X \circ f = f$, $f \circ 1_X = f$, $f \circ f = f$ — $\circ$ = And, 1_X = True, f = false

$\boxed{\text{Def}}$ A morphism $f : X \to Y$ is an isomorphism if it has an inverse $g : Y \to X$ s.t. $g \circ f = 1_X$, $f \circ g = 1_Y$

$\boxed{\text{Def}}$ A category where all morphisms are isomorphisms is called a groupoid.

Example "the groupoid of finite sets" is obtained from finset, the category with finite sets as objects and functions as morphisms, by throwing **out** all morphisms except isomorphisms (i.e. bijections).

$\boxed{\text{Def}}$ A monoid that is a groupoid is called a group oid.

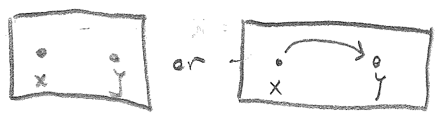
(think of usual elts of a group as morphisms)

$\boxed{\text{Def}}$ A category with only identity morphisms is a discrete category.

So any set is the set of objects of some discrete category in a unique way.

So a discrete category is "essentially the same" as a set!

$\boxed{\text{Def}}$ A preorder is a category with at most one morphism in each homset.



If there is a morphism $f: x \to y$ is a preorder "$x \leq y$"
if not we say "$x \not\leq y$"

For a preorder, the category just says:

- Composition : $x \leq y$ & $y \leq z \Rightarrow x \leq z$
- associativity is automatic
- identities : $x \leq x$ always
- left and right laws are automatic

Structure turned into the properties. Since there is at most one morphism from x to y.

we are not getting antisymmetry which says if $x \leq y$, $y \leq x$ then $x = y$

Def An equivalence relation is a preorder that is also a groupoid.

(turn the arrow $x \to y$ to $x \leftarrow y$ to get $x \leq y$ iff $x \geq y$)

Prop A preorder C is a groupoid iff $x \leq y \Rightarrow y \leq x \;\forall x, y \in C$

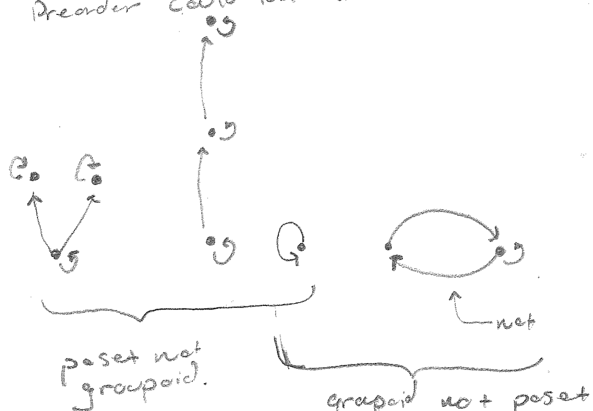
Here we will have transitivity, reflexivity, and symmetry

Prop A preorder C is skeletal, (i.e. isomorphic objects are equal) iff this extra law holds:

$$x \leq y \;\&\; x \geq y \Rightarrow x = y \quad \forall x, y \in C$$

In this case we say C is a poset.

<u>ex</u> Preorder could look like



Since Categories can be seen as mathematical objects, we should define maps between them.

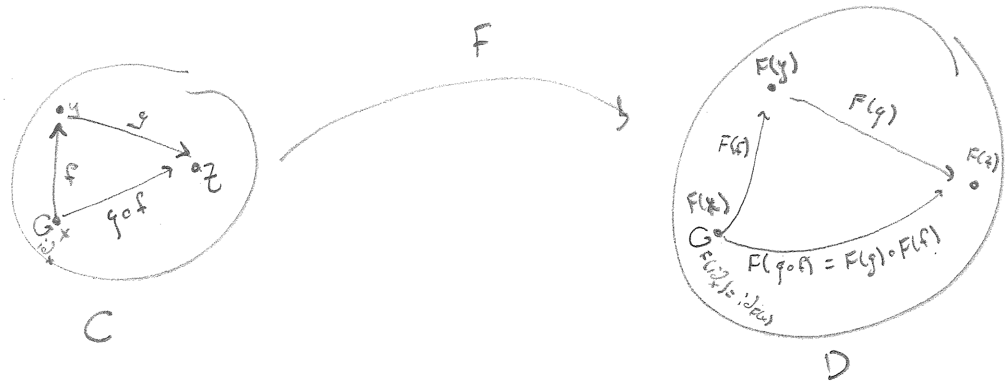
Def Given categories C, D, a functor $F: C \to D$ consists of:

- A function called $F: Ob(C) \to Ob(D)$: if $x \in C$ then $F(x) \in D$ } objects in C sent to objects in D.
- functions called $F: hom(x, y) \to hom(F(x), F(y))$: if $f: x \to y$ then $F(f): F(x) \to F(y)$ } morphisms in C to morphisms in D.

Such that

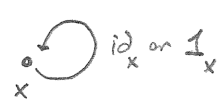
- $F(g \circ f) = F(g) \circ F(f)$ when either side is defined.
- $F(id_x) = id_{F(x)}$ $\forall x \in C$

So, a functor looks like this:



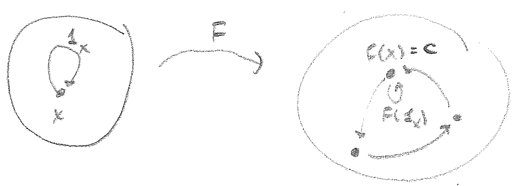
ex There is a category called "1"

It looks like this:

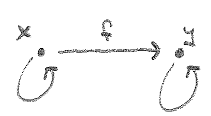


Q what is a functor $F: 1 \to C$, where C is any category

it's really just an object in C since for any $c \in C$ $\exists!$ $F: 1 \to C$ s.t. $F(x) = c$

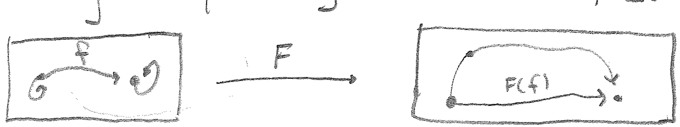


ex There is a category called "2"



Q what is a functor $F: 2 \to C$? It's a morphism (or arrow)!

For any morphism $g: c \to c'$ in C, $\exists!$ functor $F: 2 \to C$ s.t. $F(f) = g$



Prop If $F: C \to D$, $G: D \to E$ are functors, then you can compose them

$G \circ F: C \to E$ and $(H \circ G) \circ F = H \circ (G \circ F)$

Also for any category C there is an identity functor with

$1_C(x) = x \quad \forall x \in C$

$1_C(f) = f \quad \forall f: x \to y$ in C

and $F \circ 1_C = F \quad \forall F: C \to D$

$1_C \circ H = H \quad \forall H: D \to C$

Def Cat is the category whose objects are "small" categories and whose morphisms are functors.

A "small" category is one with a set of objects.

So Set or Grp or Ring are not small, but $\mathbf{1}$ and $\mathbf{2}$ are small.

Mathematics inside a category.

A lot of math is done in the Category Set, (sets and functions)

The idea is to generalize all that stuff to other categories by replacing Set by a general category C.

In Set, we have one-to-one and onto functions.

In a category C, we generalize them to epimorphisms (epis) & monomorphisms (monos)

Def A morphism $f: X \to Y$ is a mono if $\forall\ g,h: Q \to X$ we have

$f \circ g = f \circ h \Rightarrow g = h$

$Q \overset{g}{\underset{h}{\rightrightarrows}} X \xrightarrow{f} Y$

So f does not make things equal which were not before.
f is 1-1 since it does not squash things down.

Prop In Set, a morphism is monic iff it is a one-to-one function.

Similarly f is monic iff $f^{-1} \circ f = 1_X$ (Left inverse)

Notice that if we turn the arrows around in mono we get.

Def A morphism $f: Y \to X$ is a epi if $\forall\, g,h: X \to Q$

we have

$g \circ f = h \circ f \Rightarrow g = h$

$$Q \underset{h}{\overset{g}{\leftleftarrows}} X \xleftarrow{f} Y$$

{ If g and h were different on some $x \in X$, we'll only know if f hits each elt. in X.

Prop In Set, a morphism is epic iff it is an onto function.

Similarly f is epi iff $f \circ f^{-1} = I_Y$ (Right inverse) (Using Axiom of choice)

Prop a morphism f is iso iff it is both mono & epi

Prop In Ring (rings & ring homoms)

$f: \mathbb{Z} \longrightarrow \mathbb{Q}$ is mono & epi but not an iso?
$n \longmapsto n$

proof there is no ring homomorphism $g: \mathbb{Q} \to \mathbb{Z}$ since it would send $\frac{1}{2}$ to some mult. inverse of 2 in $\mathbb{Z}$ and there isn't one.

- why is f mono?

$R \underset{h}{\overset{g}{\rightrightarrows}} \mathbb{Z} \xrightarrow{f} \mathbb{Q}$ need to show if $f \circ g = f \circ h$ then $g = h$

since $f \circ g(r) = f \circ h(r)$ $\forall\, r \in R$ since f is 1-1 (as a function) we know $g(r) = h(r)$
so $g = h$

- why is f epi?

$\mathbb{Z} \xrightarrow{f} \mathbb{Q} \underset{h}{\overset{g}{\rightrightarrows}} \mathbb{R}$ $g \circ f = h \circ f \Rightarrow g = h$

i.e. if $h(n) = g(n)$ $\forall\, n \in \mathbb{Z} \subseteq \mathbb{Q}$ then $h = g$

now $g(1) = g\left(\frac{q}{q}\right) = g(q) \cdot g\left(\frac{1}{q}\right)$ so $g(q)$ is a mult. inverse of $g(1/q)$

so $g(p/q) = g(p) \cdot g\left(\frac{1}{q}\right) = g(p)/g(q)$

so g (and hence h) is determined by its values on integers; since they agree on $\mathbb{Z}$ they're equal.

Puzzle: In Top, find $f: X \to Y$ that is epic and monic but not an isomorphism.

[Limits and Colimits]

These are ways of building new objects in a category C from diagrams in C.

e.g.
$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$$

An example of a limit is:

$\boxed{\text{Def}}$ Given $X, Y \in C$, a product of them is an object Z equipped with morphisms called projections to X & Y s.t.

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

Product has universal property.

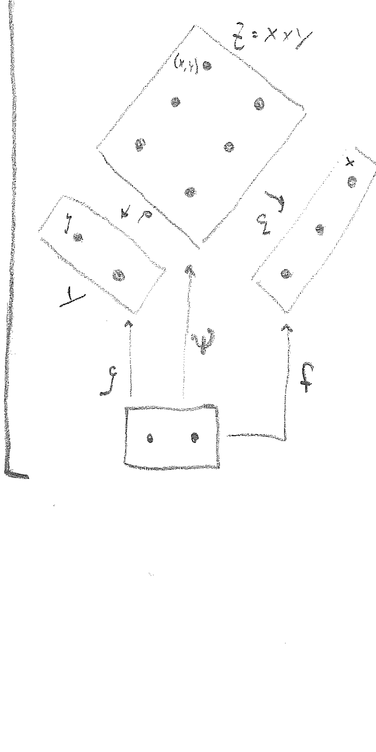
for any candidate
$$\begin{tikzcd} & Q \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

There $\exists!\ \Psi: Q \to Z$ s.t. the diagram commutes

$$\begin{tikzcd} Z \arrow[d, "p"'] \arrow[dr, "q"' near end] & Q \arrow[l, bend right, "\Psi"'] \arrow[dl, "f" near end] \arrow[d, "g"] \\ X & Y \end{tikzcd}$$

i.e. $f = p \circ \Psi$ and $g = q \circ \Psi$

Secretly we think:



The definition of Coproduct is just the same, but with all arrows reversed.

$\boxed{\text{Prop}}$ In Set, we get a product of X & Y by taking $X \times Y = \{(x,y) : x \in X, y \in Y\}$
w. $p(x,y) = x$, $q(x,y) = y$

Proof let $\psi : Q \to X \times Y$ be $\psi(q) = (f(q), g(q))$
so $p \circ \psi = f$ & $q \circ \psi = g$ & ψ is the unique map obeying these equations.

notice we could also take Z to be any set S that is isomorphic to $Z = X \times Y$.
by $\alpha : S \to X \times Y$

then we have the product but using S instead of $X \times Y$.

$$\begin{tikzcd} & S \arrow[d, "\alpha"] \arrow[ddl, bend right, "p \circ \alpha"'] \arrow[ddr, bend left, "q \circ \alpha"] & \\ & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

You must, however, remember to check that the previous diagram will commute with S replacing $X \times Y$.

So any object isomorphic to a product can also be a product.

$\boxed{\text{Prop}}$ Suppose
$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Z \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ X & & Y \end{tikzcd}$$
are both products of X & Y

then W and Z are isomorphic.

Proof we know
$$\begin{tikzcd} W \arrow[d, "p"'] \arrow[dr, "q"' near start] & Z \arrow[l, "\exists! \psi"'] \arrow[d, "q'"] \arrow[dl, "p'" near start] \\ X & Y \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "\exists! \varphi"] \arrow[d, "p"'] \arrow[dr, "q"' near start] & Z \arrow[d, "q'"] \arrow[dl, "p'" near start] \\ X & Y \end{tikzcd}$$

Suffices to show ψ and φ are inverses?

why is $\psi \circ \varphi : W \to W$ the identity?

If we can show this, an analogous argument will show $\varphi \circ \psi : Z \to Z$

But we also have

$$\begin{tikzcd} W \arrow[d, "p"'] \arrow[dr, "q" near start] & W \arrow[l, bend right, "\exists!\, 1_W"'] \arrow[d, "q"] \arrow[dl, "p"' near start] \\ X & Y \end{tikzcd}$$

So $1_W : W \to W$ makes the diagram commute

But $\psi \circ \varphi$ does the same as 1_W

$$\begin{tikzcd} W \arrow[rr, "\varphi"] \arrow[dr, "p"'] \arrow[drrr, "q" near start] & & Z \arrow[rr, "\psi"] \arrow[dl, "p'"] \arrow[dr, "q'"] & & W \arrow[dl, "q"] \arrow[dlll, "p"' near start] \\ & X & & Y & \end{tikzcd}$$

$\boxed{\text{prop}}$ if a morphism is an iso, then it's a mono and an epi.

(we've seen the converse is false)

<u>proof</u> $f : X \to Y$ has a left inverse f^{-1} so

$f \circ g = f \circ h \Rightarrow f^{-1} \circ f \circ g = f^{-1} \circ f \circ h \Rightarrow g = h \quad \forall g, h$

so f is a mono.

$f : X \to Y$ has a right inverse f^{-1} so

$g \circ f = h \circ f \Rightarrow g \circ f \circ f^{-1} = h \circ f \circ f^{-1} \Rightarrow g = h \quad \forall g, h$

$\boxed{\text{Def}}$ A morphism w. a left inverse is called a split monomorphism.

A morphism w. a right inverse is called a split epimorphism.

In Set, every mono or epi splits, but we saw this was not true in Ring or Top.

Coproducts.

$\boxed{\text{Def}}$ Given objects X & Y, a coproduct of X & Y is an object Z equipped with morphisms i, j which is universal:

$$\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & Z & \end{tikzcd}$$

In other words, for any diagram:

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Q & \end{tikzcd}$$

$\exists!\ \psi : Z \to Q$ making the diagram commute.

So

$$\begin{tikzcd} X \arrow[d, "i"'] \arrow[drr, "f"] & & Y \arrow[dll, "j"'] \arrow[d, "g"] \\ Z \arrow[rr, "\exists!\ \psi"'] & & Q \end{tikzcd}$$

and $f = \psi \circ i$
$g = \psi \circ j$

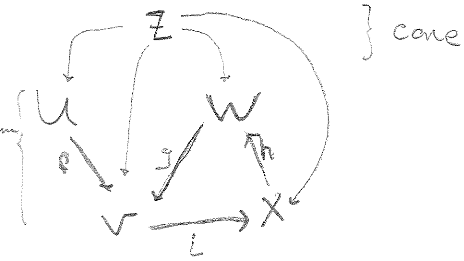
$\boxed{\text{Prop}}$ In Set, a coproduct of X & Y is their disjoint union $X + Y = X \times \{0\} \cup Y \times \{1\}$

with $i : X \to X + Y$, $x \mapsto (x, 0)$; $j : Y \to X + Y$, $y \mapsto (y, 1)$

	Products $(\times)$	coproducts $(+)$	
Set	Cartesian product $S \times T$	disjoint union $S \sqcup T = S + T$	
Top	Cartesian product $X \times Y$ w. product Top	disjoint union $X \sqcup Y$	
Gp	$G \times H$ product of groups	free product $G * H$	
AbGp	$A \oplus B$	$A \oplus B$	Abelian Categories
Vect_k	$V \oplus W = V \times W$ Direct Sum	$V \oplus W$	

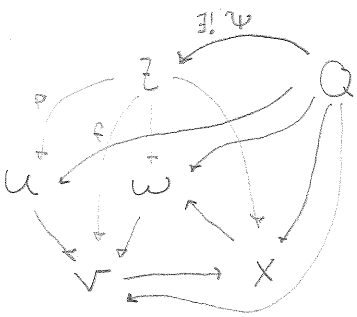
General limits and colimits

Given any diagram in a category C :



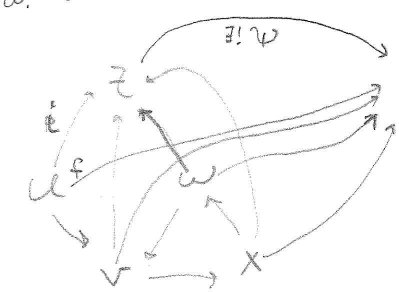
a cone over the diagram is a choice of morphisms from Z to each object in the diagram s.t. all newly formed triangles commute.

A limit of a diagram is a cone that is universal, i.e. given any competitor (candidate), i.e. another cone over the same diagram: $\exists!\ \psi: Q \to Z$ s.t. all triangles including ψ commutes; If U is an object in the diagram and $p: Z \to U$ is the morphism in the universal cone and $f: Q \to U$ is the morphism in the competitor then



$f = p \circ \psi$

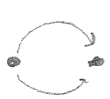
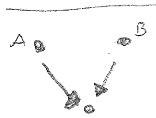
a cocone is a cone w. all new arrows turned around. So morphisms go to Z instead of from Z



{and $f = \psi \circ i$}

and a Colimit is a universal cocone!

<u>Ex</u> famous diagrams

Diagram	Limit	Colimit
	Binary Product	Binary Coproduct
	Equalizer	Coequalizer
	pullback	object C

$A \leftarrow C \rightarrow B$	object C	pushout
A •	A	B
(empty)	terminal object: 1	Initial object: 0

what is the limit of the empty diagram?

$$\begin{tikzcd} Z & Q \arrow[l, bend right, "\exists! \psi"'] \end{tikzcd}$$

what is a cone? An objects Z s.t. $\forall$ objects Q

$\exists! \ \psi : Q \to Z$

This Z is the terminal object.

- In Set, any singleton is a terminal object.
- In Vect_k, any zero dimensional vector space is a terminal object.

Similarly, an initial object Z is one s.t. for any object Q $\exists! \ \psi : Z \to Q$

- In Set, the empty set is an initial object.
- In Vect_k, any zero dimensional vector space is an initial object.

More generally, in any abelian category, initial objects are terminal.

[Equalizers]

$\boxed{\text{Def}}$ A limit of this diagram : $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$ is called an equalizer

$\boxed{\text{prop}}$ In the category of Set the equalizer of $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$

is

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & & B \end{tikzcd}$$

where $q = f\circ p = g \circ p$

with $Z = \{ a \in A : f(a) = g(a) \}$ where $p : Z \to A$ has $p(a) = a \ \forall a \in Z$

notice: q is forced to be $f\circ p = g\circ p$ so we usually will not draw q.

we'll write an equalizer like

$$\begin{tikzcd} Z \arrow[r, "p"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$

proof we need to show the cone is universal. So take a competitor and check universality property.

$$\begin{tikzcd} Z \arrow[d, "q"'] & Q \arrow[l, bend right, "\exists!\Psi\ (\text{wts})"'] \arrow[dl, "p'"] \\ A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$

we'll show $\exists!\ \Psi : Q \to Z$ making $p \circ \Psi = p'$

now $p \circ \underbrace{\Psi(q)}_{\in Z} = \Psi(q)$ since $p(a) = a \ \forall a \in Z$

so $\Psi \circ p = p'$ simply says $\Psi(q) = p'(q) \ \forall q \in Q$

Thus $\Psi = p'$ is our choice.

$\boxed{\text{Prop}}$ In AbGp or $Vect_k$, the equalizer of
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$

is $\ker(f-g) = \{a \in A ; f(a) = g(a)\}$

$= \{a \in A : f(a) - g(a) = 0\}$

proof: same as before.

$\boxed{\text{prop}}$ If $Z \xrightarrow{i} A \begin{smallmatrix}\xrightarrow{f}\\ \xrightarrow[g]{}\end{smallmatrix} B$ is an equalizer then it is monic.

Moral: Monics and limits obey lots of nice relationships as do epics with colimits.

proof say we have an equalizer. To check i is monic, we consider
$$\begin{tikzcd} Y \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & Z \arrow[r, "i"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$
if $i \circ h = i \circ k$ show $h = k$

So
$$\begin{tikzcd} Z \arrow[d, "i"'] & Y \arrow[l, bend right, "h"'] \arrow[l, "k"] \arrow[dl, "i \circ h = i \circ k"] \\ A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \\ B & \end{tikzcd}$$

Y is a competitor to Z

So since universality says $\exists ! \varphi : Y \to Z$ making everything commute, and we have two choices h, k, we have $\varphi = h = k$.

[Coequalizers]

$\boxed{\text{Def}}$ A Coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is a universal cocone over this diagram,

i.e.
$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[d, "i"] \\ & Z \end{tikzcd}$$
commuting

s.t. universality holds. i.e. $\exists! \varphi\ Z \to Q$ making everything commute.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[dl] \arrow[d] \\ Q & Z \end{tikzcd}$$

$\boxed{\text{prop}}$ In Set, the coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$

is $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{i} Z$ where $Z = B/\sim$ where $\sim$ is the coursest relation

s.t. $f(a) \sim f(a)\ \forall\ a \in A$ and $i : B \to Z$
$b \mapsto [b]$

<u>proof</u> $i \circ f = i \circ g$ with this definition. Why is it universal?

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[dl, "i'"'] \arrow[d, "i"] \\ Q & Z \arrow[l, dashed] \end{tikzcd}$$

why $\exists! \psi : Z \to Q$? we need $\varphi \circ i = i'$ $\quad \varphi(i(b)) = i'(b)$
$\Rightarrow \psi([b]) = i'(b)$

so φ is unique if it exists. To show existence we need to show it is well defined:

$[b] = [b']$ then $i'(b) = i'(b')$
$\Updownarrow$
$b \sim b'$
$\Updownarrow$

$\boxed{\text{RIP}}$

Find the good proof

$\boxed{\text{Prop}}$ In AbGp or Vect_k the coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is

the $\text{coker}(f-g) = B/C$ where $C = \text{im}(f-g)$

$\boxed{\text{Prop}}$ If $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{p} Z$ is a coequalizer, then p is epic.

proof same as the proof of the "dual" prop on P. 16

[Pullbacks] The limit of this diagram:

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is called a pullback.

$$\begin{tikzcd} A \times_C B \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

the object here is called "A times B over C" or the "fibered product". We only need to draw morphisms to A & B called projections.

we write

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

when Z is a pullback

$\boxed{\text{prop}}$ In Set the pullback of

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is $A \times_C B = \{(a,b) \in A \times B : f(a) = g(b)\}$

with $p : A \times_C B \to A$, $(a,b) \mapsto a$ $\quad q : A \times_C B \to B$, $(a,b) \mapsto b$

This is clearly a cone, to show it is universal, use the next prop. ☒

$\boxed{\text{prop}}$ Given

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

and if the product $A \times B$ exists and if this equalizer exists:

$$\begin{tikzcd} Z \arrow[r, "i"] & A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

where $i : Z \to A \times B$ is the equalizer of $A \times B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} C$

then this is the pullback

$$\begin{tikzcd} Z \arrow[r, "\pi_2 \circ i"] \arrow[d, "\pi_1 \circ i"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Back to the previous

<u>proposition</u>: In Set, a coequalizer of $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$

is the set $Z = B/\sim$ where $\sim$ is the finest equivalence relation

on B s.t. $f(a) \sim g(a)$ $\forall a \in A$

with map i: $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, "i"] & Z \end{tikzcd}$ s.t. $i(b) = [b]$.

<u>proof</u> Consider a competitor Q s.t.

$i' \circ f = i' \circ g$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right] & B \arrow[r, "i"] \arrow[dr, "i'"'] & Z \arrow[d, "\exists! \psi"] \\ & & Q \end{tikzcd}$$

we'll show $\exists! \ \psi : Z \to Q$

Lets try $\psi[b] = i'(b)$ $\forall b \in B$

If ψ is well defined, the diagram commutes since $\psi(i(b)) = i'(b)$

furthermore, ψ is unique since the formula specifies its value.

why is ψ well defined?

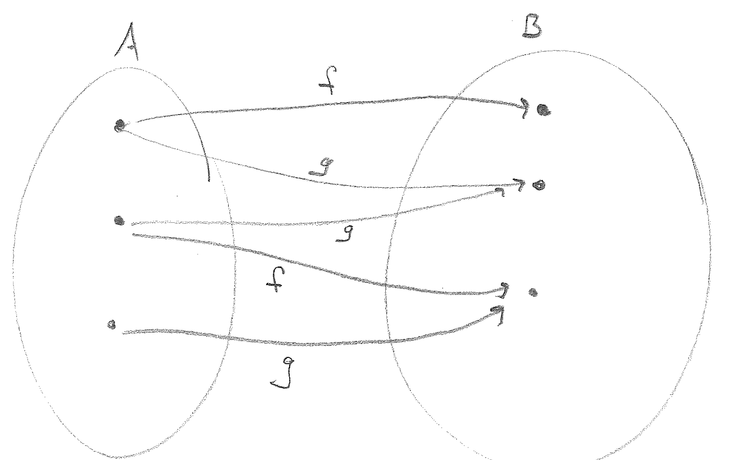
Assume $b \sim b'$, we'll show

$i'(b) = i'(b')$

If $b \sim b'$ we have

$b = b_1 \sim b_2 \sim \dots \sim b_n = b'$

where $\downarrow$



(the following are equivalent by transitivity) either $b_i = f(a)$ & $b_{i+1} = g(a)$ or $b_i = g(a)$ & $b_{i+1} = f(a)$ for some a depending on i

we need to check $i'(b_i) = i'(b_{i+1})$ for each $i = 1, \ldots, n-1$.

case 1 $b_i = f(a)$ & $b_{i+1} = g(a)$

in which case $i'(b_i) = i'(f(a)) = i'(g(a)) = i'(b_{i+1})$

case 2 $b_i = g(a)$ & $b_{i+1} = f(a)$ works similarly. ☑

Now after that detour, we return to pullbacks.

$\boxed{\text{Prop}}$ To compute a pullback of
$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$
it suffices to take a product of $A \times B$:

$$\begin{tikzcd} A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

and form the equalizer: $Z \xrightarrow{i} A \times B \overset{f\circ\pi_1}{\underset{g\circ\pi_2}{\rightrightarrows}} C$

giving the desired pullback:
$$\begin{tikzcd} Z \arrow[r, "\pi_2\circ i"] \arrow[d, "\pi_1\circ i"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Proof

Notice that the last square commutes since $f\circ\pi_1\circ i = g\circ\pi_2\circ i$, so its a candidate for being the pullback. To show it is universal, consider a competitor, Q ⤵

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!\psi"] \arrow[ddrrr, bend left, "q"] \arrow[dddrr, bend right, "p"'] & & & \\
& Z \arrow[dr, "i"] \arrow[drr, bend left, "\pi_2\circ i"] \arrow[ddr, "\pi_2\circ i"'] & & \\
& & A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\
& & A \arrow[r, "f"'] & C
\end{tikzcd}$$

How do we get $\exists!\ \psi: Q \to Z$ making the newly formed triangles commute.

By the universal property of the product, we get:

$$\begin{tikzcd}
Q \arrow[dr, "\exists!\varphi"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \\
& A &
\end{tikzcd}$$

$\exists!\ \varphi$ making the diagram commute.

By the universal property of the equalizer:

$$\begin{tikzcd}
Z \arrow[r, "i"] & A\times B \arrow[r, shift left, bend left, "f\circ\pi_1"] \arrow[r, shift right, "g\circ\pi_2"'] & C \\
& Q \arrow[ul, dashed, "\psi"] \arrow[u, "\varphi"'] &
\end{tikzcd}$$

$\exists!\ \psi: Q \to Z$

notice Q is a competitor by
$$\begin{aligned} f\circ\pi_1\circ\varphi &= f\circ p \\ &= g\circ q \\ &= g\circ\pi_2\circ\phi \end{aligned}$$

s.t. $\varphi = i\circ\psi$

why does this imply?:

① $\pi_1\circ i\circ\psi = p$

② $\pi_2\circ i\circ\psi = q$

③ $\exists!\ \psi$ making ① & ② true.

we get ① and ② by $\pi_1\circ\varphi = p$ and $\pi_2\circ\varphi = q$

③ follows from uniqueness of ψ and φ.

$\boxed{\text{Prop}}$ In Set, a pullback of

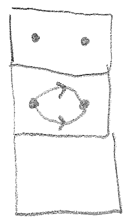
$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is $Z = \{(a,b) \in A \times B : f(a) = g(b)\}$ with obvious maps to $A \times B$.

Proof previous prop combined w. our description of products and equalizers in Set.

In fact, a category has limits for all finite diagrams iff it has:

- products $A \times B$
- equalizers
- terminal object 1



$\boxed{\text{Prop}}$ If this is a pullback:

$$\begin{tikzcd} A \times_C B \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

and g is a monomorphism, then p is a monomorphism.

proof Assume g is a mono.

Then

$$\begin{tikzcd} X \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] & \\ A \times_C B \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

need $p \circ h = p \circ k$ to imply $h = k$

we have $p \circ h = p \circ k \Rightarrow f \circ p \circ h = f \circ p \circ k$

$\Rightarrow g \circ q \circ h = g \circ q \circ k$

by g mono $\Rightarrow q \circ h = q \circ k$

notice X is a competitor to the pullback, since $p \circ h = p \circ k$ and $q \circ h = q \circ k$

so $\exists! \, \psi : X \to A \times_C B$ making the diagram commute.

$\Rightarrow h = k$ since h and k make the diagram commute.

$\boxed{\text{Prop}}$ Given

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

if α & β are pullbacks, so is the combined square $\alpha\beta$

if β and $\alpha\beta$ are pullbacks, so is α.

[Mathematics Between Categories]

Recall that given categories C & D a functor $F: C \to D$ is a map sending objects $c \in C$ to $F(c) \in D$, morphisms $f: c \to c'$ in C to morphisms $F(f): F(c) \to F(c')$ in D, preserving composition; $F(f \circ f') = F(f') \circ F(f)$ and identities: $F(1_c) = 1_{F(c)}$

$$\begin{tikzcd} \mathrm{Rng} \arrow[dr, "U_3"'] & & \mathrm{Vect}_k \arrow[dl, "U_4"] & & \\ & \mathrm{AbGp} \arrow[d, "U_2"] & & & \\ & \mathrm{Grp} \arrow[d, "U_1"] & \mathrm{Top} \arrow[dl, "U_5"'] & \mathrm{Set}^2 \arrow[dll, "U_6"] & \\ & \mathrm{Set} & & & \end{tikzcd}$$

There are many "forgetful functors" going from categories with more structure (properties) to categories with less.

eg: $U_1 : \mathrm{Grp} \to \mathrm{Set}$ sends any Group G to its underlying set, and any homomorphism to its underlying function.

eg: Given categories C & D form a category $C \times D$ where objects are ordered pairs (c, d) with $c \in C$ & $d \in D$ and morphisms (f, g) with f a morphism in C and g in D:

given $\begin{array}{l} f: c \to c' \\ g: d \to d' \end{array}$ then $(f,g) : (c,d) \to (c', d')$

we define $(f,g) \circ (f', g') = (f \circ f', g \circ g')$ and $1_{(c,d)} = (1_c, 1_d)$

In fact, $C \times D$ is the product of objects $C, D \in \mathrm{Cat}$, which is the category w.

- (small) categories as objects
- functors as morphisms.

Among other things, this means we have projections

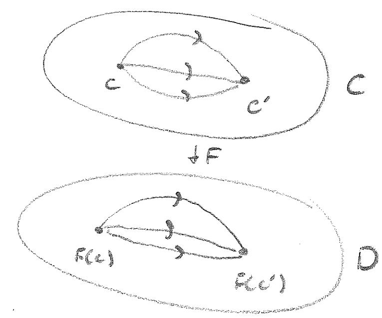
$$\begin{tikzcd} & C \times D \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C & & D \end{tikzcd}$$

Set is a large category but we can still define $Set^2 = Set \times Set$ w. pairs of sets as objects.

In the chart (P.23) let $u_6 : Set^2 \to Set$, $(S,T) \mapsto S$ be the projection onto the first component.

Functions are nice in two basic ways: 1-1 & onto.

Functors can be nice in 3 ways:



$\boxed{\text{Def}}$ A functor $F: C \to D$ is faithful if for any $c, c' \in C$

$F: \hom(c,c') \to \hom(F(c), F(c'))$ is one-to-one

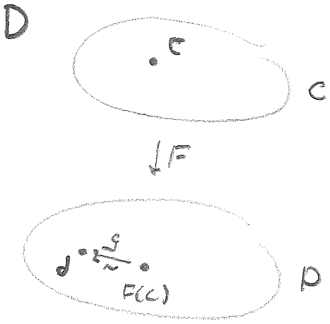
A functor $F: C \to D$ is full if for any $c, c' \in C$

$F: \hom(c,c') \to \hom(F(c), F(c'))$ is onto

A functor $F: C \to D$ is essentially surjective if for any $d \in D$

$\exists\, c \in C$ s.t. $F(c) \cong d$.

i.e. $\exists$ an isomorphism $g: F(c) \to d$ in D



e.g. Compare $FinVect_{\mathbb{R}}$ to the category C, with:

- $\{0\}, \mathbb{R}, \dots$ as objects
- all linear maps between these as morphisms.

There is a functor

$F: C \to D$ and for morphisms
$\mathbb{R}^n \to \mathbb{R}^n$ $f: \mathbb{R}^n \to \mathbb{R}^m \longmapsto f: \mathbb{R}^n \to \mathbb{R}^m$

It is clearly faithfull and full.

F is not surjective on objects, but is essentially surjective!

Later we'll define "equivalent" categories & see that if $F: C \to D$ is faithful, full, and essentially surjective.

$\boxed{\text{Def}}$ A functor $F: C \to D$ forgets nothing if it is faithful, full and essentially surjective.

$\boxed{\text{Def}}$ A functor $F: C \to D$ forgets (at most) properties if it is faithful and full.

$\boxed{\text{Def}}$ A functor $F: C \to D$ forgets structure if it is faithful

$\boxed{\text{Def}}$ A functor $F: C \to D$ in general forgets (at most) stuff

e.g: Refer to (P.23)

$U_1: Grp \to Set$ forgets structure (and therefore properties)

so it is faithful and ~~not~~ full.

given $f, f': G \to G'$ in Grp $U_1(f) = U_1(f') \Rightarrow f = f'$

It is not full since we have functions $f: U_1(G) \to U_1(G')$ that do not come from group functions.

e.g $U_2: AbGp \to Grp$ forgets (some) properties. ($ab = ba$)

So U_2 should be faithfull and full

if you have any group hom. $f: U_2(A) = U_2(A')$ then $f = U(f')$ for some hom. of abelian groups $f': A' \to A'$

But it's not essentially surjective: if G is non abelian, then $G \not\cong U_2(A)$ for any $A \in AbGp$

e.g: $U_6 : Set^2 \to Set$ forgets stuff

we can have 2 different morphisms $(f,g), (f,g') : (S,S') \to (T,T')$

but $U_6(f,g) = U_6(f,g')$ since both equal f.

In our chart, every forgetful functor $U : C \to D$ has a "left adjoint"

$F : D \to C$ which "freely creates" the stuff, structure or properties U forgot.

e.g: $F_1 : Set \to Grp$ takes a set S and forms the free group on S, $F_1(S)$

$F_2 : Grp \to Abgp$ by modding out by the commutator subgroup.

so F_2 abelianizes any group G: $F_2(G) = G/\langle xyx^{-1}y^{-1}\rangle$

$F_6 : Set \to Set^2$

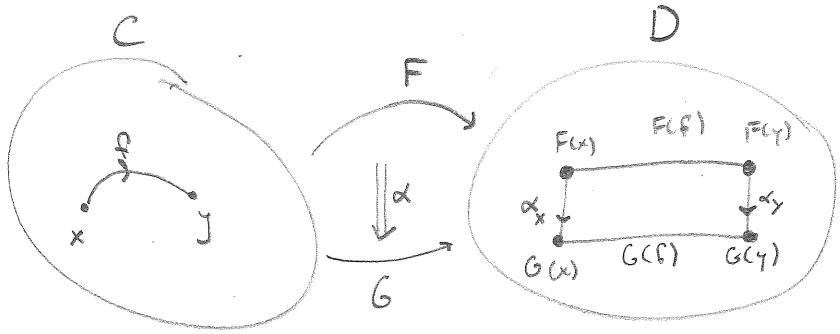
$S \mapsto (S, \emptyset)$

To define adjoint functors (and many other things)

we need Natural Transformations

Given two functors $F, G : C \to D$ we can define

a natural transformation $\alpha : F \Rightarrow G$



Naturality means the square commutes.

Def Given functors F & $G: C \to D$ a transformation $\alpha: F \Rightarrow G$ is a function sending each object $x \in C$ to a morphism

$$\alpha_x : F(x) \to G(x).$$

we say α is a natural transformation if for each morphism $f: x \to y$ in C this square commutes:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"] & F(y) \arrow[d, "\alpha_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd}$$

prop Given Categories C & D there is a category called the functor category D^C with

- objects being functors $F: C \to D$
- morphisms being natural transformations $\alpha: F \Rightarrow G$

In D^C we compose morphisms $\alpha: F \to G$, $\beta: G \to \alpha$ to get $(\beta \circ \alpha)_x: F \Rightarrow H$ is given by $\beta_x \circ \alpha_x$. In D^C $1_F: F \Rightarrow F$ is given by

$$(1_F)_x : F(x) \to F(x) \quad \forall x \in C \text{ is given by } 1_{F(x)}$$

Proof we'll check that the composite is natural given $f: x \to y$ in C, we want the following diagram to commute.

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "(\beta \circ \alpha)_x"'] & F(y) \arrow[d, "(\beta \circ \alpha)_y"] \\ H(x) \arrow[r, "H(f)"'] & H(y) \end{tikzcd}$$

$$\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\
G(x) \arrow[r, "G(f)"] \arrow[d, "\beta_x"'] & G(y) \arrow[d, "\beta_y"] \\
H(x) \arrow[r, "H(f)"] & H(y)
\end{tikzcd}$$

since the the top and bottom commute, the whole diagram commutes.

$\square$

So just as given two sets, there is a set Y^X of all functions $f: X \to Y$, given two categories, there is a category Y^X of all functors $F: X \to Y$.

Given 2 sets X & Y, there is a product $X \times Y = \{(x, y) : x \in X, y \in Y\}$

Notice $X \times Y \neq Y \times X$ but if you want to be honest $X \times Y \cong Y \times X$

and there is a specific "good" isomorphism $\alpha_{X,Y} : X \times Y \to Y \times X$
$(x, y) \mapsto (y, x)$

Its "good" becase it is natural transformation.

There are functors from Set^2 to Set:

$F : (X, Y) \mapsto X \times Y$
$G : (X, Y) \mapsto Y \times X$

and α is a natural transformation from F to G

In fact, α is a "natural Isomorphism"

$\boxed{\text{Def}}$ if $F, G : C \to D$ are functors, & $\alpha : F \Rightarrow G$ is a natural transformation we say α is a natural isomorphism if

$\alpha_x : F(x) \to G(x)$ is an isomorphism $\forall\, x \in C$

Prop $\alpha : F \Rightarrow G$ is a natural isomorphism iff it has an inverse $\alpha^{-1} : G \Rightarrow F$ in D^C

Proof Key idea define $(\alpha^{-1})_x = (\alpha_x)^{-1}$

Prop Suppose C is a category with binary product: i.e. any pair of objects $x, y \in C$ has a product. Then we can choose, for any pair $x, y \in C$, a specific product:

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_{x,y}"'] \arrow[dr, "q_{x,y}"] & \\ X & & Y \end{tikzcd}$$

and there is a functor

$$\times : C^2 \to C$$
$$(x,y) \mapsto X \times Y$$

In fact, there are two functors:

$$\times = F : C^2 \longrightarrow C$$
$$(x,y) \longmapsto X \times Y$$

$$G : C^2 \longrightarrow C$$
$$(x,y) \longmapsto Y \times X$$

and these are naturally isomorphic. $\blacksquare$

Also, products are associative up to natural isomorphism;

$$\alpha_{xyz} : (x \times y) \times z \xrightarrow{\sim} x \times (y \times z) \qquad \begin{tikzcd} C^3 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \alpha_{x,y,z}"] & C \end{tikzcd}$$

Just keep using the universal property of the product.

Def A cartesian category is a category w/ binary products and a terminal object. (i.e. it's a category where any finite set of objects has a product.

One can show in a cartesian category, that we have natural isomorphisms

$$l_x : 1 \times X \xrightarrow{\sim} x$$
$$r_x : X \times 1 \xrightarrow{\sim} x$$

All of this works similarly in a category with finite coproducts: we have natural isomorphisms

$$B_{x,y} : X+Y \xrightarrow{\sim} Y+X$$

$$\alpha_{x,y,z} : (X+Y)+Z \xrightarrow{\sim} X+(Y+Z)$$

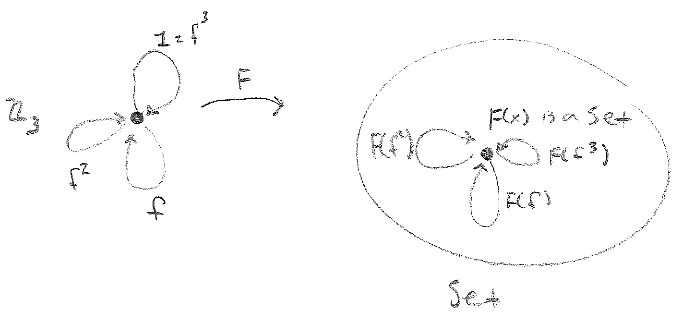
$$\ell_X : 0+X \xrightarrow{\sim}$$

$$r_X : X+0 \xrightarrow{\sim} X$$

In the case $\mathbb{C}$ = FinSet (finite sets and functions) these give familiar laws of arithmetic: $\mathbb{N}$ is the set of isomorphism classes of objects in FinSet

e.g. A group is a category G w. one object and all morphisms invertable.

A functor F is a G-set!



F picks out a set $F(x) = X$ and for each elt f it picks out a function $F(f): X \to X$

s.t. $F(ff') = F(f) \cdot F(f')$

and $F(1) = 1_X$

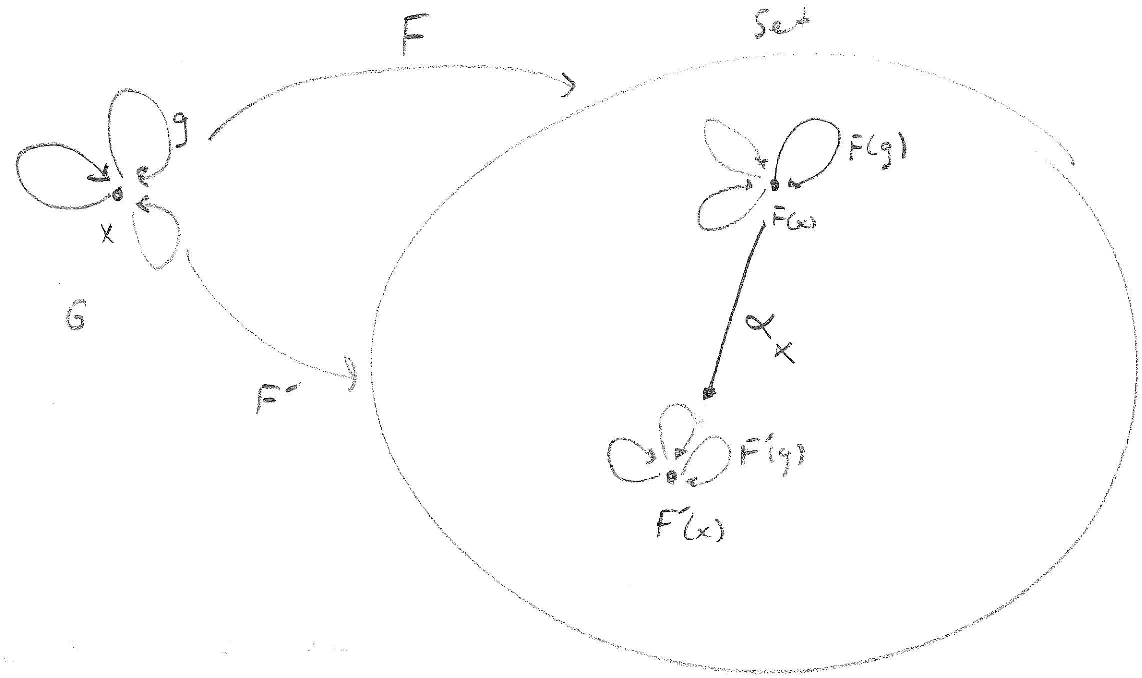
So $X = F(x)$ is a set acted on by the group action. (G-set)

Q: what is a natural transformation between 2 such functors?

Last time we saw that a 1-object category G with all morphisms invertible is a group. We also saw a functor $F: G \to \mathrm{Set}$ was a G-set: A set S with functions $F(g): S \to S$ s.t. for each $g \in G$

$$F(gg') = F(g)\,F(g')$$
$$F(1) = 1_S$$

So given 2 functors $F, F': G \to \mathrm{Set}$, what is a natural transformation $\alpha: F \Rightarrow F'$? Its a map of G-sets or ~~an~~ G-equivariant map.



$$\begin{tikzcd}
F(x) \arrow[r, "\alpha_x"] \arrow[d, "F(g)"'] & F'(x) \arrow[d, "F'(g)"] \\
F(x) \arrow[r, "\alpha_x"'] & F'(x)
\end{tikzcd}$$

So its a function $\alpha_x: F(x) \to F'(x)$, where $x \in G$ is the one object.

And each of the three squares commutes.

Another example of natural transformations.

Eg Two sets are isomorphic if $\exists$ functions $F: X \to Y$ and $G: Y \to X$

S.T. $G \circ F = I_X$ & $F \circ G = I_Y$

F is a bijection $\Leftrightarrow$ such a G exists.

we'd like to do the same with categories.

For categories, we say:

Def An equivalence of categories C & D consist of functors $F: C \to D$ $G: D \to C$ and natural isomorphisms $\alpha: G \circ F \Rightarrow 1$, $\beta: F \circ G \Rightarrow 1$

we say F & G are weak inverses and C & D are equivalent if $\exists$ an equivalence between them.

Thm Given a functor $F: C \to D$ it's a part of an equivalence (F, G, α, β) iff F is faithful, full, and essentially surjective.

Notice G may not be unique, but if G' is another, then they are naturally isomorphic.

Adjoint functors:

recall our example (p. 23) $U_1: \text{Grp} \to \text{Set}$ (sending each group to its underlying set)

$F_1: \text{Set} \to \text{Grp}$ (sending each set to the free group on that set)

we say U_1 is the right adjoint of F_1 or synonymously, F_1 is the left adjoint of U_1. Basic idea:

morphisms

$F_1 S \to G$ in Grp

are in 1-1 correspondence with morphisms

$S \to U_1 G$ in Set!

Given a function $f: S \to U_1 G$ we get a homomorphism

$\tilde{f}: F_1 S \to G$, s.t. $\tilde{f}(s) = f(s)$ for $s \in S \subseteq F_1 S$

S ↓ Generators of $F_1 S$

conversely, given a homomorphism

$h: F_1 S \to G$, we get $\underset{\sim}{h}: S \to U_1 G$ by restricting h to $S \subseteq F_1 S$

The usual picture mixes up morphisms in the category of Set & group in the same diagram.

$$\begin{tikzcd} S \arrow[d, hook] \arrow[dr, "f\ \text{given function}"] & \\ F_1 S \arrow[r, "\exists!\ \text{hom}"', "\tilde{f}"'] & G \end{tikzcd}$$

We prefer to say there is a bijection

$$\underset{\text{In grp}}{\hom(F_1 S, G)} \cong \underset{\text{In set}}{\hom(S, U_1 G)}$$

Notice F_1 is on the left and U_1 is on the right. one is mapped out of and the other is mapped into.

To define adjoint functors, we need to say that this kind of bijection is "natural".

what functors give $\hom(F_1 S, G)$ & $\hom(S, U_1 G)$? they must be two functors from Set × Grp to Set:

$$(S, G) \longmapsto \hom(F_1 S, G)$$
$$(S, G) \longmapsto \hom(S, U_1 G)$$

$\boxed{\text{Prop}}$ for any category C, there is a functor $\hom: C^{op} \times C \to \text{Set}$
$(c, c') \mapsto \hom(c, c')$
called a hom functor.

Sketch of proof: We need to say what the functor does to morphisms.

Given a morphism in $C^{op} \times C$

$\varphi : (x, y) \to (x', y')$

i.e. $f^{op} : x \to x'$ in C^{op}

$g : y \to y'$ in C

we need to define a morphism $\hom(\varphi) : \hom(x,y) \to \hom(x', y')$ in Set.
i.e. a function

So given $h \in \hom(x,y)$ what is $\hom(\varphi) h \in \hom(x', y')$

$$\begin{tikzcd} x \arrow[r, "h"] & y \arrow[d, "g"] \\ x' \arrow[u, "f"] \arrow[r] & y' \end{tikzcd}$$

So its $g \circ h \circ f = \hom(\varphi) h$

Thus the hom-functor from $C^{op} \times C \to \text{Set}$ not only can describe the homsets, but also composition in C!

Then remember to check its a functor.

Now given functors $F : C \to D$, $U : D \to C$, how can we say the isomorphism $\hom(Fc, d) \cong \hom(c, Ud)$ is natural.

$$\begin{tikzcd} & \underset{(F(c),d)}{D^{op} \times D} \arrow[dr, "\hom"] \arrow[dd, phantom, "\Downarrow \alpha"] & \\ \underset{(c,d)}{C^{op} \times D} \arrow[ur, "F^{op} \times id_D"] \arrow[dr, "1 \times U"'] & & \text{Set} \\ & \underset{(c,U(d))}{C^{op} \times C} \arrow[ur] & \end{tikzcd}$$

$\in \hom(F(c), d)$

$\downarrow \alpha_{c,d}$

$\in \hom(c, U(d))$

Here $F^{op} : C^{op} \to D^{op}$ is just F in disguise $F^{op}(f^{op}) = (F(f))^{op}$

Given functors U and F

$$\begin{tikzcd} D \arrow[d, bend left, "U"] \\ C \arrow[u, bend left, "F"] \end{tikzcd}$$

We say F is a left adjoint of U or U is a right adjoint of F if there is a natural isomorphism

$$\begin{tikzcd} & D^{op} \times D \arrow[dr, "hom"] \arrow[dd, phantom, "\Downarrow \alpha"] & \\ C^{op} \times D \arrow[ur, "F^{op} \times 1"] \arrow[dr, "1 \times U"'] & & Set \\ & C^{op} \times D \arrow[ur, "hom"'] & \end{tikzcd}$$

So we have bijections $\alpha_{c,d} : hom(Fc, d) \xrightarrow{\sim} hom(c, Ud) \quad \forall c \in C, d \in D$

which are natural.

At first we'll downplay naturality and just try to understand the bijections.

<u>ex</u>:

$$\begin{tikzcd} Grp \arrow[d, bend left, "U"] \\ Set \arrow[u, bend left, "F"] \end{tikzcd}$$

where UG is the underlying set of $G \in Grp$ and FS is the free group on $S \in Set$.

In this case, the bijection lets us turn any function $f : S \to UG$ into a group homomorphism.

$$\tilde{f} = \alpha^{-1}_{S,G}(f) : FS \to G$$

and conversely any homomorphism $h : FS \to G$ comes from a function

$$\underset{\sim}{h} = \alpha_{S,G}(h) : S \to UG$$

ex: Does the forgetful functor $\mathcal{U}$ have a left adjoint?

$$\begin{tikzcd} \mathrm{Vect}_k \arrow[d] \\ \mathrm{Set} \end{tikzcd}$$

well, is there a functor $F: \mathrm{Set} \to \mathrm{Vect}$?

For any set S, there is a vector space $F(S)$ whose basis is S. Notice $F(S) = \left\{ \sum_{s_i \in S} c_i s_i : c_i \in k, \text{ only finitely many non zero } c_i \right\}$

what does F do to a morphism?

It should give us a linear map $Ff: FS \to FT$

via $Ff(\sum c_i s_i) = \sum c_i f(s_i) \in FT$

Lets check $F(g \circ f) = F(g) \circ F(f)$ and $F(1_S) = 1_{F(S)}$

why is F left adjoint to $\mathcal{U}$?

We need bijections $\hom(FS, V) \cong \hom(S, \mathcal{U}V)$ $\forall S \in \mathrm{Set}$ & $\forall V \in \mathrm{Vect}_k$

Given $f: S \to \mathcal{U}V$ we can define $\tilde{f}: FS \to V$ in some natural way.

Try $\tilde{f}(\sum c_i s_i) = \sum c_i f(s_i)$

Conversely given $\ell: FS \to V$ can we convert it into a function $\underset{\sim}{\ell}: S \to \mathcal{U}V$?

Try $\underset{\sim}{\ell}(s) = \ell(s)$

Now check these are inverses: $\underset{\sim}{(\tilde{f})} = f$ and $\widetilde{(\underset{\sim}{\ell})} = \ell$

So we'll get a bijection between $\hom(FS, V) \cong \hom(S, \mathcal{U}V)$

Sometimes a functor has both a left and right adjoint.

ex:
$$\begin{tikzcd} \mathrm{Top} \arrow[d, "U"] \\ \mathrm{Set} \end{tikzcd}$$
to get a left adjoint, think of ways to turn S into a topological space. One is the discrete topology and the other is indiscrete topology.

The left adjoint of U, say $L : \mathrm{Set} \to \mathrm{Top}$ must have

$$\hom(LS, X) \cong \hom(S, UX)$$

i.e. here continuous maps $f : LS \to X$ are in 1-1 correspondence as functions $f : S \to UX$. To make this work you use the discrete topology.

LS is S w. the discrete topology.

The right adjoint of U, say $R : \mathrm{Set} \to \mathrm{Top}$ must have

$$\hom(UX, S) \cong \hom(X, RS)$$

i.e. continuous maps from any space $h : X \to RS$ are in 1-1 correspondence with functions $\underline{h} : UX \to S$. Now we need to make any map into RS is continuous, so we use the indiscrete topology and RS is S w. the indiscrete topology.

ex say C is any category. There is always a functor called the diagonal functor.

$\Delta : C \to C \times C$ with $\Delta(c) = (c, c)$

and $\Delta(f) : \Delta c \to \Delta c'$ by $\Delta f = (f, f) : (c, c) \to (c', c')$

Prop If C has binary products then the functor:

$\times : C \times C \to C$ is the right adjoint of Δ.

Prop ctd. In fact, the converse is true! So Δ has a right adjoint iff C has binary products! So products are just right adjoints to the diagonal!

Proof (Sketch)

We need bijections between $\hom(\Delta c, (c', c'')) \cong \hom(c, c' \times c'')$

Left side: $\hom(\Delta c, (c', c'')) = \hom((c, c), (c', c''))$
$\cong \hom(c, c') \times \hom(c, c'')$

so we need $\hom(c, c') \times \hom(c, c'') \cong \hom(c, c' \times c'')$

Indeed, the universal property of the product says

$$\begin{tikzcd} & c \arrow[d, "\exists! \psi"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & c' \times c'' \arrow[dl, "p"] \arrow[dr, "q"'] & \\ c' & & c'' \end{tikzcd}$$

commutes and $f = p \circ \psi$ and $g = q \circ \psi$, so we have a bijection

$\hom(c, c') \times \hom(c, c'') \cong \hom(c, c' \times c'')$
$(f, g) \longmapsto \psi$

$\boxed{\text{Prop}}$ If C is a category then C has a binary coproduct $(+ : C \times C \to C)$ iff $\Delta : C \to C \times C$ has a left adjoint.

Proof (Sketch) For starters, we need bijections between

$\hom(c' + c'', c) \cong \hom((c', c''), \Delta c)$

and $\hom((c', c''), \Delta c) = \hom((c', c''), (c, c)) \cong \hom(c', c) \times \hom(c'', c)$

So we need $\hom(c' + c'', c) \cong \hom(c', c) \times \hom(c'', c)$. By universal property of coproducts:

$$\begin{tikzcd} c' \arrow[dr, "q"] \arrow[ddr, bend right, "f"'] & & c'' \arrow[dl, "p"'] \arrow[ddl, bend left, "g"] \\ & c' + c'' \arrow[d, "\exists! \psi"] & \\ & c & \end{tikzcd}$$

and our bijection sends $\psi \longmapsto (f, g)$

Slogan: A product (an example of a limit) is an example of a right adjoint - it's easy to describe morphisms going into it.

A coproduct (an example of a colimit) is an example of a left adjoint - its easy to describe morphisms going out of it.

Next we'll see all limits are left adjoints and all colimits are right adjoints.

Notice $\oplus : \mathrm{Vect}_k^2 \to \mathrm{Vect}$ is both left and right adjoint to $\Delta : \mathrm{Vect}_k \to \mathrm{Vect}_k^2$

In fact, if a category has limits, these limits give a right adjoint to some functor: "limits are right adjoints". Similarly "colimits are left adjoints."

We often think about the limit of a diagram in a category C. What is a "diagram in C", really?

Naively, it is a collection of objects and morphisms between them.

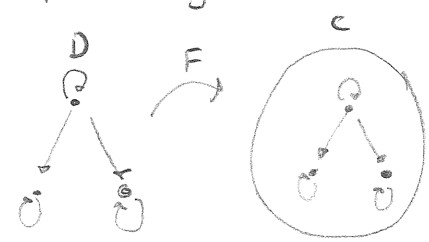
$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ c' & & c'' \arrow[d, "f''"] \\ & & c''' \end{tikzcd}$$

We can make it into a category by adding morphisms to make it a subcategory of C.

we're often interested in diagrams of some shape, like pullbacks.

But shapes can be interpreted as categories

Let D be any category; we'll take this as an "diagram shape". What is a D-shaped diagram in some category C? Its a functor $F: D \to C$

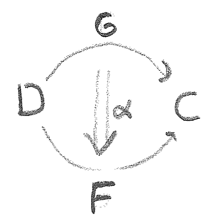


when we take the limit of this diagram, we get an object (defined up to isomorphism) $\lim F \in C$.

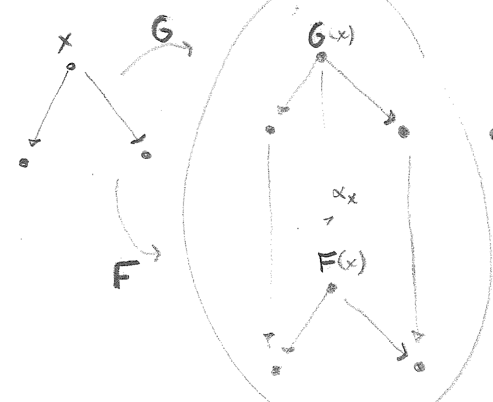
What is the process that takes us from $F: D \to C$ to $\lim F \in C$

The key: there is a category C^D with:

- objects being functors $F: D \to C$
- morphisms being natural transformations

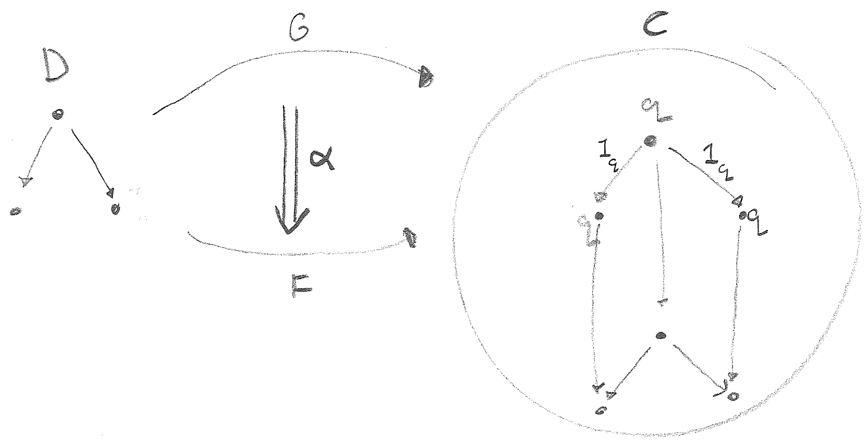


these morphisms look like:



and all squares commute.

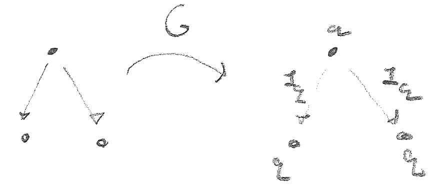
But, were actually interested in cones over F, which is a natural transformation $\alpha: G \Rightarrow F$ that sends every object of D to some object in C and every morphism of D to some identity morphism in that object.



Here $G: D \to C$ was determined by the object $q \in C$.

We turned an object $q \in C$ into an object $G \in C^D$. So G is a functor, but its an object in the category of functors.

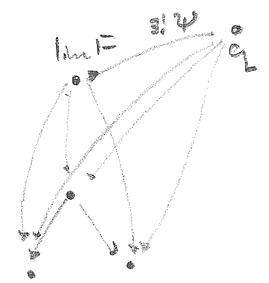
$\Delta_D : C \to C^D$ and $\Delta_D(q)$ is the diagram



Here $G = \Delta_D(q)$

Then a cone over F with apex $q \in C$ is a natural transformation $\alpha : \Delta_D(q) \Rightarrow F$

Recall the limit of a diagram is a universal cone.



Recall also that U is a right adjoint of F if:

$$\hom(Fx, y) \cong \hom(x, Uy).$$

So adjoint functors are about converting one kind of morphism into another in a bijective way and that's what were doing when we are stating the universal property:

- morphisms $\psi : q \to \lim F$ in C
- Cones over F with apex q i.e. natural transformations $\alpha : \Delta_D(q) \Rightarrow F$, morphisms α from $\Delta_D(q)$ to F in C^D.

So: $\hom(\Delta_D(q), F) \cong \hom(q, \lim F)$

So $\lim : C^D \to C$ had better be right adjoint to $\Delta_D : C \to C^D$.

we need to check that the bijection we came up with is natural.

$\boxed{\text{Thm}}$ If C has all limits of D-shaped diagrams, then we have a functor

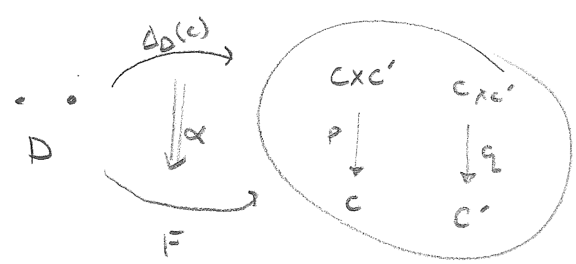
$$\lim : C^D \to C$$
$$F \mapsto \lim F$$

which is a right adjoint to

$$\Delta_D : C \to C^D$$

The converse holds as well.

ex what choice of D gives the case of binary products (a special case of limits)?



So D has 2 objects and only identity morphisms. C^D is C^2 and

$\times : C^2 \to C$ are right adjoint to

$\Delta_2 = \Delta : C \to C^2$

$\boxed{\text{Thm}}$ if a category C has colimits of all D-shaped diagrams, there is a functor

$\text{colim}: C^D \to C$ which is left adjoint to

$\Delta_D : C \to C^D$

with $\hom(\text{colim}\, F, q) \cong \hom(F, \Delta_D\, q)$

Note: $\alpha \in \hom(F, \Delta_D\, q)$ is a cocone:

$$\begin{tikzcd}
 & \bullet \arrow[dl] \arrow[dr] \arrow[dd] & \\
\bullet \arrow[dr] & & \bullet \arrow[dl] \\
 & \underset{q}{\bullet} &
\end{tikzcd}$$

$\boxed{\text{Thm}}$ Left adjoints preserve colimits and right adjoints preserve limits.

Proof Lets show if $F: C \to D$ is a left adjoint to $U: D \to C$, then F preserves colimits.

For concreteness, we'll show F preserves pushouts. The general case is analogous. Say we have a pushout in C:

$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] \arrow[dd] & \\
b \arrow[dr] & & c \arrow[dl] \\
 & x &
\end{tikzcd}$$

Here x is the apex of the cocone on the diagram we're taking the colimit of, and the universal property holds.

Claim Applying F to the universal cocone gives a universal cocone in D.

$$\begin{tikzcd}
 & F(a) \arrow[dl] \arrow[dr] \arrow[dd] \arrow[ddrr, bend left] & & \\
F(b) \arrow[dr] \arrow[drr] & & F(c) \arrow[dl] \arrow[d] & \\
 & F(x) \arrow[r, dashed, "\exists!\,\psi"'] & Q &
\end{tikzcd}$$

So we choose a competitor cocone w. apex Q. We must show $\exists!\,\psi : F(x) \to Q$ making the newly formed triangles commute.

we can look at $\mathcal{U}(Q)$ in $\mathcal{C}$

Notice $\hom(F(X), Q) \cong \hom(X, \mathcal{U}(Q))$ so to get $\psi: F(X) \to Q$ we'll get

$\varphi: X \to \mathcal{U}(Q)$. $\mathcal{U}(Q)$ becomes a competitor due to adjointness of F and $\mathcal{U}$.

The commutivity of triangles in $\mathcal{D}$ implies the commutivity of triangles in $\mathcal{C}$ since there is a bijection between homsets.

By universality, we get $\exists!\ \varphi: X \to \mathcal{U}(Q)$ making all newly formed triangles commute. Now by the bijective correspondence, we get $\psi: F(X) \to Q$. You just need to check the newly formed triangles commute.

$\square$

ex $F: \mathrm{Set} \to \mathrm{Grp}$ preserves colimits e.g coproducts, so $F(S+T) \cong F(S) + F(T)$

Here $S+T$ is the disjoint union of S & T. $F(S+T)$ is the free group on the set.

$F(S) + F(T)$ is the free product $F(S) * F(T)$ of two free groups.

ex $\mathcal{U}: \mathrm{Grp} \to \mathrm{Set}$ preserves limits (e.g. products) : $\mathcal{U}(G \times H) \cong \mathcal{U}(G) \times \mathcal{U}(H)$

where $G \times H$ is the usual product of groups G & H. In Set, the product is the usual cartesian product.

$\boxed{\text{Thm}}$ The composite of left adjoints is a left adjoint. The composite of right adjoints is a right adjoint.

Proof say $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{F'} \mathcal{E}$ and say F, F' are left adjoints to $\mathcal{U}$ and $\mathcal{U}'$

we'll show $F' \circ F: \mathcal{C} \to \mathcal{E}$ is the left adjoint to $\mathcal{U} \circ \mathcal{U}': \mathcal{E} \to \mathcal{C}$

we want a natural isomorphism $\hom(F' \circ F(c), e) \cong \hom(c, \mathcal{U} \circ \mathcal{U}'(e))$

Here's how we get it.

$$\hom(F' \circ F(c), e) \cong \hom(F(c), U'(e)) \quad \text{by } F' \text{ left adjoint to } U'$$
$$\cong \hom(c, U(U'(e)))$$

Now make sure you can compose natural isomorphisms and get a natural isomorphisms.

__Ex__

$$\begin{tikzcd} \mathrm{Ring} \arrow[d, bend left, "U'"] \\ \mathrm{Abgp} \arrow[u, bend left, "F'"] \arrow[d, bend left, "U"] \\ \mathrm{Set} \arrow[u, bend left, "F"] \end{tikzcd}$$

were saying we can form the free abelian ring by going through $F' \circ F$. It's left adjoint to $U \circ U'$ from ring to Set.

Starting from $\emptyset$ (initial set) we get the trivial group $F(\emptyset) = 0$ (the initial ab. group) then the free ring on it is $\mathbb{Z}$, since we start with $\{0, 1\}$ and build. $\mathbb{Z}$ is the initial ring.

Starting from a one element set $\{x\}$ we get $F(\{x\}) \cong \mathbb{Z}$

then $F'(F(\{x\})) = \mathbb{Z}[X]$, the ring of polynomials in X with integer coefficients.

Units and counits of adjunctions (a pair of adjoint functors)

Suppose we have
$$\begin{tikzcd} D \arrow[d, bend left, "U"] \\ C \arrow[u, bend left, "F"] \end{tikzcd}$$
with F left adjoint to U.

$\leadsto \hom(Fc, d) \cong \hom(c, Ud)$

we can apply this bijection to an identity morphism and get something interesting.

Let $d = Fc$ then $\hom(Fc, Fc) \xrightarrow[\sim]{\varphi} \hom(c, UFc)$

$1_{Fc} \in \hom(Fc,Fc)$ $\quad \varphi(1_{Fc}) \in \hom(c, UFc)$ $\leadsto$ its called the unit $\eta_c : c \to UFc$

or apply φ^{-1} to an identity if $c = Ud$

then $\hom(FUd, d) \xleftarrow[\sim]{\varphi^{-1}} \hom(Ud, Ud)$

$\varphi^{-1}(1_{Ud}) \qquad \ni 1_{Ud}$ and $\varphi^{-1}(1_{Ud})$ is called the counit $\varepsilon_d : FUd \to d$

ex $F: Set \to Grp$

$U: Grp \to Set$

given any set S, we get a unit $\eta_S : S \to U(F(S))$

This is the inclusion of the generators: elts of S are generators of $F(S)$.

Given any group, we get a counit $\varepsilon_G : FUG \to G$. This is the conversion of the formal product into actual ones.

Recall an adjunction is a pair of categories C, D, and a pair of functors:

$$\begin{tikzcd} D \arrow[d, bend left, "U"] \\ C \arrow[u, bend left, "F"] \end{tikzcd}$$

and a natural isomorphism $\alpha_{c,d} : \hom(Fc, d) \cong \hom(c, Ud)$

So F is left adjoint to U and U is right adjoint to F.

what does the naturality of α actually say?

we're taking $(c,d) \in C^{op} \times D$ and applying two functors to it, getting objects $\hom(Fc, d)$ and $\hom(c, Ud) \in Set$. α is a natural isomorphism between these functors, so we get a commuting square for each morphism

$(f, g) : (c, d) \to (c', d')$ in $C^{op} \times D$.

i.e. $f : c \to c'$ in C

$g : d \to d'$ in D.

The commuting square is as follows: $h \in \hom(Fc,d)$

$$\begin{tikzcd} Fc \arrow[r, "h"] & d \arrow[d, "g"] \\ Fc' \arrow[u, "Ff"] \arrow[r, dashed] & d' \end{tikzcd} \qquad \begin{tikzcd} h \arrow[d, mapsto] & \hom(Fc,d) \arrow[r, "\alpha_{c,d}"] \arrow[d, "\hom(Ff,g)"'] & \hom(c,Ud) \arrow[d, "\hom(f,Ug)"] & k \arrow[d, mapsto] \\ g\circ h \circ Ff & \hom(Fc',d') \arrow[r, "\alpha_{c',d'}"] & \hom(c',Ud') & Ug\circ k \circ f \end{tikzcd}$$

here $\hom(Ff,g)$ is just a name for the map

$$\hom(Fc,d) \longrightarrow \hom(Fc',d')$$
$$h \longmapsto g\circ h \circ Ff$$

now

$$\begin{tikzcd} c \arrow[r, "k"] & Ud \arrow[d, "Ug"] \\ c' \arrow[u, "f"] \arrow[r, dashed] & Ud' \end{tikzcd} \quad \Rightarrow \quad \begin{aligned} \hom(f,Ug) : \hom(c,Ud) &\longrightarrow \hom(c',Ud') \\ k &\longmapsto Ug\circ k \circ f. \end{aligned}$$

So saying that α is natural says the square commutes, i.e.

$$\alpha_{c',d'}(g\circ h\circ Ff) = Ug \circ \alpha_{c,d}(h)\circ f$$

Lets use naturality to prove something about the unit and counit of a adjunction.

recall $\alpha_{c,Fc} : \hom(Fc, Fc) \xrightarrow{\sim} \hom(c, UFc)$

$$1_{Fc} \longmapsto \eta_c$$

where $\eta_c = \alpha_{c,Fc}(1_{Fc})$ is the unit of the adjunction: $\eta_c : c \to UFc$

Also $\quad \hom(FUd,d) \xleftarrow{\sim} \hom(Ud,Ud) \quad : \alpha^{-1}_{Ud,d}$

$$\varepsilon_d \longleftarrow\!\!\shortmid 1_{Ud}$$

where $\varepsilon_d = \alpha^{-1}_{Ud,d}(1_{Ud}) \qquad \varepsilon_d : FUd \to d \quad$ is the counit.

If $\begin{tikzcd} \mathrm{Vect} \arrow[d, bend left, "U"] \\ \mathrm{Set} \arrow[u, bend left, "F"] \end{tikzcd}$ is the adjunction

$\eta_S : \mathrm{Set} \to UF_S$ is the inclusion of the basis elements in the whole vector space that it is a basis of.

$\varepsilon_V : FUV \to V$ sends formal linear combinations to "actual" linear combos in V

$$c_1[v_1] + \dots + c_k[v_k] \longmapsto c_1v_1 + \dots + c_kv_k$$

Thm: Given an adjunction we can recover the natural isomorphism $\alpha_{c,d} : \hom(Fc, d) \to \hom(c, Ud)$ and its inverse from the unit and counit.

i.e. we can find a formula for α in terms of the unit and α^{-1} in terms of the counit.

$$\begin{tikzcd}
 & f \arrow[r, mapsto] & \alpha_{c,d}(f) & Uf \circ k \circ \eta_c \\
f \circ h \circ F1_c & \hom(Fc, d) \arrow[r, "\alpha_{c,d}"] & \hom(c, Ud) & \\
h \arrow[u, mapsto] & \hom(Fc, Fc) \arrow[u, "\hom(F1_c, f)"] \arrow[r] & \hom(c, UFc) \arrow[u, "\hom(1_c, Uf)"'] & k \arrow[uu, mapsto] \\
 & 1_{Fc} \arrow[r, mapsto] & \eta_c &
\end{tikzcd}$$

So we get:

$$\begin{tikzcd}
f \arrow[r, mapsto] & \alpha_{c,d}(f) = Uf \circ \eta_c \\
1_{F_c} \arrow[u, mapsto] \arrow[r, mapsto] & \eta_c \arrow[u, mapsto]
\end{tikzcd}$$

and $\alpha_{c,d}(f) = Uf \circ \eta_c$

We can recover α from the unit.

Similarly:

$$\begin{tikzcd}
 & \alpha^{-1}_{c,d}(g) & g \arrow[l, mapsto] & \\
1_d \circ h \circ Fg & \mathrm{hom}(Fc,d) & \mathrm{hom}(c,Ud) \arrow[l, "\alpha^{-1}_{c,d}"'] & 1_{Ud} \circ k \circ f \\
h \arrow[u, mapsto] & \mathrm{hom}(FUd,d) \arrow[u] & \mathrm{hom}(Ud,Ud) \arrow[l, "\alpha^{-1}_{Ud,d}"] \arrow[u] & k \arrow[u, mapsto] \\
 & \varepsilon_d & 1_{Ud} \arrow[l, mapsto] &
\end{tikzcd}$$

So we get

$$\begin{tikzcd}
\alpha^{-1}_{c,d}(g) = \varepsilon_d \circ Fg & g \arrow[l, mapsto] \\
\varepsilon_d \arrow[u, mapsto] & 1_{Ud} \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$

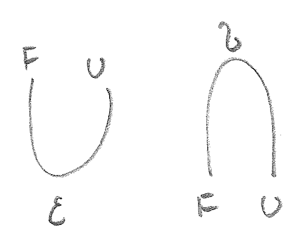
and $\alpha^{-1}_{c,d}(g) = \varepsilon_d \circ Fg$

$\boxed{\text{VI}}$

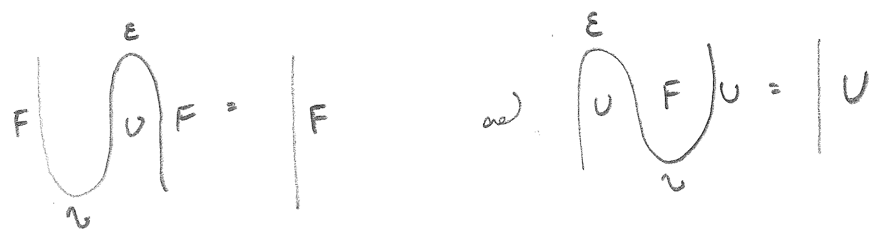
This hints that there is another way to think about adjoint functors that says $F: C \to D$, $U: D \to C$ are adjoint if we have natural transformations:

$\eta: I_C \Rightarrow UF$ (giving $\eta_c : c \to UFc$)

$\varepsilon: FU \Rightarrow I_D$ $\qquad \varepsilon_d : FUd \to d$



obeying two equations saying:



These are called zigzag equations. These pictures make sense as part of a 2D diagramatic notation for:

- categories
- functors
- natural transformations.

} an example of a two category.

[Towards Topos Theory]

A topos is a category that is enough like **Set** that you can do "all mathematics" in it.

In brief, a topos is a cartesian closed category with finite limits and a subobject classifier.

Recall: A category is cartesian if it has binary products $\times$ and a terminal object I, i.e. it has finite products,

A category has finite limits if you can take the limit of any finite sized diagram. In fact, a category w. finite limits is the same as a cartesian category with equalizers.

Roughly, a category is cartesian closed if it has objects like X^Y.

(In Set, these are sets of functions)

In Set, subsets $S \subseteq T$ are in 1-1 correspondence with characteristic functions: $\chi : T \to \{0,1\}$ or $\{\text{False}, \text{True}\}$

$\{F, T\}$ is the subobject classifier in Set.

Any topos has its own subobject classifier Ω.

[Cartesian Closed Categories] (ccc's)

We've studied addition: $X+Y$ · coproduct: Left adjoint to Δ

multiplication: $X\times Y$ · product: Right adjoint to Δ

we've shown or maybe just claimed that $+$ & $\times$ are associative and commutative up to isomorphism.

We saw there was a canonical morphism

$$X\times Z + Y\times Z \to (X+Y)\times Z$$

that is not always an isomorphism. If it was we'd say a category with $+, \times$ is distributive.

Now exponentiation!

In set, $X^Y = \{f: Y\to X\}$ and $|X^Y| = |X|^{|Y|}$

Can we define X^Y in some other categories?

we want to make X^Y an object in our category.

In any category C, we have $\hom(X,Y) = \{f: Y\to X\} \in Set$ and this is the same as X^Y in Set. We want to define

$X^Y \in C$ whenever possible. Notice this is not the set of morphisms from Y to X but the object of morphisms from Y to X or hom-object. (Also called exponential or internal hom)

One goal is to free category theory from the domination of Set. One step is to work with $X^Y \in C$ as a substitute for $\hom(Y, X) \in \mathrm{Set}$.

Prop In Set, there is a functor

$$-\times Y : \mathrm{Set} \longrightarrow \mathrm{Set}$$
$$X \longmapsto X \times Y$$

which has a right adjoint

$$-[\ ]^Y : \mathrm{Set} \longrightarrow \mathrm{Set}$$
$$X \longmapsto X^Y$$

So $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$

proof sketch:

The isomorphism turns $f : X \times Y \to Z$ to $\underset{\sim}{f} : X \to Z^Y$

computer scientists call $f \mapsto \underset{\sim}{f}$ "currying"

$\underset{\sim}{f}(x)(y) = f(x,y)$

$x \in X$
$\underset{\sim}{f}(x) \in Z^Y$

Next we explain what $-\times Y$, $-[\ ]^Y$ do to morphisms. Check its a functor and show its an isomorphism.

Def A cartesian closed category or CCC is a cartesian category C s.t. for every $Y \in C$, the functor $-\times Y : C \to C$ has a right adjoint $[\ ]^Y : C \to C$ so we have a natural isomorphism:

$$\hom(X \times Y, Z) \cong \hom(X, Z^Y)$$

e.g. ① Set is a CCC.

② FinSet (The category of finite sets and functions between them) is also a CCC.

③ G-sets (The category of sets with a group action and functions preserving the action)

↑

also CCC

↓

④ Graph (category where object is a graph i.e. a pair of sets E, V and functions $s, t : E \to V$ $\bullet \xrightarrow{e} \bullet$
$s(e)$ $t(e)$

A morphism from $s,t : E \to V$ to $s',t' : E' \to V'$ is a pair

$\varphi_0 : V \to V'$

$\varphi_1 : E \to E'$

s.t.

$$\begin{tikzcd} E \arrow[r, "\varphi_1"] \arrow[d, "{s,t}"'] & E' \arrow[d, "{s',t'}"] \\ V \arrow[r, "\varphi_0"'] & V' \end{tikzcd}$$

commutes

Given graphs X & Y there is a graph X^Y whose vertices are maps of graphs $\varphi : Y \to X$ but where there also are edges that correspond to transformations.

⑤ Cat (a category is a graph with the ability to compose edges and a functor is a map of graphs that preserves composition)

is also CCC.

Given $X, Y \in$ Cat, we get X^Y with

- functors : $F : Y \to X$ as objects
- Natural transformations $\begin{tikzcd} Y \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & X \end{tikzcd}$ α as morphism from F to G.

Puzzle: Show

$$\hom(X \times Y, Z) \cong \hom(X, Z^Y)$$
$$f \longmapsto \underset{\sim}{f}$$

where $\underset{\sim}{f}(x)(y) = f(x,y)$ but what does $\underset{\sim}{f}$ do to morphisms?

Non-examples:

(6) Grp
(7) $Vect_k$
(8) Ring.

} not ccc for if we have groups G, H, we do not have G^H some 1-1 correspondence between homomorphisms

$$K \times H \to G \quad K \to G^H$$

$\boxed{\text{Prop}}$ Any ccc that has coproducts is distributive.

proof Sketch

we'll show $(X+Y) \times Z \cong X \times Z + Y \times Z$

this follows if we know a functor $- \times Z : C \to C$ preserves coproducts.

we know left adjoints preserve colimits. Since C is ccc, $\times Z$ is the left adjoint of $-[\ ]^Z : C \to C$ so $\hom(X \times Z, Y) \cong \hom(X, Y^Z)$

$\boxed{\text{prop}}$ In a ccc, • $(X \times Y)^Z \cong (X^Z \times Y^Z)$

• $1^Z \cong 1$

proof • $(X \times Y)^Z \cong (X^Z \times Y^Z)$ says exponentiation preserves products.
right adjoints preserve limits e.g products, and $-[\ ]^Z$ is a right adjoint to $- \times Z$.
• The terminal object 1 is the limit of the empty diagram.
Right adjoints preserve 1, so $1^Z \cong 1$ $\blacksquare$

We knew $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$.

Is the analogue true where we use "hom-objects" instead of hom sets?

is $Z^{X \times Y} \cong (Z^Y)^X$?

It's true in any ccc, but why?

Next time: evaluation morphism

$$ev: X^Y \times Y \to X$$

$$(f, y) \mapsto f(y) \quad \text{in Set,}$$

but it can be generalized to any ccc.

$$Z^Y \times Y^X \to Z^X$$

$$(f, g) \longmapsto f \circ g \quad \text{in Set}$$

can also be generalized to any ccc.

eg If $C = Cat$, $\hom(X, Y)$ is the set of functors $F: X \to Y$, while Y^X is the category of functors $F: X \to Y$ and natural transformations between them!

In general, you can get $\hom(X, Y)$ from Y^X but not vice versa. We call $\hom(X, Y)$ the external hom since it lives in Set, which is outside C. Y^X is called the exponential or internal hom since Y^X is in C.

Internalization is the process of taking math that lives in Set and moving it into some category.

Eg In Set, we can define a group to be an object $G \in Set$ with morphisms

$m: G \times G \to G$ mult

$inv: G \longrightarrow G$ inverses

$id: I \longrightarrow G$ identity assigning map.

S.T.

①
$$\begin{tikzcd} G\times G\times G \arrow[r, "I_G \times m"] \arrow[d, "m \times I_G"'] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"'] & G \end{tikzcd}$$
commutes (Associativity)

②
$$\begin{tikzcd} I\times G \arrow[d, "i \times I_G"'] & G \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "I_G"] & G\times I \arrow[d, "I_G \times i"] \\ G\times G \arrow[r, "m"] & G & G\times G \arrow[l, "m"'] \end{tikzcd}$$
(left & right inverses)

③
$$\begin{tikzcd} G\times G \arrow[dd, "inv \times I_G"'] & G \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\varepsilon!"] & G\times G \arrow[dd, "I_G \times inv"] \\ & I \arrow[d, "i"] & \\ G\times G \arrow[r, "m"] & G & G\times G \arrow[l, "m"'] \end{tikzcd}$$

To talk about a group internal to another category we need our category to have a few properties.

We need binary products and a terminal object. This, however, gives us all finite products and thus a cartesian category.

We can thus define a group internal to C or group object in C or group in C when C is a cartesian category.

eg $C = Top$ then a group in C is called a topological group.

$C = Diff$ (smooth manifolds and smooth maps) then a group internal to C is a lie group.

$C =$ algebraic variety, a group in C is called an algebraic group.

Recall a cartesian category C is ccc if for any $Y \in C$
$- \times Y$ has a right adjoint.

Any adjunction
$$\begin{tikzcd} D \arrow[d, bend left, "U"] \\ C \arrow[u, bend left, "F"] \end{tikzcd}$$
has a unit and counit:

$\eta_X : X \to UFX \qquad X \in C$

$\varepsilon_Y : FUY \to Y \qquad Y \in D$

Now we have an adjunction w. C:
$$\begin{tikzcd} C \arrow[d, bend left, "{[\,]^Y}"] \\ C \arrow[u, bend left, "- \times Y"] \end{tikzcd}$$

we get:

$$\eta_X : X \longrightarrow (X \times Y)^Y$$

(Coevaluation: In Set $\eta_X(x) : Y \to X \times Y$
$\eta_X(x)(y) = (x, y)$

$$\varepsilon_X : X^Y \times Y \to X$$

(Evaluation: In set $\varepsilon_X : X^Y \times Y \to X$
$(f, y) \mapsto f(y)$

we have analogues of these in any ccc.

Free book
Emily Riehl
- Categories in context

Next: in any category we have composition:

$$\hom(Y, Z) \times \hom(X, Y) \longrightarrow \hom(X, Z)$$
$$(f, g) \longmapsto f \circ g$$

How do we internalize this? In a ccc, can we internalize this to define internal composition?

$$\circ : Z^Y \times Y^X \longrightarrow Z^X$$

$$\circ \in \hom(Z^Y \times Y^X, Z^X) \cong \hom(Z^Y \times Y^X \times X, Z)$$

So we can get $\circ$ from a morphism

$\tilde{\circ} : Z^Y \times Y^X \times X \to Z$ which indeed we have in any ccc

$$Z^Y \times Y^X \times X \xrightarrow{1_{Z^Y} \times \varepsilon} Z^Y \times Y \xrightarrow{\varepsilon} Z$$ where ε is the evaluation.

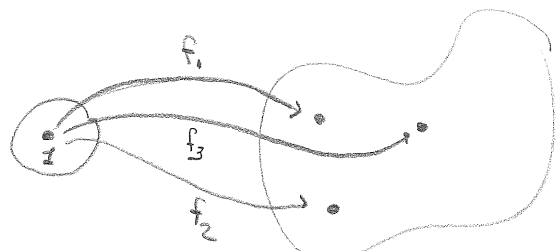
So this is just an internalized way of saying the old definition of composition: $(f \circ g)(x) = f(g(x))$

Elements

In set we have elements, but what about objects in other categories?

Elements of a set X are in 1-1 correspondance with functions

$f: 1 \to X$ where 1 is a terminal object in Set.



Def If C is a category with a terminal object, an element of an object $X \in C$ will be defined as a morphism $f: 1 \to X$

we define the set $elt(X)$ to be $hom(1, X)$.

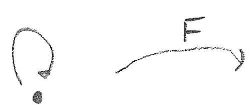
eg $C = Top$, then $elt(X) = \{\text{continuous maps } f: \{*\} \to X\}$

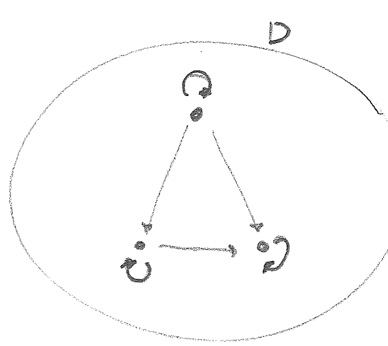
$\{*\}$ is the one-point space. In fact, $elt(X)$ is in 1-1 correspondance with the points of X.

$C = Grp$ $elt(G) = \{\text{homomorphisms } f: 1 \to G\}$ 1 is the trivial group.

so $elt(G)$ has only 1 element. (Since 1 is also initial)

$C = Cat$ $elt(D) = \{\text{functors } F: 1 \to D\}$, 1 is the terminal category: 





so F maps to the objects in D.

As in $\mathcal{C} = \mathrm{Grp}$, we have elt forgetting information,

eg $\mathrm{elt}\left(\overset{\circlearrowleft}{\bullet} \longrightarrow \overset{\circlearrowleft}{\bullet}\right) \cong \mathrm{elt}\left(\overset{\circlearrowleft}{\bullet} \quad \overset{\circlearrowleft}{\bullet}\right)$

$\boxed{\text{prop}}$ Suppose C is a category with a terminal object $1 \in C$

Then there is a functor

$$\mathrm{elt} : C \to \mathrm{Set}$$

with $\mathrm{elt}(X) = \{ f : 1 \to X \}$ $\forall X \in C$

and given any morphism $g : X \to Y$ in C

$\mathrm{elt}(g) : \mathrm{elt}(X) \to \mathrm{elt}(Y)$ is defined as follows:

$$\begin{tikzcd}
1 \arrow[r, "f"] \arrow[dr, "{\mathrm{elt}(g) f = g \circ f}"'] & X \arrow[d, "g"] \\
& Y
\end{tikzcd}$$

proof:

elt preserves composition: given $X \xrightarrow{g} Y \xrightarrow{h} Z$

we need $\mathrm{elt}(h \circ g) = \mathrm{elt}(h) \circ \mathrm{elt}(g)$

$$\begin{tikzcd}
1 \arrow[r, "f"] & X \arrow[d, "g"] \\
& Y \arrow[d, "h"] \\
& Z
\end{tikzcd}$$

given $f \in \mathrm{elt}(X)$ we have

$$\begin{aligned}
\mathrm{elt}(h \circ g) f &= (h \circ g) \circ f \\
&= h \circ (g \circ f) \\
&= h \circ (\mathrm{elt}(g) f) \\
&= \mathrm{elt}(h)(\mathrm{elt}(g) f).
\end{aligned}$$

Similarly, $\mathrm{elt}(1_X)f = 1_X \circ f$
$= f$

So $\mathrm{elt}(1_X) = 1_{\mathrm{elt}(X)}$

Note: $\mathrm{elt}: C \to \mathrm{Set}$ may not be faithful, i.e. we can have two different morphisms $g, g': X \to Y$ in C with $\mathrm{elt}(g) = \mathrm{elt}(g')$

ex $C = \mathrm{grp}$, then $\mathrm{elt}(G)$ is just the one element set. So any homomorphism $h: G \to G'$ must be sent to a function $\mathrm{elt}(h): 1 \to 1$ but there is only one!

$\boxed{\text{Prop}}$ if C is a cartesian category, $\mathrm{elt}: C \to \mathrm{Set}$ preserves finite products.

Proof It's easy to show elt preserves the terminal object:
if $1 \in C$ then
$$\mathrm{elt}(1) = \{f: 1 \to 1\}$$
is a one element set, so its terminal in Set.

why does elt preserve binary products?
Say $X, Y \in C$; then their product is ~~thus~~ a universal cone

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

To show elt preserves products, we need to show this cone is universal in Set:

$$\begin{tikzcd} & \mathrm{elt}(X \times Y) \arrow[dl, "\mathrm{elt}(p)"'] \arrow[dr, "\mathrm{elt}(q)"] & \\ \mathrm{elt}(X) & & \mathrm{elt}(Y) \end{tikzcd}$$

choose a competitor:

$$\begin{tikzcd} Q \arrow[r, dashed, "\exists! \psi"] \arrow[dr, "f"'] \arrow[drrr, bend left=60, "g"] & \mathrm{elt}(X \times Y) \arrow[d, "\mathrm{elt}(p)"'] \arrow[drr, "\mathrm{elt}(q)"] & & \\ & \mathrm{elt}(X) & & \mathrm{elt}(Y) \end{tikzcd} \quad (\ast)$$

we want to show $\exists!\ \psi : Q \to \mathrm{elt}(X \times Y)$ making the new triangles commute.

$f : Q \to \mathrm{elt}(X)$ sending a point $a \in Q$ to a point $f(a) \in \mathrm{elt}(X) = \{h : 1 \to X\}$

so $f(a) : 1 \to X$. Similarly $g(a) : 1 \to Y$

we want to define $\psi : Q \to \mathrm{elt}(X \times Y)$; this will send any $a \in Q$ to $\psi(a) : 1 \to X \times Y$

But $X \times Y$ is universal!

$$\begin{tikzcd} 1 \arrow[dr, dashed] \arrow[ddr, bend right, "f(a)"'] \arrow[ddrr, bend left, "g(a)"] & & \\ & X \times Y \arrow[d, "p"'] \arrow[dr, "q"] & \\ & X & Y \end{tikzcd}$$

So for each $a \in Q$ we get $\exists!\ \psi(a) : 1 \to X \times Y$ s.t. the diagram to the left commutes!

So define ψ this way and check that $(\ast)$ commutes and that forces ψ to be unique. $\boxtimes$

What if C is ccc? Then

$$\hom(X,Y) \cong \hom(1\times X, Y) \cong \hom(1, Y^X)$$
$$= \mathrm{elt}(Y^X) \quad \text{Since } X \cong 1\times X$$

thus:

$$\begin{tikzcd} 1\times X \arrow[r, "\alpha"] \arrow[dr, "f\circ\alpha"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha^{-1}"] \arrow[dr, "g\circ\alpha^{-1}"'] & 1\times X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

gives us a bijection

$$\hom(X,Y) \cong \hom(1\times X, Y)$$
$$f \longmapsto f\circ\alpha$$
$$g\circ\alpha^{-1} \longleftarrow g$$

Moral: we can convert the hom-object $Y^X \in C$ to the homset $\hom(X,Y) \in \mathrm{Set}$ by taking elements.

Given $f: X\to Y$ in $\hom(X,Y)$, we can convert it into an element of Y^X, called the name of f:

$$\ulcorner f \urcorner : 1 \to Y^X$$

Conversely, any element of Y^X is the name of a unique morphism $f: X\to Y$

In functional programming, objects are data types, morphisms are programs and any program has a "name" which is an $\ulcorner f\urcorner \in \mathrm{elt}(Y^X)$

[Subobjects]

$\boxed{\text{Def}}$ In any category C, a subobject of an object $X \in C$.

It's an equivalence class of monomorphisms

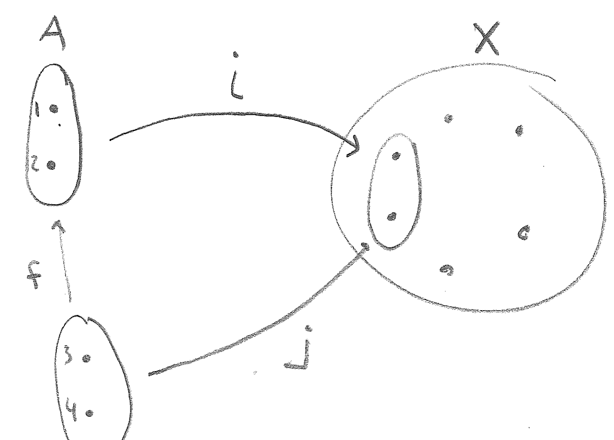
$$i : A \to X$$

where monos $i : A \to X$ $j : B \to X$ are equivalent if there is an isomorphism $f : A \to B$ s.t.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

commutes.

ex if $C = \text{Set}$, then subobjects are subsets! They correspond in a 1-1 way.



Given a mono $i : A \to X$ we get a subset $\text{im}(i) \subseteq X$.

Any subset $S \subseteq X$ arises in this way via the inclusion:

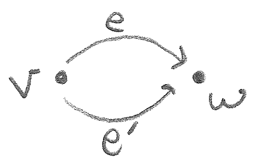
$$i : S \to X$$
$$s \mapsto s \in X$$

finally, given monos $i ; A \to X$, $j : B \to X$ that define the same subset, $\text{im}(i) = \text{im}(j)$ there exists a bijection $f : A \to B$ s.t. $i = j \circ f$

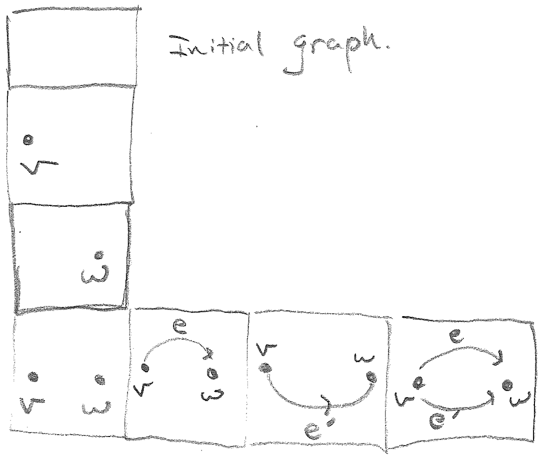
or $f = (j|_{\text{im}(j)})^{-1} \circ i$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

<u>ex</u> In the category of graphs, how many subobjects does the following have?



They are



Initial graph.

$\boxed{\text{Prop}}$ In Set, subobjects of $S \in Set$ are in 1-1 correspondence with functions $\chi : S \to 2$, $2 = \{F, T\}$

<u>Proof</u> subobjects of S are just subsets of it. So any subset $A \subseteq S$ has a characteristic function $\chi : S \to 2$

$$\chi(s) = \begin{cases} F, & s \notin A \\ T, & s \in A \end{cases}$$

Conversely, given $\chi : S \to 2$, let $A = \chi^{-1}(T) = \{s \in S : \chi(s) = T\}$ ☒

Roughly, a subobject classifier in a category C is an object $\Omega \in C$ that plays the role of $2 = \{F, T\}$, in that subobjects of any object $S \in C$ are in 1-1 correspondence with morphisms $\chi : S \to \Omega$

Set has a subobject classifier $2 = \{F, T\}$. What does this really mean? First there is a function called true:

$t : 1 \to 2$ from $1 = \{*\}$ to 2 given by $t(*) = T \in 2$

For any set A we have a unique function

$!_A : A \to 1$ since 1 is the terminal object.

Claim given a monomorphism $i : A \to X$ $\exists!$ function $\chi_i : X \to 2$

called the characteristic function of i s.t.

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "i"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi_i"'] & 2 \end{tikzcd}$$

is a pullback. χ_i in more familiar terms will be the characteristic function of the subset $\mathrm{im}(i) \subseteq X$. We call it the characteristic function of the mono i.

Proof of claim

first we show χ_i makes the square a pullback. Notice $\chi_i(x) = \begin{cases} T & x \in \mathrm{Im}\, i \\ F & x \notin \mathrm{Im}\, i \end{cases}$

Let Q be a competitor:

$$\begin{tikzcd} Q \arrow[drr, bend left, "!_Q"] \arrow[dr, dotted, "\psi"] \arrow[ddr, bend right, "f"'] & & \\ & A \arrow[r, "!_A"] \arrow[d, "i"] & 1 = \{*\} \arrow[d, "t"] \\ & X \arrow[r, "\chi_i"'] & 2 = \{T, F\} \end{tikzcd}$$

We show $\exists!\ \psi : Q \to A$ making the newly formed triangles commute. Since Q is a competitor,

$$\begin{aligned} \chi_i(f(q)) &= t(!_Q(q)) \\ &= t(*) \\ &= T \end{aligned}$$

$\Rightarrow f(q) \in \mathrm{Im}(i)$

Since i is 1-1 we define $\psi(q) = i^{-1}(f(q))$

Uniqueness follows from i being 1-1 and ψ obviously makes the triangle commute. Notice the upper triangle commutes since 1 has a single element.

You should check χ_i is the only such morphism that makes this square a pullback.

Generalizing, we define the following:

$\boxed{\text{Def}}$ Given a category with a terminal object, a subobject classifier is an object $\Omega \in C$ with a morphism $t: 1 \to \Omega$ s.t. for any mono morphism $i: A \to X$, $\exists!\ \chi_i : X \to \Omega$

s.t.
$$\begin{tikzcd}
A \arrow[r, "!_A"] \arrow[d, "i"'] & 1 \arrow[d, "t"] \\
X \arrow[r, "\chi_i"'] & \Omega
\end{tikzcd}$$
is a pullback.

$\boxed{\text{Def}}$ A (elementary) <u>topos</u> is a cartesian closed category with finite limits and a subobject classifier.

Brief History: Grothendieck introduced a concept of topos now called a Grothendieck topos which is a special case of an elementary topos. He was proving the Weil hypothesis in number theory. Later in the 60-70's Lawvere simplified and generalized the concept of topos to define an elementary topos.

examples of elementary topoi:

① **Set**: category of sets and functions.

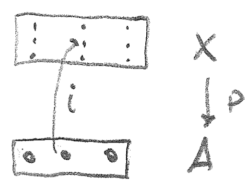
② Finset: category of finite sets and functions.
- this doesn't have all limits, only finite limits.

③ Set′: category of sets and functions as defined using Zermelo-Fraenkel without using the axiom of choice.

The axiom of choice for a category theorist is equivalent to:

for every epimorphism $p: X \to A$ $\exists$ a monomorphism $i: A \to X$

s.t. $p \circ i = 1_A$



when this happens we say an epimorphism splits.

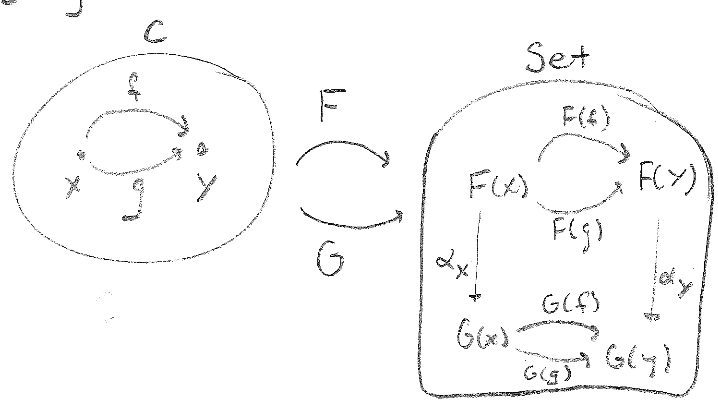
This need not hold in general.

④ Graph: category of graphs.

⑤ Previous example is a special case of a category Set^C, the category of functors from C to Set and natural transformations between them. C can be any category. These are called presheaf categories when we write them as $Set^{(D^{op})}$

if
$$C = \begin{tikzcd} x \arrow[loop left, "1_x"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & y \arrow[loop right, "1_y"] \end{tikzcd}$$

$Set^C \cong Graph$



A functor $F: C \to Set$ is a graph with $E = F(x)$ $V = F(y)$ $s = F(f)$ $t = F(g)$

So a graph is an object in Set^C for this particular choice of C.

A morphism in Set^C is a map between graphs (eg. α_X, α_Y)

⑥ Another example of a presheaf category is called the category of simplicial sets.

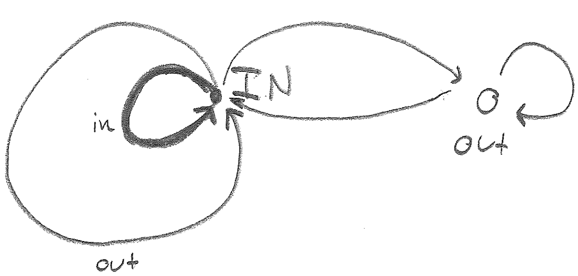
The subobject classifier in Graph.

This is some graph Ω s.t. subgraphs A of any graph X correspond to morphisms of graphs $\chi : X \to \Omega$ in such a way that

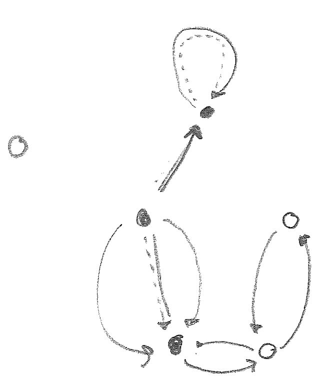
$$\begin{tikzcd} A \arrow[r, "!A"] \arrow[d, "i \text{ mono}"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

is a pullback.

Ω looks like this:



we'll draw X then a subgraph $i : A \to X$. Filled in dots and dashed arrows represent the subgraph.



So filled vertices go to In, hollow to out and edges go to obvious edges. Thats how χ works

Conversely, any morphism of graphs $\chi : X \to \Omega$ determines a subgraph consisting of edges and vertices that are darkened in Ω.

The terminal graph, 1, looks like: 

The darkened subgraph of Ω is a copy of 1.

We get this from the morphism $t : 1 \to \Omega$ which you have in any topos. A vertex or edge of X will be mapped to this subgraph in Ω <u>iff</u> the vertex or edge is in A.

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "i"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

Graph X

This diagram commutes and it's also a pullback. This allows you to reconstruct A if you know about χ

The most important properties of topoi:

$\boxed{\text{Prop}}$ A topos has finite colimits, meaning it has colimits of finite sized diagrams.

$\boxed{\text{Prop}}$ Any morphism $f : X \to Y$ in a topos has an epi-mono factorization

i.e.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & A \arrow[ur, "i"'] & \end{tikzcd}$$

there is an epi $p : X \to A$ and $i : A \to Y$ a mono s.t. the triangle commutes.

<u>Ex</u> In Set, you take $p = f|_{im(f)}$ then i the inclusion.

So each $f = i \circ p$ with p onto and i 1-1.

$\boxed{\text{Prop}}$ In a topos, the epi-mono factorization of any morphism $f: X \to Y$ is unique up to isomorphism: given two epi-mono factorizations;

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] \arrow[ddr, bend right, "p'"'] & Y \\ & A \arrow[u, "i"'] \arrow[d, "g"] \\ & A' \arrow[uu, bend right, "i'"'] \end{tikzcd}$$

$\exists!$ isomorphism $g: A \to A'$ making the resulting diagram commute.

$\boxed{\text{Def}}$ Given an epi-mono factorization, we call A the image of f. we denote it $\text{im}(f)$

Lets generalize $\subseteq, \cap, \cup$ to any topos. From here on assume C is a topos.

$\boxed{\text{Def}}$ Given $X \in C$, define $\text{Sub}(X)$ to be the set of all subobjects of X / equivalence classes of monomorphisms $i: A \to X$ where $i: A \to X$, $j: B \to X$ are equivalent if $\exists$ an iso $g: A \to B$ s.t.

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "g", "\sim"'] & & X \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

commutes.

<u>Note</u> $\text{Sub}(X) \cong \text{hom}(X, \Omega)$ since Ω is the subobject classifier.

$\boxed{\text{Prop}}$ $\text{Sub}(X)$ is a poset where we say the equivalence class of one mono is contained in (or $\subseteq$) the equivalence class of $j: B \to X$ if $\exists$ $f: A \to B$ making the diagram commute

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "f"'] & & X \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

Notice f must be a mono and if it exists, it's unique.

Lets say $[i] \subseteq [j]$ in this case.

Proof

We need to check

① if $[i] \subseteq [j]$ & $[j] \subseteq [k] \Rightarrow [i] \subseteq [k]$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "i"] & \\ B \arrow[r, "j"] \arrow[d, "g"'] & X \\ C \arrow[ur, "k"'] & \end{tikzcd} \qquad \curvearrowright \qquad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g \circ f"'] & X \\ C \arrow[ur, "k"'] & \end{tikzcd}$$

② $[i] \subseteq [i]$ is easy.

③ if $[i] \subseteq [j]$ & $[j] \subseteq [i]$ then $[i] = [j]$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i"] & X \\ B \arrow[u, "g"] \arrow[ur, "j"'] & \end{tikzcd}$$

We'll show g is the inverse of f.

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "i"] & \\ B \arrow[r, "j"] \arrow[d, "g"'] & X \\ A \arrow[ur, "i"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "g"'] \arrow[dr, "j"] & \\ A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

So $i \circ g \circ f = i \circ 1_A$ and $j \circ f \circ g = j \circ 1_B$ but i and j are monic, so we can left cancel.

$\Rightarrow \quad g \circ f = 1_A$ and $f \circ g = 1_B$.

Next time well define $\cup$ for Subobjects, and this makes $Sub(X)$, which is a poset and thus a category, into a category with coproduct: $\cup$ is the coproduct in $Sub X$. Similarly $\cap$ is the product in $Sub(X)$

[Set theory, Topos, and Logic]

In **Set**, every subset of $X \in Set$ corresponds to a predicate on elements of X. $\chi : X \to \{T, F\}$ i.e. a characteristic function.

χ determines a subset $A \subseteq X$ via $A = \{x \in X : \chi(x) = T\}$ and conversely any subset $A \subseteq X$ determines $\chi : X \to \{T, F\}$ via $\chi(x) = \begin{cases} T, & x \in A \\ F, & x \notin A \end{cases}$

In a topos, we get a similar bijection between $Sub(X)$ and $hom(X, \Omega)$

The concepts of $\cup$ and $\cap$ for subsets correspond to the operations of $\vee$ (or) & $\wedge$ (and) on predicates:

$$\{x \in X : \chi(x) = T\} \cup \{x \in X : \varphi(x) = T\} = \{x \in X : (\chi \vee \varphi)(x) = T\}$$

and similarly for $\cap$ & $\wedge$.

$\boxed{\text{Prop}}$ In **Set**, $Sub(X)$ for $X \in$ **Set** is a poset via $\subseteq$, and thus a category where there exists a unique morphism from A to B iff $A \subseteq B$ $(A, B \subseteq X)$. In this category, $A \cap B$ is the product of A and B while $A \cup B$ is the coproduct.

Proof Sketch: We have

$$\begin{tikzcd} & A \cap B \arrow[dl, phantom, "\subseteq" sloped] \arrow[dr, phantom, "\subseteq" sloped] & \\ A & & B \end{tikzcd}$$

and this cone is universal:

$$\begin{tikzcd} Q \arrow[dr, dashed, "\exists!"] \arrow[ddr, bend right] \arrow[ddrrr, bend left] & & & \\ & A \cap B \arrow[d, phantom, "\cong" sloped] \arrow[drr, phantom, "\subseteq" sloped] & & \\ & A & & B \end{tikzcd}$$

Since $Q \subseteq A$ and $Q \subseteq B \Rightarrow Q \subseteq A \cap B$ we're done. The coproduct is the same with inclusions turned around. $\blacksquare$

In fact, in **Set**, $Sub(X)$ has all finite limits and all finite colimits.

A category has all finite limits if it has:

- Binary products
- Terminal object

$\Big\} \Rightarrow$ All finite products exist (Cartesian)

- Equalizer

$Sub(X)$ has binary products $(\cap)$, terminal object (X, since $A \subseteq X$ $\forall A \in Sub(X)$), and equalizers:

$$B \underset{g}{\overset{f}{\rightrightarrows}} C \quad \text{is really} \quad B \underset{f}{\overset{f}{\rightrightarrows}} C \quad \text{in a Poset.}$$

and the equalizer is: $B \xrightarrow{1} B \underset{f}{\overset{f}{\rightrightarrows}} C$

Similarly, in Set, Sub(X) has all finite colimits since it has:

- binary coproducts ($A \cup B$ is the coproduct of A & B)
- initial object ($\emptyset$, since $\emptyset \subseteq A \ \forall A \in Sub(X)$)
- coequalizers (turn arrows around in previous argument.)

$\boxed{\text{Def}}$ A lattice is a poset with all finite limits and colimits.

(This is equivalent to more standard definitions)

$\subseteq$	$\Rightarrow$	$\longrightarrow$	
Set Theory	Logic	Category Theory	
$\cap$	$\wedge$	Binary Product	Finite limits
X	T	Terminal Object	
$\cup$	$\vee$	Binary Coproduct	Finite Colimits
$\emptyset$	F	Initial Object	
$B \cup A^c$	$Q \vee \neg P$ or "P implies Q"	Exponentiation	Cartesian Closedness

Note: $X = \{x \in X : T = T\}$ $\quad \emptyset = \{x \in X : F = T\}$

In fact, the poset Sub(X) is cartesian closed. In general this means $hom(B \times C, D) \cong hom(B, D^C)$. In Sub(X) this means that these sets have no elements or one.

Also, the product is intersection. So this says

$$B \cap C \subseteq D \text{ iff } B \subseteq \text{“}D^C\text{”} \quad \text{so } D^C = D \cup C^c$$

or in terms of logic $\quad P \wedge Q \Rightarrow R$ iff $P \Rightarrow R \vee \neg Q$

or “Q implies R”

$\boxed{\text{Thm}}$ In any topos, for any object X the poset $Sub(X)$ is

a Heyting algebra:

it's a poset with finite limits, finite colimits, and

is Cartesian closed.

i.e. It's a Cartesian closed lattice.

Proof Sketch.

Given two subobjects $[i], [j]$ of X, we wish to form

$[i] \cup [j]$ and $[i] \cap [j]$. Taking the pullback we get intersection.

$$\begin{tikzcd} A \cap B \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, dotted] & B \arrow[d, "j"] \\ A \arrow[r, "i"'] & X \end{tikzcd}$$

(Pullback is happening in the topos)

From previous work, since i, j are monic, so are f and g.

Finally, $i \circ f = j \circ g$ is also monic and we get a new subobject of X.

This is what we call $[i] \cap [j]$.

For unions, we start with the coproduct.

But we get $\varphi : A+B \to X$ from the universal property of the coproduct.

$$\begin{tikzcd} A+B \arrow[dr, "\exists! \varphi"] & B \arrow[l, "g"'] \arrow[d, "j"] \\ A \arrow[u, "f"] \arrow[r, "i"'] & X \end{tikzcd}$$

We'd like a mono. We'll use the fact that φ has an epi mono factorization:

$$\begin{tikzcd} A+B \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & X \\ & \mathrm{Im}\,\varphi \arrow[ur, "k"'] & \end{tikzcd}$$

where p is epic and k is monic.

k gives a new subobject of X. We'll call it $[i] \cup [j]$.

Where does topos theory go from here?

- Using the Mitchell-Benabou language, we can reason inside any topos. We can write things like

$$\{ x \in A \cap B : \forall y \in Y \ \exists z \in Z, f(x,y) = y \}$$ and prove things about them using the logic internal to the topos, and "generalized elements".

- There are also maps between topoi:

$$\begin{tikzcd} C \arrow[r, bend left] & D \arrow[l, bend left] \end{tikzcd}$$ consisting of certain nice adjunctions. They're called "geometric morphisms."

There's a topos called $\mathrm{Th}(\mathrm{Grp})$. "the theory of a group." A geometric morphism from some topos C to $\mathrm{Th}(\mathrm{Grp})$ turns out to be the same as a group object in C.

This idea works for tons of mathematical structures or concepts.