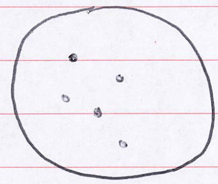
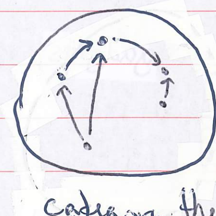
Category Theory Course 1/5/16

Category Theory:
- unifies mathematics
- studies the mathematics of mathematics
- moves toward higher-dimensional algebra ("homotopifying" mathematics)

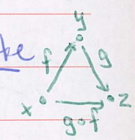


set theory
0-dimensional

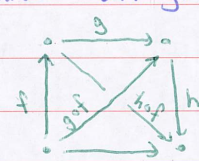


category theory
1-dimensional

Def A <u>category</u> C consists of:
- a class $Ob(C)$ of <u>objects</u>
  (If $x \in Ob(C)$ we write simply $x \in C$)
- Given $x, y \in C$ there's a set $\hom(x,y)$, called a <u>homset</u>, whose elements are called <u>morphisms</u> or <u>arrows</u> from x to y. If $f \in \hom(x,y)$ we write $f : x \longrightarrow y$.
- Given $f : x \longrightarrow y$ and $g : y \longrightarrow z$ there is a morphism called their <u>composite</u> $g \circ f : x \longrightarrow z$

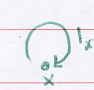


- Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$ if either side is well-defined.



$h \circ (g \circ f) = (h \circ g) \circ f$

- For any $x \in C$, there is an <u>identity morphism</u> $1_x : x \longrightarrow x$



- We have the <u>left and right unit laws</u>:
  $1_x \circ f = f$ for any $f : x' \longrightarrow x$
  $f \circ 1_x = f$ for any $f : x \longrightarrow x'$

Examples of categories

Categories of mathematical objects
For any kind of mathematical object, there's a category with objects of that kind & morphisms being the structure-preserving maps between objects of that kind.

- **Set** is the category with sets as objects & functions as morphisms.
- **Grp** is the category of groups and homomorphisms
- For k any field, Vect_k of vector spaces over k and linear maps.
- **Ring** is the category of rings & ring homomorphisms

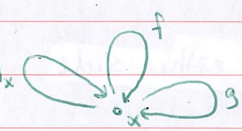
These are categories of "algebraic" objects, namely:
- a set] stuff
- with operations] structure
- obeying equations] properties

with morphisms being functions that preserve the operations.
All this is formalized in "universal algebra", using "algebraic theories".

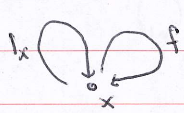
There are also categories of non-algebraic gadgets:
- **Top**, the category of topological spaces & continuous maps.
- **Met**, the category of metric spaces & continuous maps.
- **Meas**, the category of measurable spaces & measurable maps

Categories as mathematical objects
There are lots of small, manageable categories:

Def A <u>monoid</u> is a category with one object.
(Then $\hom(x,x)$ for this object x is a set with associative product & unit.)



Ex



with $1_x \circ f = f$
$f \circ 1_x = f$
$f \circ f = 1_x$ is usually called $\mathbb{Z}/2$

Or we could take $f \circ f = f$, and this gives another famous monoid:

1_x = TRUE		1_x = FALSE
f = FALSE	or	f = TRUE
$\circ$ = AND		$\circ$ = OR

Def A morphism $f: x \to y$ is an <u>isomorphism</u> if it has an <u>inverse</u> $g: y \to x$, i.e. a morphism with

$$g \circ f = 1_x$$
$$f \circ g = 1_y$$

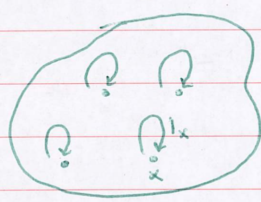
If there exists an isomorphism between 2 objects $x, y \in C$, we say they're <u>isomorphic</u>.

Def A category where all morphisms are isomorphisms is called a <u>groupoid</u>.

Ex "the groupoid of finite sets" is obtained by taking FinSet, with finite sets as objects and functions as morphisms, and then throwing out all morphisms except isomorphisms (i.e. bijections), getting a groupoid.

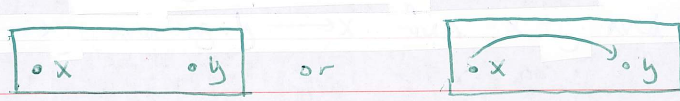
Def A monoid that is a groupoid is called a <u>group</u>.
(The usual "elements" of a group are now the morphisms.)

Def A category with only identity morphisms is a <u>discrete</u> category.



So any set is the set of objects of some discrete category in a unique way.
So a discrete category is "essentially the same" as a set.

Def A <u>preorder</u> is a category with at most one morphism in each hom set.



is hom(x,y)

If there is a morphism $f: x \to y$ in a preorder we say "$x \le y$";
if not we say "$x \not\le y$".
For a preorder, the category axioms just say
- composition: $x \le y$ & $y \le z \Rightarrow x \le z$
- associativity is automatic
- identities: $x \le x$ always
- left & right unit laws are automatic

We're not getting antisymmetry:
$$x \le y \ \&\ y \le x \Rightarrow x = y$$

Categories as Mathematical Object, cont.

$\boxed{\text{Def}}$ A <u>preorder</u> is a category C where for all $x, y \in C$ there is at most one morphism $f : x \to y$.

We write "$x \leq y$" iff $\exists\ f : x \to y$.

We know what C is if we know this relation on objects (if C is a preorder), & then the category axioms simply say:

- there's a class of objects
- there's a relation $\leq$ on objects
- $x \leq y$ & $y \leq z \Rightarrow x \leq z$ (composition) $\quad \forall x, y, z \in C$
- $x \leq x$ (identities) $\quad \forall x \in C$

$\boxed{\text{Def}}$ An <u>equivalence relation</u> is a preorder that's also a groupoid.

$\boxed{\text{Prop}}$ A preorder C is a groupoid iff this extra law holds:

$x \leq y \Rightarrow y \leq x \quad \forall x, y \in C.$

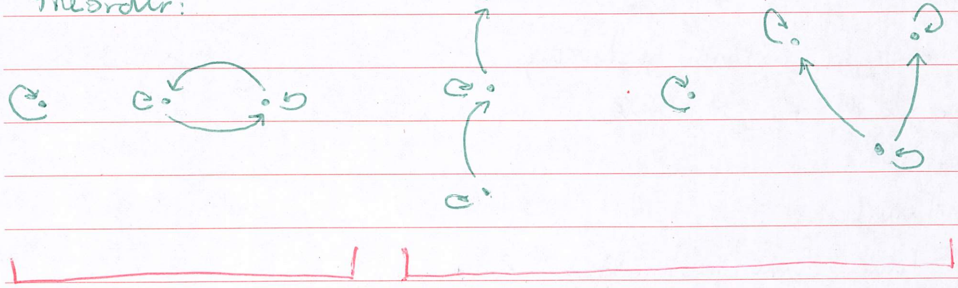
Here we have transitivity, reflexivity, & symmetry of "$\leq$" so we usually call this relation $\sim$.

$\boxed{\text{Prop}}$ A preorder is <u>skeletal</u>, i.e. isomorphic objects are equal, iff this extra law holds:

$x \leq y$ & $y \leq x \Rightarrow x = y \quad \forall x, y \in C$

In this case we say C is a <u>poset</u>.

Preorder:



this part is a groupoid but not a poset

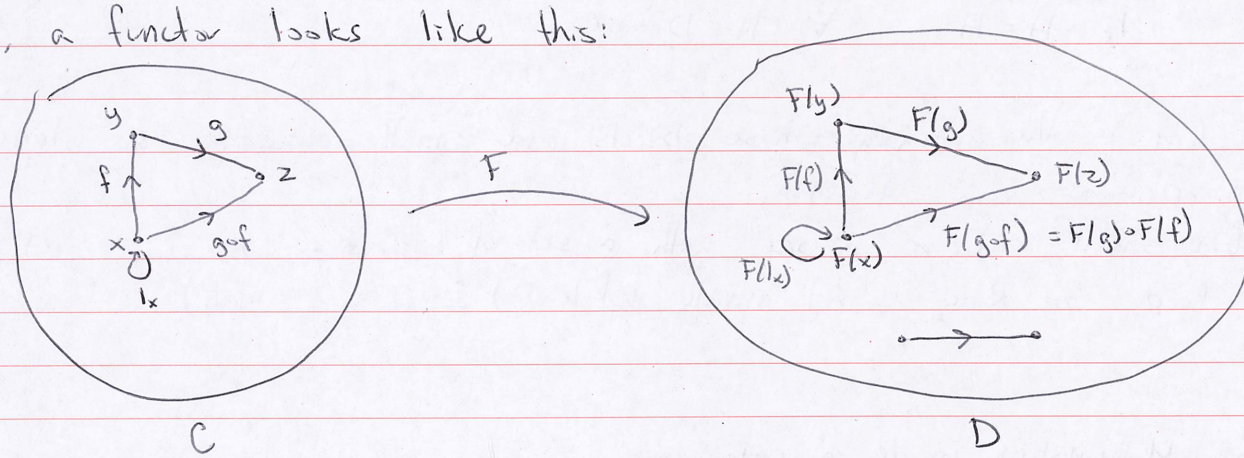
this part is a poset but not a groupoid

Since categories can be seen as mathematical objects, we should define maps between them:

Def Given categories C & D, a <u>functor</u> $F: C \longrightarrow D$ consists of:
- a function called F from $Ob(C)$ to $Ob(D)$: if $x \in C$ then $F(x) \in D$
- functions called F from $hom(x,y)$ $(\forall x,y \in C)$ to $hom(F(x), F(y))$: if $f: x \longrightarrow y$ then $F(f): F(x) \longrightarrow F(y)$

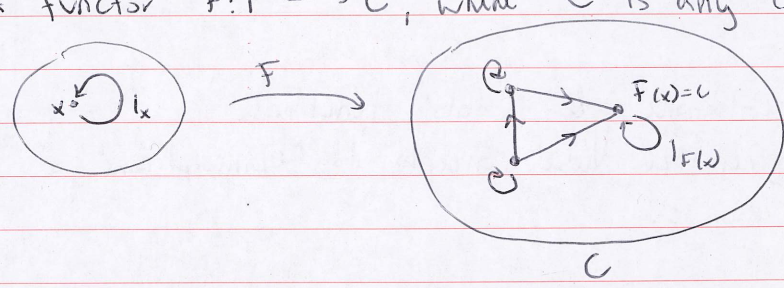
such that:
- $F(g \circ f) = F(g) \circ F(f)$ whenever either side is well-defined.
- $F(1_x) = 1_{F(x)}$ $\forall$ $x \in C$

So, a functor looks like this:



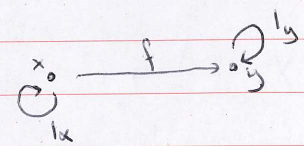
Ex There's a category called "1". It looks like this: $x \circlearrowleft 1_x$

What is a functor $F: 1 \longrightarrow C$, where C is any category?



The answer is: "an object in C", since for any $c \in C$ $\exists!$ $F: 1 \longrightarrow C$ s.t. $F(x) = c$.

Ex There's a category called "2". (a poset)



What is a functor $F : 2 \to C$?

It's just a morphism or arrow in C!

For any morphism $g : c \to c'$ in C, $\exists!$ functor $F : 2 \to C$ s.t. $F(f) = g$.

$\boxed{\text{Prop}}$ If $F : C \to D$ & $G : D \to E$ are functors then you can define a functor $G \circ F : C \to E$, and $(H \circ G) \circ F = H \circ (G \circ F)$.

Also, for any category C there's an <u>identity functor</u> $1_C : C \to C$ with

$1_C(x) = x \quad \forall x \in C$

$1_C(f) = f \quad \forall f : x \to y$ in C

and $F \circ 1_C = F \quad \forall F : C \to D$

$1_C \circ H = H \quad \forall H : D \to C$

$\boxed{\text{Def}}$ Cat is the category whose objects are "small" categories & whose morphisms are functors.

(A "small" category is one with a set of objects --- so e.g. Set or Grp or Ring is not small, while 1 & 2 are small.)

Doing Mathematics inside a category.

A lot of math is done in Set, the category of sets & functions. Let's try to generalize all that stuff to other categories: replace Set by a general category C.

In Set we have "one-to-one" & "onto" functions.

In a category C we generalize these concepts to "epimorphisms" or "epis" & "monomorphisms" or "monos".

$\boxed{\text{Def}}$ A morphism $f : X \to Y$ is a <u>mono</u> if $\forall g, h : Q \to X$ we have $f \circ g = f \circ h \Rightarrow g = h$

$$Q \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

$\boxed{\text{Prop}}$ In Set, a morphism is monic iff it's a one-to-one function.

Turning around the arrows in the definition of mono, we get:

$\boxed{\text{Def}}$ A morphism $f: Y \longrightarrow X$ is an <u>epi</u> if $\forall$ $g, h: X \longrightarrow Q$ we have $g \circ f = h \circ f \Rightarrow g = h$.

$$Y \xrightarrow{f} X \underset{h}{\overset{g}{\rightrightarrows}} Q$$

$\boxed{\text{Prop}}$ In Set, a morphism is epic iff it's an onto function.

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$\boxed{\text{Def}}$ A morphism $f: X \to Y$ is an <u>iso</u> if $\exists\ f^{-1}: Y \to X$ that's a <u>left inverse</u> $f^{-1} \circ f = 1_X$
& a <u>right inverse</u> $f \circ f^{-1} = 1_Y$

$\boxed{\text{Prop}}$ In Set, $f: X \to Y$ is a mono iff it has a left inverse, and an epi iff it has a right inverse (using the Axiom of Choice).
Thus f is an iso iff it's a mono & epi.

assume we have identity

$\boxed{\text{Prop}}$ In Ring (rings & ring homomorphisms) $f: \mathbb{Z} \longrightarrow \mathbb{Q}$, $n \longmapsto n$
is a mono and an epi but not an iso; in fact it has neither left nor right inverse.

Pf:
There's no ring homomorphism $g: \mathbb{Q} \to \mathbb{Z}$ since it would send $\frac{1}{2}$ to some multiplicative inverse of 2.

Why is f a mono?
Need:
$f \circ g = f \circ h \Rightarrow g = h$ $\qquad R \underset{h}{\overset{g}{\rightrightarrows}} \mathbb{Z} \xrightarrow{f} \mathbb{Q}$
If $(f \circ g)(r) = (f \circ h)(r)\ \forall r \in R$,
since f is 1-1 $g(r) = h(r)\ \forall r$ (as a function)
$\Rightarrow g = h$

Why is f epi?
Need:
$g \circ f = h \circ f \Rightarrow g = h$ $\qquad \mathbb{Z} \xrightarrow{f} \mathbb{Q} \underset{h}{\overset{g}{\rightrightarrows}} R$
We know $g(p) = h(p)$ & $g(q) = h(q)$
So $g(1) = g(\frac{q}{q}) = g(q) \cdot g(\frac{1}{q})$, so we can write $g(\frac{1}{q}) = \frac{1}{g(q)}$.
So $g(\frac{p}{q}) = g(p) \cdot g(\frac{1}{q}) = \frac{g(p)}{g(q)}$
So g (& similarly h) is determined by its values on integers; since they agree on $\mathbb{Z}$ they're equal

Puzzle: In Top, find $f: X \to Y$ that is epic & monic but not an isomorphism.

Limits & Colimits

These are ways of building new objects in a category C from diagrams in C,

e.g.:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$$

An example of a limit is:

$\boxed{\text{Def}}$ Given objects $X, Y \in C$, a product of them is an object Z equipped with morphisms

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

called projections to X & Y,

s.t. for any candidate Q

$$\begin{tikzcd} & Q \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

there $\exists !\ \Psi : Q \to Z$ s.t.

this diagram commutes

$$\begin{tikzcd} Z \arrow[d, "p"'] \arrow[dr, "q"' near start] & Q \arrow[l, bend right, "\Psi"'] \arrow[dl, "f" near start] \arrow[d, "g"] \\ X & Y \end{tikzcd}$$

i.e.

$f = p \circ \Psi$

$g = q \circ \Psi$

The definition of coproduct is just the same but with all arrows reversed.

$\boxed{\text{Prop}}$ In Set, we get a product of X & Y by taking $X \times Y = \{(x,y) : x \in X,\ y \in Y\}$ with $p(x,y) = x$ and $q(x,y) = y$.

Pf:

Given

$$\begin{tikzcd} & Q \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

let $\Psi : Q \longrightarrow X \times Y$ be $\Psi(q) = (f(q), g(q))$

We indeed get $p \circ \Psi = f$, $q \circ \Psi = g$ & Ψ is the unique map obeying these equation.

But we could also take as our product any set S that's isomorphic to $X \times Y$, via some iso. $\alpha : S \to X \times Y$

$$\begin{tikzcd} X \times Y \arrow[d, "p"'] \arrow[dr, "q"] & S \arrow[l, "\alpha"', "\sim"] \arrow[d, "q \circ \alpha"] \arrow[dl, "p \circ \alpha"] \\ X & Y \end{tikzcd}$$

Use $p \circ \alpha$ & $q \circ \alpha$ as projections; then you can check

$$\begin{tikzcd} & S \arrow[dl, "p \circ \alpha"'] \arrow[dr, "q \circ \alpha"] & \\ X & & Y \end{tikzcd}$$

is also a product of X & Y.

So "any object isomorphic to a product can also be a product."

$\boxed{\text{Prop}}$ Suppose

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Z \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ X & & Y \end{tikzcd}$$

are both a product of X & Y.

Then W & Z are isomorphic

"Products are unique up to isomorphism."

Pf:

Since W is the product

(*)
$$\begin{tikzcd} W \arrow[d, "p"'] \arrow[dr, "q"] & Z \arrow[l, "\exists ! \psi"'] \arrow[d, "q'"] \arrow[dl, "p'"] \\ X & Y \end{tikzcd}$$

$\exists ! \ \psi : Z \to W$ making this commute

Since Z is the product

(**)
$$\begin{tikzcd} W \arrow[r, "\exists !"] \arrow[d, "p"'] \arrow[dr, "q"] & Z \arrow[d, "q'"] \arrow[dl, "p'"] \\ X & Y \end{tikzcd}$$

$\exists ! \ \varphi : W \to Z$ s.t. the diagram commutes

Suffices to show ψ & φ are inverse. Why is $\psi \circ \varphi : W \to W$ the identity? If we can show this, the same arguement will show $\varphi \circ \psi = 1_Z$.

$$\begin{tikzcd} W \arrow[d, "p"'] \arrow[dr, "q"] & W \arrow[l, bend right, "\exists !"'] \arrow[d, "q"] \arrow[dl, "p"] \\ X & Y \end{tikzcd}$$

There is a unique arrow making this commute since W is the product. $1_W : W \to W$ does the job, but also $\varphi \circ \psi$ does the job.

$$\begin{tikzcd}
W \arrow[r, "\varphi"] \arrow[d, "p"'] \arrow[dr, "q"] & Z \arrow[r, "\psi"] \arrow[dl, "p'"'] \arrow[d, "q'"] & W \arrow[dll, "p"'] \arrow[dl, "q"] \\
X & Y &
\end{tikzcd}$$

This commutes since * & ** do

so this commutes:

$$\begin{tikzcd}
W \arrow[r, "\psi\circ\varphi"] \arrow[d, "p"'] \arrow[dr, "q"] & W \arrow[dl, "p'"'] \arrow[d, "q'"] \\
X & Y
\end{tikzcd}$$

so $\psi \circ \varphi = 1_W$ by uniqueness.

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Whoops:

$\boxed{\text{Prop}}$ If a morphism is an iso, it's both a mono & an epi.
(We've seen the converse is false)

Pf:

If $f: X \to Y$ has a left inverse f^{-1}, it's a mono:

$$f \circ g = f \circ h \Rightarrow f^{-1} \circ f \circ g = f^{-1} \circ f \circ h \Rightarrow g = h \qquad \forall g,h$$

Similarly, if f has a right inverse f^{-1}, it's an epi

$$g \circ f = h \circ f \Rightarrow g \circ f \circ f^{-1} = h \circ f \circ f^{-1} \Rightarrow g = h \qquad \forall g,h$$

$\boxed{\text{Def}}$ A morphism with a left inverse is called a <u>split monomorphism</u>; a morphism with a right inverse is called a <u>split epimorphism</u>.

In Set every mono (or epi) splits, but we saw not in Ring (or Top).

Coproducts

$\boxed{\text{Def}}$ Given objects X & Y, a <u>coproduct</u> of X & Y is an object Z equipped with morphisms

$$\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & Z & \end{tikzcd}$$

(where i & j are called <u>inclusions</u>),

which is <u>universal</u>: for any diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Q & \end{tikzcd}$$

$\exists ! \ \Psi: Z \to Q$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[d, "i"'] \arrow[dr, "f" near start] & Y \arrow[d, "g"] \arrow[dl, "j"' near start] \\ Z \arrow[r, "\exists ! \Psi"'] & Q \end{tikzcd}$$

i.e. $f = \Psi \circ i$

$g = \Psi \circ j$

$\boxed{\text{Prop}}$ In Set, a coproduct of X & Y is their <u>disjoint union</u>

$X + Y = X \times \{0\} \cup Y \times \{1\}$ with $i: X \to X+Y$, $x \mapsto (x, 0)$; $j: Y \to X+Y$, $y \mapsto (y, 1)$

	PRODUCTS $\times$	COPRODUCTS $+$
Set	cartesian product $S \times T$	disjoint union $S \sqcup T = S + T$
Top	cartesian product $X \times Y$ w/ product topology	disjoint union $X \sqcup Y = X + Y$
Gp	product of groups $G \times H$	free product $G * H$
AbGp	$A \oplus B = A \times B$ product of abelian groups	$A \oplus B$
Vect_k	$V \oplus W = V \times W$ direct sum of vector spaces	$V \oplus W$

abelian categories

The free product $G * H$ consists of equivalence classes of words

$x_1 x_2 \cdots x_n$ where $x_i \in G \cup H$

where $x_1 \cdots x_{i-1} \cdot 1 \cdot x_{i+1} \cdots x_n \sim x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$ where 1 is the identity in G or H

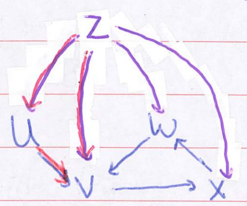
and $x_1 \cdots x_i x_{i+1} \cdots x_n = x_1 \cdots x_{i-1} y x_{i+2} \cdots x_n$ if $x_i, x_{i+1} \in G$ or $x_i, x_{i+1} \in H$ and $y = x_i x_{i+1}$

General limits & colimits

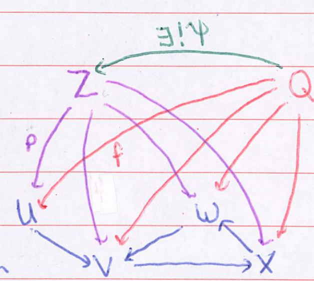
Given any diagram in a category C:

$$\begin{tikzcd} U \arrow[dr, "f"] & & W \\ & V \arrow[ur, leftarrow, "g"] \arrow[r, "i"] & X \arrow[ul, "h"'] \end{tikzcd}$$

a cone over the diagram is:



a choice of morphisms from Z to each object in the diagram, such that all newly formed triangles commute.



A limit of the diagram is a cone that's universal: i.e., given any competitor (another candidate), another cone over the same diagram:

$\exists!\, \psi : Q \to Z$ s.t. all triangles including ψ commute:

if U is any object in the diagram and $p : Z \to U$ is the morphism in the universal cone, and $f : Q \to U$ is the morphism in the competitor then $f = p \circ \psi$.

A *cocone* is like a cone, but with morphisms to Z instead of from:

$$\begin{tikzcd}
U \arrow[dr] \arrow[ddrr] & & W \arrow[dd] & \\
& V \arrow[ur] \arrow[rr] \arrow[dr] & & X \arrow[ul] \arrow[dl] \\
& & Z &
\end{tikzcd}$$

A *colimit* is the 'universal cocone':

$$\begin{tikzcd}
U \arrow[dr] \arrow[ddr, "i"'] \arrow[ddrrr, "f"] & & W \arrow[ddl] \arrow[ddr] & \\
& V \arrow[ur] \arrow[rr] \arrow[d] \arrow[drr] & & X \arrow[ul] \arrow[dll] \arrow[d] \\
& Z \arrow[rr, "\exists!\psi"'] & & Q
\end{tikzcd}$$

$f = \psi \circ i$

Examples of different diagrams

Diagrams	LIMITS	COLIMITS
$\bullet \quad \bullet$	(binary) product	(binary) coproduct
$\bullet \rightrightarrows \bullet$	equalizer	coequalizer
$A \to C \leftarrow B$	pullback	C
$A \leftarrow C \to B$	C	pushout
$\bullet A$	A	A
$A \to B$	A	B
	terminal object 1	initial object 0

What's a limit of the empty diagram:

$$Z \xleftarrow{\exists!\Psi} Q$$

It's an object Z s.t. for all objects Q, $\exists!\ \Psi: Q \to Z$.
This is called a terminal object.

In Set, any 1-element set is a terminal object.
In Vect_k, any 0-dim. vector space is a terminal object.

Similarly, an initial object Z is one s.t. for any object Q, $\exists!\ \Psi: Z \to Q$.

In Set, the empty set is an initial object.
In Vect_k, any 0-dim. vector space is an initial object.

In any abelian category, initial objects are terminal & vice versa.

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Equalizer

Def A limit of this diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

is called an *equalizer*.

Prop In Set, the equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd} \qquad q = f\circ p = g\circ p$$

with

$Z = \{a \in A : f(a) = g(a)\}$

where $p: Z \to A$ has $p(a) = a$ for all $a \in Z$. (It's an inclusion)

q is forced to be $f\circ p = g\circ p$.

Note Since q is determined by p, we usually don't draw it, & write an equalizer like $Z \xrightarrow{p} A \overset{f}{\underset{g}{\rightrightarrows}} B$

Similarly for lots of other limits and colimits.

Pf:

We need to check that this cone is universal, so take a competitor:

$$\begin{tikzcd} Z \arrow[dr, "p"'] & Q \arrow[d, "p'"] & \\ & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

and we want to show $\exists!\ \psi: Q \to Z$ making everything commute: $p\circ\psi = p'$

$(p\circ \overbrace{\psi)(q}^{\in Z}) = \psi(q) \quad \forall q \in Q$ since $p(a) = a \quad \forall a \in Z$.

Thus $\psi\circ p = p'$ simply says $\psi(q) = p'(q) \quad \forall q \in Q$

Thus $\exists!\ \psi$ making everything commute, namely $\psi = p'$.

Prop In Gp, AbGp, or Vect_k, the equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is $\ker(f-g)$.

Note $\ker(f-g) = \{a \in A : f(a) = g(a)\}$

Pf: the same as before

Prop If $Z \xrightarrow{i} A \underset{g}{\overset{f}{\rightrightarrows}} B$ is an equalizer then i is monic.

Moral monics and limits get along well;
epics and colimits do too.

Pf:

Assume we have an equalizer

To check that i is monic we consider $Y \underset{k}{\overset{h}{\rightrightarrows}} Z \xrightarrow{i} A \underset{g}{\overset{f}{\rightrightarrows}} B$

and show $i \circ h = i \circ k \Rightarrow h = k$

Y is a competitor to Z.

Since Z is universal, $\exists ! \, \Psi : Y \to Z$ making everything commute, so

$\Psi = h = k$ ∎

Coequalizers

Def A coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is a universal cocone over this diagram,

i.e.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[d, "i"] \\ & Z \end{tikzcd}$$

commutes s.t. if we have a competitor

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[dl, "i'"'] \arrow[d, "i"] \\ Q & Z \end{tikzcd}$$

$\exists ! \, \Psi : Z \to Q$ making everything commute.

Prop In Set, the coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{i} Z$

where $Z = B/\sim$ where $\sim$ is the finest equivalence relation s.t. $f(a) \sim g(a) \; \forall a \in A$

and $i : B \to Z$

$b \mapsto [b]$ ← its equivalence class

Pf:

$i \circ f = i \circ g$ with this definition, so this is a cocone. Why is it universal?

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[dl, "i'"'] \arrow[d, "i"] \\ Q & Z \arrow[l, "\Psi"] \end{tikzcd}$$

Why $\exists ! \, \Psi : Z \to Q$ making this commute?

To commute, we need

$\Psi \circ i = i'$

$\Psi(i(b)) = i'(b) \quad \forall b \in B$

$\Psi([b]) = i'(b)$

This shows Ψ is unique if it exists; to show it exists need to check it's well-defined:

(continued)

Pf: (continued)

$$[b] = [b'] \overset{\text{need}}{\Longrightarrow} i'(b) = i'(b')$$

$$\Updownarrow$$

$$b \sim b'$$

$$\Updownarrow$$

??? Either $b, b' \in im(f) \cap im(g)$ and $b = b'$

or $b, b' \in im(f) \cap im(g)$ and ~~

Find the good proof!

$\boxed{\text{Prop}}$ In AbGp or Vect_k, the coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is $\text{coker}(f-g) = B/im(f-g)$

$\boxed{\text{Prop}}$ If $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{p} Z$ is a coequalizer, p is epic.

Pf:

Same as proof of the "dual" proposition for equalizers.

Pullbacks

$\boxed{\text{Def}}$ The limit of this diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

is called a pullback,

& denoted

$$\begin{tikzcd} A \times_C B \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

The object here "A times B over C", or the fibred product, and we only need to draw its morphisms to A & B called projections.

We write

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

when Z is a pullback.

$\boxed{\text{Prop}}$ In Set, the pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is

$A \times_C B = \{(a,b) \in A \times B : f(a) = g(b)\}$ with $p: A \times_C B \longrightarrow A$, $(a,b) \longmapsto a$; $q: A \times_C B \longrightarrow B$, $(a,b) \longmapsto b$

Pf:

This is clearly a cone; to show it's universal use the next Prop. ∎

Prop Given
$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

if the product $A \times B$ exists and if this equalizer exists:

$$\begin{tikzcd} Z \arrow[dr, "i"] & & \\ & A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

where $i : Z \to A \times B$ is the equalizer of $A \times B \overset{f \circ \pi_1}{\underset{g \circ \pi_2}{\rightrightarrows}} C$

then this is a pullback:

$$\begin{tikzcd} Z \arrow[r, "\pi_2 \circ i"] \arrow[d, "\pi_1 \circ i"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

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$\boxed{\text{Prop}}$ In Set, a coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is the set $Z = B/\sim$ where $\sim$ is the finest equiv. relation on B s.t. $f(a) \sim g(a)$ $\forall a \in A$. with map $i: A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{i} Z$ s.t. $i(b) = [b]$. Note: $i \circ f = i \circ g$

Pf:

Consider a competitor

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "i"] \arrow[dr, "i'"'] & Z \\ & & Q \end{tikzcd}$$

with $i' \circ f = i' \circ g$

Want: $\exists!\ \Psi: Z \to Q$

Try $\Psi([b]) = i'(b)$ $\forall b \in B$

If Ψ is well-defined, then the diagram commutes since $\Psi(i(b)) = i'(b)$, and Ψ is unique since this formula specifies it.

Why is Ψ well-defined?

Assume $b \sim b'$, need to show $i'(b) = i'(b')$.

If $b \sim b'$, then $b = b_1 \sim b_2 \sim b_3 \sim \cdots \sim b_n = b'$ where for each i either $b_i = f(a)$ & $b_{i+1} = g(a)$ OR $b_i = g(a)$ & $b_{i+1} = f(a)$ for some a (depending on i).

Need to check $i'(b_i) = i'(b_{i+1})$ for each $i = 1, \dots, n-1$.

Either $b_i = f(a)$ & $b_{i+1} = g(a)$ — in which case $i'(b_i) = i'(f(a)) = i'(g(a)) = i'(b_{i+1})$

OR $b_i = g(a)$ & $b_{i+1} = f(a)$, which works similarly. ∎

Pullbacks & Pushouts

$\boxed{\text{Prop}}$ To compute a pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ it suffices to take a product of A & B:

$$\begin{tikzcd} A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

and then form the equalizer of: $Z \xrightarrow{i} A \times B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} C$

giving the desired pullback:

$$\begin{tikzcd} Z \arrow[r, "\pi_2 \circ i"] \arrow[d, "\pi_1 \circ i"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

(Pf on next page)

Pf.

Note the last square commutes since $f\circ\pi_1\circ i = g\circ\pi_2\circ i$, so it's a candidate for being the pullback.
To show it's universal, consider a competitor:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!\Psi"] \arrow[drrr, bend left, "q"] \arrow[ddrr, bend right, "p"'] & & & \\
& Z \arrow[r, "i"] \arrow[rr, bend left] \arrow[dr] & A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\
& & A \arrow[r, "f"] & C
\end{tikzcd}$$

only little square does not commute

How do we show $\exists!\ \Psi: Q \longrightarrow Z$ making the newly formed triangle commute?
By the universal property of the product, we get

$$\begin{tikzcd}
Q \arrow[dr, "\varphi"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \\
& A &
\end{tikzcd}$$

making this commute

Why is Q a competitor?
Need: $f\circ\pi_1\circ\varphi = g\circ\pi_2\circ\varphi$

$$\begin{aligned} f\circ\pi_1\circ\varphi &= f\circ p \\ &= g\circ q \\ &= g\circ\pi_2\circ\varphi \end{aligned}$$ (by various comm. diagrams)

By the universal property of the equalizer, $\exists!\ \Psi: Q \longrightarrow Z$ making this diagram commute:

$$\begin{tikzcd}
Z \arrow[r, "i"] & A\times B \arrow[r, shift left, "f\circ\pi_1"] \arrow[r, shift right, "g\circ\pi_2"'] & C \\
Q \arrow[u, "\Psi"] \arrow[ur, "\varphi"'] & &
\end{tikzcd}$$

In particular, $\varphi = i\circ\Psi$.
Why does this imply:

(1) $\pi_1\circ i\circ\Psi = p$

(2) $\pi_2\circ i\circ\Psi = q$

(3) a unique Ψ making (1) & (2) true.

(continued)

Pf: (continued)

For (1) & (2), suffices to show $\pi_1 \circ \varphi = p$ and $\pi_2 \circ \varphi = q$, but we already had this by the universal property of the product.

Exercise: check (3) ∎

"Category theory makes trivial things trivially trivial." – Michael Barr

"I'm content to let them be trivial." – Timothy Gowers

Prop In Set, a pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is $Z = \{(a,b) \in A \times B : f(a) = g(b)\}$ with obvious maps to A & B.

Pf:

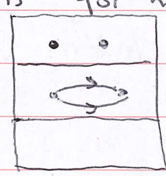
Previous Prop. combined with our description of products & equalizers in Set. ∎

In fact, a category has limits for all finite diagrams iff it has:

products

equalizers

terminal object 1



Prop If this is a pullback:

$$\begin{tikzcd} A \times_C B \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

and g is mono, then p is a mono.

Pf:

Assume g is a mono. Show p is a mono:

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & A \times_C B \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

Need: $p \circ h = p \circ k \Rightarrow h = k$.

$$\begin{aligned} p \circ h = p \circ k &\Rightarrow f \circ p \circ h = f \circ p \circ k \\ &\Rightarrow g \circ q \circ h = g \circ q \circ k \\ &\rightarrow q \circ h = q \circ k \quad \text{since } g \text{ is mono} \end{aligned}$$

(continued)

Pf: (continued)

Note X is a competitor to the pullback

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists ! \psi"] \arrow[drr, bend left, "q \circ h = q \circ k"] \arrow[ddr, bend right, "p \circ h = p \circ k"'] & & \\
& A \times_C B \arrow[r, "q"] \arrow[d, "p"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"] & C
\end{tikzcd}$$

$f \circ p \circ h = g \circ q \circ h$
$= g \circ q \circ k$

So $\exists ! \psi : X \longrightarrow A \times_C B$ making this commute.
Both h & k do make it commute.
So $h = k$ $\blacksquare$

$\boxed{\text{Prop}}$ Given

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathcal{A}"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathcal{B}"] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

1) If $\mathcal{A}$ & $\mathcal{B}$ are pullbacks, so is the combined square $\mathcal{AB}$.
2) If $\mathcal{B}$ & $\mathcal{AB}$ are pullbacks, so is $\mathcal{A}$.

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Mathematics Between Categories

Recall that given categories C & D, a <u>functor</u> $F: C \to D$ is a map sending objects $c \in C$ to objects $F(c) \in D$, morphisms $f: c \to c'$ in C to morphisms $F(f): F(c) \to F(c')$ in D, preserving composition $F(f' \circ f) = F(f') \circ F(f)$ & identities $F(1_c) = 1_{F(c)}$.

$$\begin{tikzcd}
\mathrm{Ring} \arrow[dr, "U_3"'] & & \mathrm{Vect}_k \arrow[dl, "U_4"] & & & \\
& \mathrm{AbGrp} \arrow[d, "U_2"'] & & & & \\
& \mathrm{Grp} \arrow[dr, "U_1"'] & & \mathrm{Top} \arrow[dl, "U_5"] & & \mathrm{Set}^2 \arrow[dlll, "U_6"] \\
& & \mathrm{Set} & & &
\end{tikzcd}$$

($U :=$ underlying)

There are many "forgetful functors" going from categories of "fancy" mathematical gadgets to categories of less fancy ones, forgetting some extra properties, structure, or stuff.

$\boxed{\text{Ex}}$ $U_1: \mathrm{Grp} \to \mathrm{Set}$ sends any group G to its <u>underlying</u> set, and any homomorphism $f: G \to G'$ to its <u>underlying</u> function.

$\boxed{\text{Ex}}$ Given categories C & D, there's a category $C \times D$, where objects are ordered pairs (c,d) with $c \in C$, $d \in D$, and morphisms are ordered pairs (f,g) with f a morphism in C, g a morphism in D: given $f: c \to c'$ in C and $g: d \to d'$ in D then $(f,g): (c,d) \to (c',d')$. We define $(f',g') \circ (f,g) = (f' \circ f, g' \circ g)$ and $1_{(c,d)} = (1_c, 1_d)$.

In fact $C \times D$ is the product of the objects $C, D \in \mathrm{Cat}$, which is the category with
- (small) categories as objects
- functors as morphisms

Among other things this means we have projections

$$\begin{tikzcd} & C \times D \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C & & D \end{tikzcd}$$

Set is a large category, but we can still define $Set^2 = Set \times Set$, with pairs of sets as objects.

In the chart, let $U_6 : Set^2 \longrightarrow Set$ be the projection onto the first component.
$$(S,T) \longmapsto S$$

Functions can be nice in two ways: one-to-one & onto

Functors can be nice in three ways:

Def A functor $F: C \to D$ is faithful if for any $c, c' \in C$
$F: \hom(c,c') \longrightarrow \hom(F(c), F(c'))$ is one-to-one.

Def A functor $F: C \to D$ is full if for any $c, c' \in C$
$F: \hom(c,c') \longrightarrow \hom(F(c), F(c'))$ is onto.

Def A functor $F: C \to D$ is essentially surjective if for any $d \in D$, there exists $c \in C$ such that $F(c) \cong d$ meaning there exists an isomorphism $g: F(c) \to d$ in D.

Ex Compare $FinVect_{\mathbb{R}}$ to this category C, with
(finite dim'l vector spaces)

- $\{0\}, \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$ as objects
- all linear maps between these as morphisms

$F: C \longrightarrow FinVect$
$\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and similarly for morphisms: $f: \mathbb{R}^n \to \mathbb{R}^m \longmapsto f: \mathbb{R}^n \to \mathbb{R}^m$

This is faithful & full, not surjective on objects, but essentially surjective.

Later we'll define "equivalent" categories & see that if $F: C \to D$ is faithful, full, & essentially surjective then C & D are equivalent.

We say:

$\boxed{\text{Def}}$ • A functor $U: C \to D$ forgets nothing if it's faithful, full, & ess. surj.

• U forgets (at most) properties if it's faithful & full.

• U forgets (at most) structure if it's faithful.

• In general we say U forgets (at most) stuff.

$\boxed{\text{Ex}}$ $U_1: \mathrm{Grp} \to \mathrm{Set}$ forgets (at most) structure.

It's faithful: given $f, f': G \to G'$ in Grp, $U_1(f) = U_1(f') \Rightarrow f = f'$.

It's not full: there are usually functions $f: U_1(G) \to U_1(G')$ that don't come from group homomorphism, e.g.: $f(gh) \neq f(g)f(h)$ or $f(1) \neq 1$.

$\boxed{\text{Ex}}$ $U_2: \mathrm{AbGrp} \to \mathrm{Grp}$ forgets (at most) properties: the comm. law is forgotten.

This is faithful and also full: if you have any group homomorphism $f: U_2(A) \to U_2(A')$ then $U(f')$ for some homomorphism of abelian groups $f': A \to A'$

But it's not ess. surjective: if G is nonabelian, $G \not\cong U_2(A)$ for some $A \in \mathrm{AbGrp}$.

$\boxed{\text{Ex}}$ $U_6: \mathrm{Set}^2 \to \mathrm{Set}$

forgets stuff: $U_6(S, S') = S$ It forgets the second set in the pair.

Technically, its not faithful:

we can have 2 different morphisms $(f,g), (f,g'): (S,S') \to (T,T')$

with $U_6(f,g) = f = U_6(f,g')$

In our chart, every forgetful functor $U: C \to D$ has a "left adjoint" $F: D \to C$ which "freely creates" the stuff, structure or properties that U forgets.

$\boxed{\text{Ex}}$ $F_1: \mathrm{Set} \to \mathrm{Grp}$ takes a set S and forms the free group on S, $F_1(S)$.

$F_2: \mathrm{Grp} \to \mathrm{AbGrp}$ abelianizes any group G, forming

$$F_2(G) = \frac{G}{\langle xyx^{-1}y^{-1} \rangle}$$

normal subgp gen. by these elements

$\boxed{\text{Ex}}$ $L_6: \mathrm{Set} \to \mathrm{Set}^2$

$S \mapsto (S, \emptyset)$

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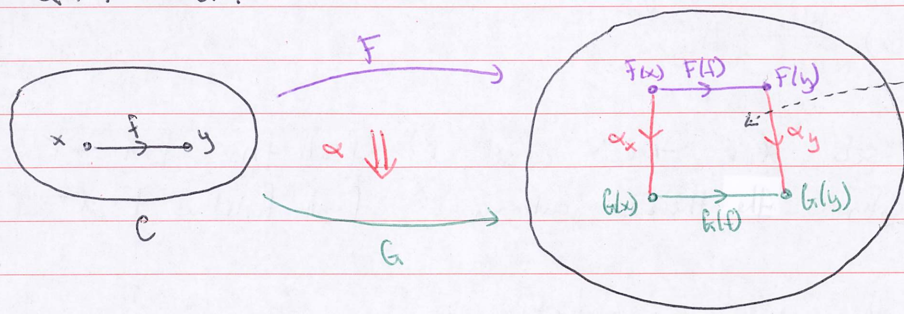
To define adjoint functors (and many other things) we need...
Natural Transformations

Given 2 functors $F, G: C \to D$ we can define a natural transformation $\alpha: F \Rightarrow G$!



this square commutes: this is a natural transformation

$\boxed{\text{Def}}$ Given functors $F, G: C \to D$ a transformation $\alpha: F \Rightarrow G$ is a function sending each object $x \in C$ to a morphism $\alpha_x: F(x) \to G(x)$.
We say α is a natural transformation if for each morphism $f: x \to y$ in C this square commutes:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd}$$

$\boxed{\text{Prop}}$ Given categories C & D there's a category, the functor category D^C, with:

- objects being functors $F: C \to D$
- morphisms being natural transformations $\alpha: F \Rightarrow G$

In D^C we compose $\alpha: F \Rightarrow G$, $\beta: G \Rightarrow H$ to get $\beta\circ\alpha: F \Rightarrow H$ as follows:
$(\beta\circ\alpha)_x: F(x) \to H(x)$ $\forall x \in C$ is given by $\beta_x \circ \alpha_x$.
In D^C the identity $1_F: F \Rightarrow F$, $(1_F)_x: F(x) \to F(x)$ $x \in C$ is given by $1_{F(x)}$.

Pf:

Check that the composite $\beta\circ\alpha$ is natural:
given $f: x \to y$ in C, want this to commute:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "(\beta\circ\alpha)_x"'] & F(y) \arrow[d, "(\beta\circ\alpha)_y"] \\ H(x) \arrow[r, "H(f)"'] & H(y) \end{tikzcd}$$

(continued)

Pf: (continued)

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"] \arrow[dd, bend right=60, "(\beta\circ\alpha)_x"'] & F(y) \arrow[d, "\alpha_y"'] \arrow[dd, bend left=60, "(\beta\circ\alpha)_y"] \\ G(x) \arrow[r, "G(f)"] \arrow[d, "\beta_x"] & G(y) \arrow[d, "\beta_y"'] \\ H(x) \arrow[r, "H(f)"] & H(y) \end{tikzcd}$$

α natural $\Rightarrow$ this commutes

β natural $\Rightarrow$ this commutes

∎

So just as given 2 sets X, Y there's a set Y^X of all fns $f: X \to Y$
2 categories X, Y there's a category Y^X of all functors $F: X \to Y$

Given 2 sets X & Y, they have a product:

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

Notice $X \times Y \neq Y \times X$

but $X \times Y \cong Y \times X$

and there's a specific "good" isomorphism

$$\alpha_{X,Y} : X \times Y \longrightarrow Y \times X$$
$$(x,y) \longmapsto (y,x)$$

It's "good" because it's natural, in the sense we just defined.

There are 2 functors from Set^2 to Set,

$F: (X,Y) \longmapsto X \times Y$

$G: (X,Y) \longmapsto Y \times X$

and α is a natural transformation from F to G.

In fact it's a "natural isomorphism":

$\boxed{\text{Def}}$ If $F, G: C \to D$ are functors and $\alpha: F \Rightarrow G$ is a nat. tran., we say α is a natural isomorphism if $\alpha_x: F(x) \to G(x)$ is an isomorphism $\forall x \in C$.

$\boxed{\text{Prop}}$ $\alpha: F \Rightarrow G$ is a natural isomorphism iff it has an inverse $\alpha^{-1}: G \Rightarrow F$ in D^C.

Pf:

Key idea: $(\alpha^{-1})_x = (\alpha_x)^{-1}$ ∎

Prop Suppose C is a category <u>with binary products</u>: any pair of objects $x, y \in C$ has a product. Then we can choose, for any pair $x, y \in C$, a specific product:

$$\begin{tikzcd} & x \times y \arrow[dl, "p_{x,y}"'] \arrow[dr, "q_{x,y}"] & \\ x & & y \end{tikzcd}$$

and then there is a functor $\times : C^2 \longrightarrow C$

$$(x,y) \longmapsto x \times y$$

In fact there are 2 functors:

$F : C^2 \longrightarrow C$ (this is the functor $\times$)

$(x,y) \longmapsto x \times y$

$G : C^2 \longrightarrow C$

$(x,y) \longmapsto y \times x$

and these are naturally isomorphic.

We say "products are commutative up to natural isomorphism."

Also, products are associative up to natural isomorphism:

$\alpha_{x,y,z} : (x \times y) \times z \xrightarrow{\sim} x \times (y \times z)$

$$\begin{tikzcd} C^3 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

(Just keep using universal property of product.)

Def A <u>cartesian</u> category is a category with binary products and a terminal object. (I.e. it's a category where any finite set of objects has a product — a <u>finite products</u> category).

One can show that in a cartesian category we have natural isomorphisms

$l_x : 1 \times x \xrightarrow{\sim} x$

$r_x : x \times 1 \xrightarrow{\sim} x$

All this works similarly in a cat. w/ finite coproducts:

$B_{x,y} : x + y \xrightarrow{\sim} y + x$

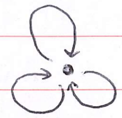
$\alpha_{x,y,z} : (x+y)+z \xrightarrow{\sim} x + (y+z)$

$l_x : 0 + x \xrightarrow{\sim} x$

$r_x : x + 0 \xrightarrow{\sim} x$

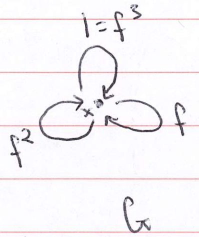
In the case $C = \text{FinSet}$ (finite sets & functions) these give familiar laws of arithmetic: $\mathbb{N}$ is the set of isomorphism classes of objects in FinSet.

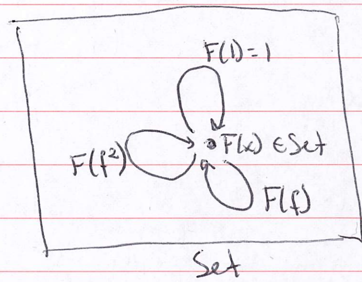
Another example: A group is a category G with one object and with all morphisms invertible:



$\mathbb{Z}/3$

What's a functor $F: G \longrightarrow \text{Set}$?





F picks out a set $X = F(x)$ and for each group element f it picks out a function $F(f): X \longrightarrow X$ s.t. $F(ff') = F(f)F(f')$ & $F(1) = 1_X$.

So: X is a set acted by the group G, or a G-set.

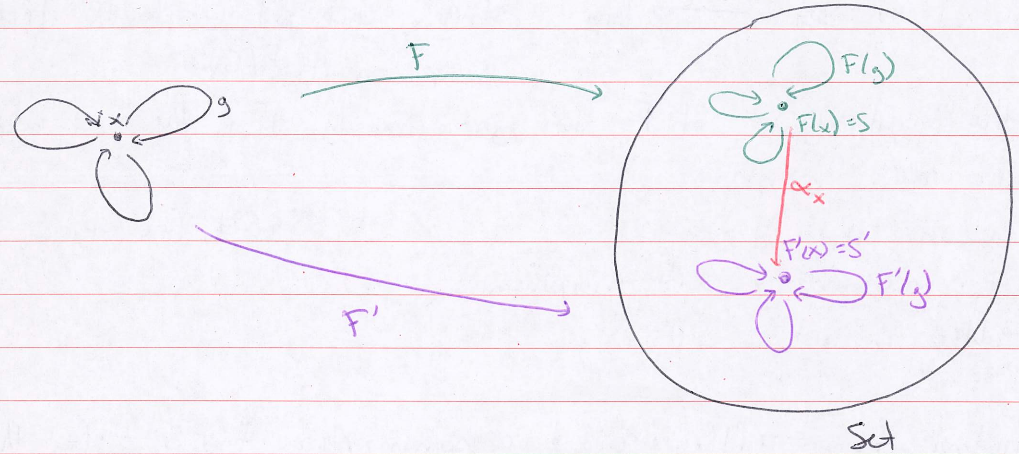
So: a functor $F: G \longrightarrow \text{Set}$ is a G-Set. What's a natural transformation between 2 such functors?

2/2/16

We saw that a 1-object category G with all morphisms invertible is a group.
We saw that a functor $F: G \to \text{Set}$ is a <u>G-set</u>: a set with functions $F(g): S \to S$ for each $g \in G$ s.t.

$$F(gg') = F(g) \circ F(g')$$
$$F(1) = 1_S$$

Given 2 functors $F, F': G \to \text{Set}$, what's a natural transformation $\alpha: F \Rightarrow F'$?
It's called a <u>map of G-sets</u> or <u>G-equivariant</u> map, but let's draw one:



It's a function $\alpha_x: F(x) \to F'(x)$, where $x \in G$ is the one object, $F(x) = S$ is our first G-set, and $F'(x) = S'$ is our second, s.t. all squares like this commute:

$$\begin{tikzcd} F(x) \arrow[r, "F(g)"] \arrow[d, "\alpha_x"'] & F(x) \arrow[d, "\alpha_x"] \\ F'(x) \arrow[r, "F'(g)"] & F'(x) \end{tikzcd}$$

so $F'(g) \circ \alpha_x = \alpha_x \circ F(g)$

Another example of natural transformations

$\boxed{\text{Ex}}$ Two sets are isomorphic if there are functions $F: X \to Y$, $G: Y \to X$ s.t. $G \circ F = 1_X$ & $F \circ G = 1_Y$.
Given F, when can you find such a G? Iff f is 1-1 & onto.
For categories we say:

$\boxed{\text{Def}}$ An <u>equivalence</u> of categories C & D consists of functors $F: C \to D$, $G: D \to C$ and natural isomorphisms $\alpha: G \circ F \overset{\sim}{\Rightarrow} 1_X$, $\beta: F \circ G \overset{\sim}{\Rightarrow} 1_Y$. We say that F & G are <u>weak inverses</u>. We say C & D are equivalent if there exists an equivalence between them.

Thm Given a functor $F: C \to D$, it's part of an equivalence (F, G, α, β) iff F is faithful, full, & essentially surjective. If such a G exists, it may not be unique, but if G' was another one, it's naturally isomorphic to G.

Another example: adjoint functors

Recall an example: $U: \mathrm{Grp} \longrightarrow \mathrm{Set}$ sending each group G to its underlying set $U(G)$

$F: \mathrm{Set} \longrightarrow \mathrm{Grp}$ sending each set S to the free group on it $F(S)$.

We say U is the "right adjoint" of F, or synonymously, F is the "left adjoint" of U. The basic idea: morphisms

$FS \to G$ in Grp $\qquad\qquad S \in \mathrm{Set},\ G \in \mathrm{Grp}$

are in 1-1 correspondence with morphisms

$S \to UG$ in Set

Why?

Given a function $f: S \to UG$ we get a homomorphism $\tilde{f}: FS \to G$, the unique one s.t. $\tilde{f}(s) = f(s)$ for $s \in S \subseteq \underbrace{FS}_{\text{the underlying set}}$

Conversely,

given a homomorphism $h: FS \to G$, we get $\underset{\sim}{h}: S \to UG$ by restricting h to $S \subseteq FS$.

The usual picture:

$$\begin{tikzcd} S \arrow[d, "\text{inclusion function}"'] \arrow[dr, "\text{given a function } f"] & \\ FS \arrow[r, "\exists! \text{ homomorphism } \tilde{f}"'] & G \end{tikzcd}$$

mixes up morphisms in Set & Grp in the same diagram.

We prefer to say there's a bijection

$$\underset{\uparrow \text{in Grp}}{\hom(FS, G)} \cong \underset{\uparrow \text{in Set}}{\hom(S, UG)}$$

Note F is on the left of $\hom(F-, -)$,

U is on the right of $\hom(-, U-)$

To define adjoint functors, we need to say that this kind of bijection is "natural".

What functors give $\hom(FS, G)$ & $\hom(S, UG)$?

They must be 2 functors from Set × Grp to Set:

on objects, these do:

$$(S,G) \longmapsto \hom(FS,G) \in \mathrm{Set}$$
$$(S,G) \longmapsto \hom(S,UG) \in \mathrm{Set}$$

What's the "hom" doing here?

$\boxed{\text{Prop}}$ For any category there's a functor called the <u>hom functor</u>.

$$\hom : C^{op} \times C \longrightarrow \mathrm{Set}$$
$$(c, c') \longmapsto \hom(c, c')$$

Here C^{op} is the <u>opposite</u> of C: the category with one morphism $f^{op} : y \to x$ for each $f : x \to y$ in C, and $f^{op} \circ g^{op} = (g \circ f)^{op}$ with same identity morphisms.

Sketch of Pf:

Need to define $\hom : C^{op} \times C \to \mathrm{Set}$ on morphisms.

Given a morphism in $C^{op} \times C$:

$$\varphi : (x, y) \longrightarrow (x', y')$$

i.e. a pair of morphisms

$f^{op} : x \to x'$ in C^{op} i.e. $f : x' \to x$ in C

$g : y \to y'$ in C

we need to define a morphism

$$\hom(\varphi) : \hom(x,y) \longrightarrow \hom(x', y')$$

in Set, i.e. a function.

Given $h \in \hom(x,y)$, what $\hom(\varphi)h \in \hom(x', y')$?

It's $g \circ h \circ f$

$$\begin{tikzcd} x \arrow[r, "h"] & y \arrow[d, "g"] \\ x' \arrow[u, "f"] \arrow[r, "g \circ h \circ f"'] & y' \end{tikzcd}$$

$g \circ h \circ f = \hom(\varphi) h$

Thus the hom-functor $\hom : C^{op} \times C \to \mathrm{Set}$ will not only describe the hom sets, but also composition in C.

Then: check it's really a functor — e.g. check it preserves composition.

Given functors $F: \mathcal{C} \to \mathcal{D}$, $U: \mathcal{D} \to \mathcal{C}$, how can we say the isomorphism $\hom(Fc, d) \cong \hom(c, Ud)$ is natural?

$$(F(c), d) \in \mathcal{D}^{op} \times \mathcal{D}$$

$$\begin{tikzcd}
 & \mathcal{D}^{op} \times \mathcal{D} \arrow[dr, "\hom"] \arrow[dd, phantom, "\cong \Downarrow \alpha"] & \\
\mathcal{C}^{op} \times \mathcal{D} \arrow[ur, "F^{op} \times 1_{\mathcal{D}}"] \arrow[dr, "1 \times U"'] & & \mathbf{Set} \\
 & \mathcal{C}^{op} \times \mathcal{C} \arrow[ur, "\hom"'] &
\end{tikzcd}$$

$$(c, d) \in \mathcal{C}^{op} \times \mathcal{D} \qquad (c, U(d)) \in \mathcal{C}^{op} \times \mathcal{C}$$

$$\hom(F(c), d) \in \mathbf{Set} \qquad \hom(F(c), d) \xrightarrow[\cong]{\alpha_{c,d}} \hom(c, U(d))$$

Here $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is just F in disguise $F^{op}(f^{op}) = (F(f))^{op}$ and α is a natural isomorphism.

2/4/16

Given functors $\begin{tikzcd} D \arrow[d, shift left, "U"] \\ C \arrow[u, shift left, "F"] \end{tikzcd}$ we say F is a left adjoint of U or U is a right adjoint of F if there's a natural isomorphism

$$\begin{tikzcd} & D^{op}\times D \arrow[dr, "\hom"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\ C^{op}\times D \arrow[ur, "F^{op}\times 1"] \arrow[dr, "1\times U"'] & & Set \\ & C^{op}\times C \arrow[ur, "\hom"'] & \end{tikzcd}$$

So we have bijections

$\alpha_{c,d}: \hom(Fc, d) \xrightarrow{\sim} \hom(c, Ud)$ for all $c \in C$, $d \in D$

which are natural, i.e. making certain squares commute.

At first let's downplay the naturality condition & look at examples, focussing on the bijections.

Ex $\begin{tikzcd} Grp \arrow[d, shift left, "U"] \\ Set \arrow[u, shift left, "F"] \end{tikzcd}$ where UG is the underlying set of $G \in Grp$, FS is the free group on $S \in Set$.

The bijection lets us turn any function $f: S \longrightarrow UG$ into a homomorphism $\tilde{f} = \alpha^{-1}_{S,G}(f): FS \longrightarrow G$

& conversely: any homomorphism $h: FS \longrightarrow G$ comes from a function $\underset{\sim}{h} = \alpha_{S,G}(h): S \longrightarrow UG$

Ex Does the forgetful functor

$\begin{tikzcd} Vect_k \arrow[d, "U"] \\ Set \end{tikzcd}$ have a left adjoint?

Is there some famous functor $F: Set \longrightarrow Vect$?

Yes, for any set S there's a vector space FS whose basis is S:

$FS = \{ \sum_{s_i \in S} c_i s_i \; : c_i \in k, \text{ only finitely many nonzero} \}$

where the sums are formal expressions.

What does $F: Set \longrightarrow Vect_k$ do to a morphism $f: S \to T$ in Set?

It should give a linear map $Ff: FS \longrightarrow FT$. What is it?

$Ff\left(\sum_{s_i \in S} c_i s_i\right) = \sum_{s_i \in S} c_i f(s_i) \in FT$

Check F is a functor:

$F(g\circ f) = F(g)\circ F(f)$

$F(1_S) = 1_{F(S)}$

(continued)

Why is F left adjoint to U? Need bijections:

$$\hom(FS, V) \cong \hom(S, UV) \qquad \forall S \in \mathrm{Set} \quad \forall V \in \mathrm{Vect}_k$$

(and then check they're natural)

Need: given a function $f : S \longrightarrow UV$

we can define a linear map $\tilde{f} : FS \longrightarrow V$

in some "natural" way. Try

$$\tilde{f}\Big(\sum_{s_i \in S} c_i s_i\Big) = \sum_{s_i \in S} c_i f(s_i)$$

Conversely given a linear map $l : FS \to V$

need a function $\underset{\sim}{l} : S \longrightarrow UV$

Try: $\underset{\sim}{l}(s) = l(s)$

Check these maps are inverses: $\underset{\sim}{(\tilde{f})} = f$ and $\widetilde{(\underset{\sim}{l})} = l$.

So, we have a bijection $\hom(FS, V) \cong \hom(S, UV)$

Sometimes a functor has both a left & right adjoint.

$\boxed{\text{Ex}}$

$$\begin{tikzcd} \mathrm{Top} \arrow[d, "U"] \\ \mathrm{Set} \end{tikzcd}$$

To dream up a left adjoint, think of ways to turn a set S into a topological space.

One is the <u>discrete topology</u>: here you give S as many open sets as possible, so every subset is open.

Another is the <u>indiscrete topology</u>: here you give S as few open sets as possible, only $\emptyset$ & S are open.

The left adjoint of $U : \mathrm{Top} \to \mathrm{Set}$, say $L : \mathrm{Set} \longrightarrow \mathrm{Top}$ must have

$$\hom(LS, X) \cong \hom(S, UX) \qquad S \in \mathrm{Set}, \quad X \in \mathrm{Top}$$

i.e. here continuous maps $\tilde{f} : LS \longrightarrow X$ are "the same" as functions $f : S \to UX$

To make this true, LS should have as many open sets as possible, so LS is S with discrete topology.

The right adjoint of U, say $R : \mathrm{Set} \longrightarrow \mathrm{Top}$, has

$$\hom(UX, S) \cong \hom(X, RS)$$

i.e. continuous maps $h : X \to RS$ are "the same" as functions $\underset{\sim}{h} : UX \to S$

To make this true, RS should have as few open sets as possible, i.e. it should be S with indiscrete topology.

Suppose C is any category. There's always a functor $\Delta : C \to C \times C$ called the diagonal with

$$\Delta(c) = (c,c) \qquad c \in C$$

& if $f : c \to c'$

$$\Delta f : \Delta c \to \Delta c' \quad \text{is given by} \quad \Delta f = (f,f) : (c,c) \to (c',c')$$

$\boxed{\text{Prop}}$ If C has binary products then the functor $\times : C \times C \to C$ is the right adjoint of $\Delta : C \to C \times C$.

(In fact, the converse is true: Δ has a right adjoint iff C has binary products, & then it's $\times$.)

Sketch of Pf:

For starter we need bijections

$$\hom(\Delta c, (c', c'')) \cong \hom(c, c' \times c'') \qquad c \in C \qquad (c', c'') \in C \times C$$

The left side:

$$\hom(\Delta c, (c', c'')) = \hom((c,c), (c', c''))$$
$$\cong \hom(c, c') \times \hom(c, c'')$$

since a morphism from (c,c) to (c', c'') is a pair $f : c \to c'$, $g : c \to c''$

So we need:

$$\hom(c, c') \times \hom(c, c'') \cong \hom(c, c' \times c'')$$

Indeed, the universal property of the product says:

$$\begin{tikzcd}
 & c \arrow[d, "\exists! \psi"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
 & c' \times c'' \arrow[dl, "p"'] \arrow[dr, "q"] & \\
c' & & c''
\end{tikzcd}$$

So (f,g) gives ψ & conversely ψ gives $f = p \circ \psi$ & $g = q \circ \psi$, so we have a bijection:

$$\hom(c, c') \times \hom(c, c'') \xrightarrow{\sim} \hom(c, c' \times c'')$$
$$(f,g) \longmapsto \psi$$

∎

$\boxed{\text{Prop}}$ If C has binary coproducts, $\Delta : C \to C \times C$ has a left adjoint, $+ : C \times C \to C$ assigning to each pair (c', c'') their coproduct $c' + c''$.

(Conversely, if Δ has a left adjoint, C has binary coproducts & left adjoint is $+$.)

(Sketch of Pf on next page)

Sketch of Pf:

For starters we need a bijection

$$\hom(c'+c'', c) \cong \hom((c', c''), \Delta c).$$

Note: $\hom((c', c''), \Delta c) \cong \hom((c', c''), (c, c))$

$$\cong \hom(c', c) \times \hom(c'', c)$$

So need:

$$\hom(c'+c'', c) \cong \hom(c', c) \times \hom(c'', c)$$

and indeed the definition of coproducts gives

$$\begin{tikzcd}
c' \arrow[dr, "i"] \arrow[ddr, bend right, "f"'] & & c'' \arrow[dl, "j"'] \arrow[ddl, bend left, "g"] \\
& c'+c'' \arrow[d, "\exists! \psi"'] & \\
& c &
\end{tikzcd}$$

So our bijection sends $\psi \mapsto (f, g)$ $\boxtimes$

A product (an example of a limit) is an example of a right adjoint – it's easy to describe morphisms going into it.

A coproduct (an example of a colimit) is an example of a left adjoint – it's easy to describe morphisms going out of it.

Last time we saw that if C has products, the functor $\times: C^2 \longrightarrow C$, $(c,c') \longmapsto c \times c'$
is a right adjoint to the diagonal functor $\Delta: C \longrightarrow C^2$, $c \longmapsto (c,c)$

& similarly $+: C^2 \longrightarrow C$, if C has coproducts, is a left adjoint to Δ.

(Thus $\oplus: \mathrm{Vect}_k^2 \longrightarrow \mathrm{Vect}_k$ is both left & right adjoint to $\Delta: \mathrm{Vect}_k \to \mathrm{Vect}_k^2$.)

In fact if a category has limits, these limits give a right adjoint to some functor: "limits are right adjoints". Similarly "colimits are left adjoints".

We often think about the limit of a diagram in a category C. What's a "diagram in C", really?

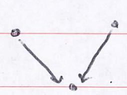
$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ c' & & c'' \arrow[d, "f''"] \\ & & c''' \end{tikzcd}$$

Namely, it's a collection of objects & morphisms between them.
We can make it into a category:

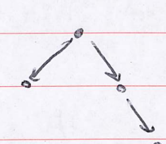
$$\begin{tikzcd} & c \arrow[loop above, "1_c"] \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[ddr, bend right, "f''\circ f'"'] & \\ c' \arrow[loop left, "1_{c'}"] & & c'' \arrow[loop above, "1_{c''}"] \arrow[d, "f''"] \\ & & c''' \arrow[loop right, "1_{c'''}"] \end{tikzcd}$$

Now it's a subcategory of C.

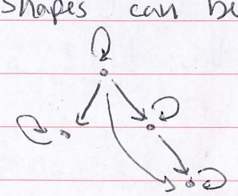
We're often interested in diagram of some shape, like pullbacks



or

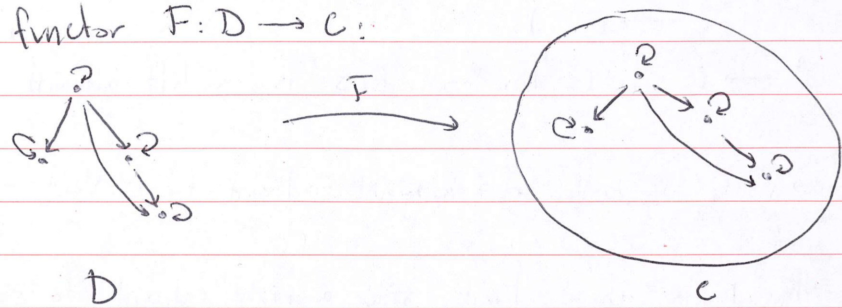


These "shapes" can be interpreted as categories:



Let D be any category: we'll take this as our "diagram shape". What is a D-shaped diagram in some category C?

It's a functor $F: D \to C$:



When we take the limit of this diagram, we get an object (defined up to isomorphism) $\lim F \in C$.

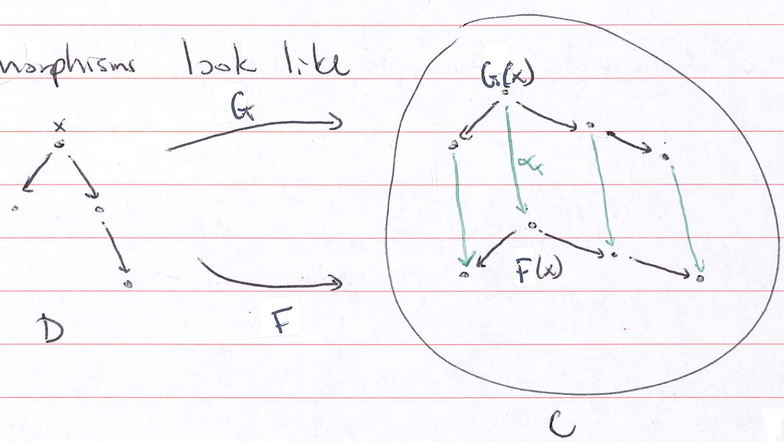
What's the process that takes us from $F: D \to C$ to $\lim F \in C$?

The key: there's a category C^D with:

- objects being functors $F: D \to C$
- morphisms being natural transformations

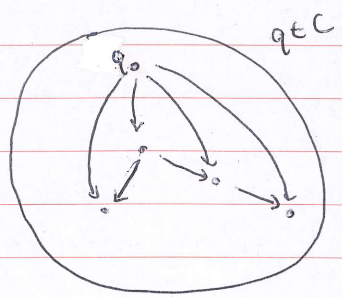
$$\begin{tikzcd} D \arrow[r, bend left, "G"{name=U}] \arrow[r, bend right, "F"'{name=L}] & C \arrow[from=U, to=L, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

These morphisms look like

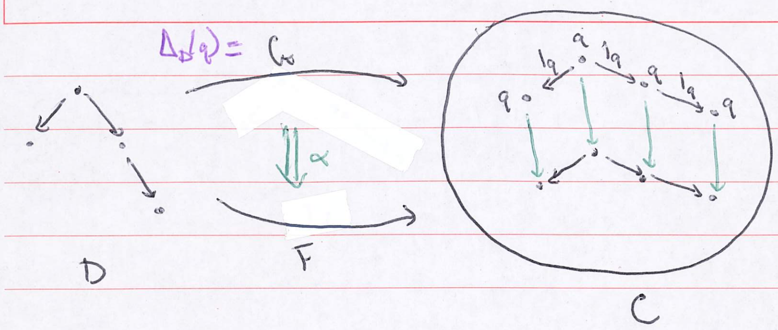


where all the squares commute

When we take a limit of $F: C \to D$, we study cones over F:



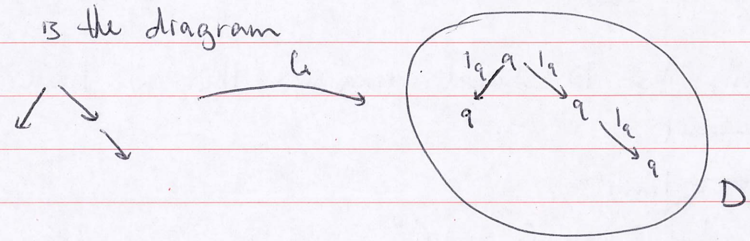
A cone over F is a natural transformation $\alpha: G \Rightarrow F$ where G sends every object of D to some object C & G sends every morphism of D to the identity morphism of that object.



Here $G: D \to C$ was determined by the object $q \in C$, via the above recipe. It turns an object $q \in C$ into an object $G \in C^D$.

So this recipe should be a functor $\Delta_D : C \to C^D$

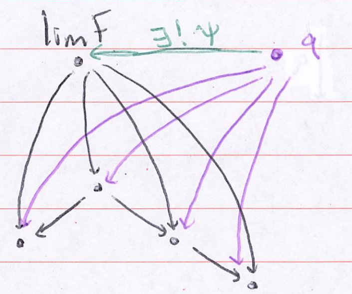
$\Delta_D(q)$ is the diagram



Here $G = \Delta_D(q)$

So: a cone over F with apex $q \in C$ is a natural transformation
$\alpha : \Delta_D(q) \Rightarrow F$

What's the limit of a diagram? If $F \in C^D$



It's a universal cone over that diagram.

Remember U is the right adjoint of F if:

$\hom(Fx, y) \cong \hom(x, Uy)$

So adjoint functors are about converting one kind of morphism into another in a bijective way, & that's what we're doing when we're stating the universal property:

- morphisms $\psi : q \to \lim F$ in C
- cones over F with apex q, i.e. natural transformations $\alpha : \Delta_D(q) \Rightarrow F$ morphisms α from $\Delta_D(q)$ to F in C^D.

So: $\hom(\Delta_D(q), F) \cong \hom(q, \lim F)$

So it looks like we have

$$\lim : C^D \longrightarrow C$$

which is right adjoint to

$$\Delta_D : C \longrightarrow C^D$$

This is true – you need to check that

$$\hom(\Delta_D(q), F) \cong \hom(q, \lim F)$$

is a <u>natural</u> bijection to finish the proof of:

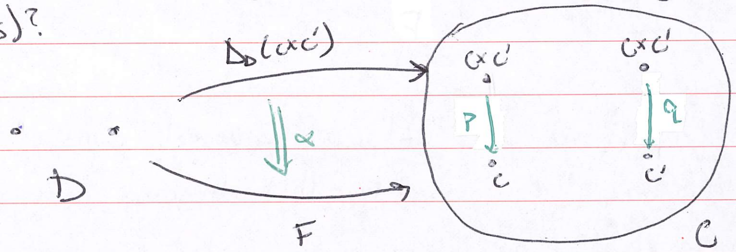
$\boxed{\text{Thm}}$ If C has all limits for D-shaped diagrams, then we have a functor

$$\lim : C^D \longrightarrow C$$
$$F \longmapsto \lim F$$

which is right adjoint to $\Delta_D : C \longrightarrow C^D$.

The converse is true to: if $\Delta_D : C \longrightarrow C^D$ has a right adjoint, then this gives limits of D-shaped diagrams in C.

What choice of D gives the case of binary products (a special case of limits)?



Here D has 2 objects & only identity morphisms, so we could call it 2, so $C^D = C^2$ & $\times : C^2 \longrightarrow C$ is right adjoint to $\Delta_2 = \Delta : C \longrightarrow C^2$.

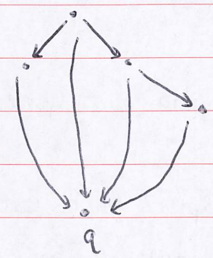
Similarly,

$\boxed{\text{Thm}}$ If a category C has colimits of all D-shaped diagrams, there's a functor $\operatorname{colim} : C^D \to C$ left adjoint to $\Delta_D : C \longrightarrow C^D$ & conversely.

So $\hom(\operatorname{colim} F, q) \cong \hom(F, \Delta_D q)$.

Note

$\alpha \in \hom(F, Dq)$ is a cocone:



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$\boxed{\text{Thm}}$ Left adjoints preserve colimits; right adjoints preserve limits.

Pf: (sketch)

Let's show that if $F: C \to D$ is a left adjoint to $U: D \to C$, then F preserves colimits.

For concreteness, let's show F preserves pushouts – general case is analogous.

So suppose we have a pushout in C:

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] \arrow[dd] & \\ b \arrow[dr] & & c \arrow[dl] \\ & x & \end{tikzcd}$$

Here x is the apex of a cocone on the diagram we're taking a colimit of, & the universal property holds.

The claim is that applying F to this universal cocone gives a universal cocone in D:

$$\begin{tikzcd} & F(a) \arrow[dl] \arrow[dr] \arrow[dd] \arrow[ddrr] & & \\ F(b) \arrow[dr] \arrow[drrr] & & F(c) \arrow[dl] \arrow[dr] & \\ & F(x) \arrow[rr, dashed, "\substack{\text{want} \\ \exists ! \psi}"'] & & Q \end{tikzcd}$$

Choose a competitor cocone with apex Q. Need to show $\exists ! \ \psi: F(x) \to Q$ making the newly formed triangle commute.

We can look at $U(Q) \in C$

Note $\hom(F(x), Q) \cong \hom(x, U(Q))$

So to get $\psi: F(x) \to Q$, let's find $\varphi: x \to U(Q)$.

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] \arrow[dd] \arrow[ddrr] & & \\ b \arrow[dr] \arrow[drrr] & & c \arrow[dl] \arrow[dr] & \\ & x & & U(Q) \end{tikzcd}$$

$U(Q)$ becomes a competitor due to the adjointness of F & U, e.g. $\hom(F(a), Q) \cong \hom(a, U(Q))$

For same reason, the triangles involving $U(Q)$ commute since those involving Q commute.

So $U(Q)$ is a competitor.

Thus $\exists ! \ \varphi: x \to U(Q)$ making the newly formed triangles commute.

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] \arrow[dd] \arrow[ddrr] & & \\ b \arrow[dr] \arrow[drrr] & & c \arrow[dl] \arrow[dr] & \\ & x \arrow[rr, dashed, "\exists ! \varphi"'] & & U(Q) \end{tikzcd}$$

This gives us $\psi: F(x) \to Q$, check it makes its newly formed triangle commute & is unique (since φ is).

Ex $F: \text{Set} \longrightarrow \text{Grp}$ preserves colimits, e.g. coproducts, so
$F(S+T) \cong F(S) + F(T)$

Here $S+T$ is the disjoint S & T, $F(S+T)$ is the free group with elements of $S+T$ as generators, and $F(S)+F(T) = F(S) * F(T)$ is the "free product" of $F(S)$ & $F(T)$.

Ex $U: \text{Grp} \longrightarrow \text{Set}$ preserves limits, e.g. products:
$U(G \times H) \cong U(G) \times U(H)$
where $G \times H$ is the usual product of groups G & H.

$\boxed{\text{Thm}}$ The composite of left adjoints is a left adjoint. The composite of right adjoints is a right adjoint.

Pf:

Suppose we have functors $C \xrightarrow{F} D \xrightarrow{F'} E$
and F & F' are left adjoints of functors U, U'. $C \xleftarrow{U} D \xleftarrow{U'} E$

We'll show that $F' \circ F : C \longrightarrow E$ is the left adjoint of $U \circ U' : E \longrightarrow C$.

Want a natural isomorphism $\hom(F' \circ F(c), e) \cong \hom(c, U \circ U'(e))$

Here's how we get it:

$$\begin{aligned} \hom(F' \circ F(c), e) &\cong \hom(F(c), U'(e)) &&\text{, since } F' \text{ is left adjoint to } U' \\ &\cong \hom(c, U \circ U'(e)) &&\text{, since } F \text{ is left adjoint to } U \quad \blacksquare \end{aligned}$$

Ex

$$\begin{tikzcd} \text{Ring} \arrow[d, shift left, "U'"] \arrow[dd, bend left=60, "U \circ U'"] \\ \text{Ab Grp} \arrow[u, shift left, "F'"] \arrow[d, shift left, "U"] \\ \text{Set} \arrow[u, shift left, "F"] \arrow[uu, bend left=60, "F' \circ F"] \end{tikzcd}$$

$F' \circ F$ is left adjoint to the forgetful functor $U \circ U'$ from Ring to Set.

Starting from $\emptyset$ (the initial set) we get $F(\emptyset) = \{0\}$ (the trivial abelian group, which is the initial abelian group) & then $F'(F(\emptyset)) = \mathbb{Z}$ (the ring of integers, which is the initial ring).

Starting from a one-element set $\{x\}$, we get
$F(\{x\}) = \{\ldots, -x, 0, x, x+x, \ldots\} \cong \mathbb{Z}$
& then $F'(F(x)) = \mathbb{Z}[x]$ the ring of polynomials in x with integer coefficients.

Units and counits of adjunctions (= pair of adjoint functors)

Suppose we have
$$\begin{tikzcd} D \arrow[d, bend left, "U"] \\ C \arrow[u, bend left, "F"] \end{tikzcd}$$
with F left adjoint to U.

$$\hom(Fc, d) \cong \hom(c, Ud) \qquad \forall\ c \in C,\ \forall d \in D$$

We can apply this bijection to an identity morphism & get something interesting. We can do this if $d = Fc$.

$$\begin{array}{ccc} \hom(Fc, Fc) & \xrightarrow[\sim]{\varphi} & \hom(c, UFc) \\ \Psi & & \Psi \\ 1_{Fc} & & \varphi(1_{Fc}) \end{array}$$

$\varphi(1_{Fc})$ is called the <u>unit</u>, η_c: $\quad \eta_c : c \longrightarrow UFc$

We can also apply φ^{-1} to an identity if $c = Ud$.

$$\begin{array}{ccc} \hom(FUd, d) & \xleftarrow[\sim]{\varphi^{-1}} & \hom(Ud, Ud) \\ \Psi & & \Psi \\ \varphi^{-1}(1_{Ud}) & & 1_{Ud} \end{array}$$

$\varphi^{-1}(1_{Ud})$ is called the <u>counit</u>, ε_d: $\quad \varepsilon_d : FUd \longrightarrow d$

These give various famous morphisms.

$\boxed{\text{Ex}}$ $F: \text{Set} \longrightarrow \text{Grp}$
$U: \text{Grp} \longrightarrow \text{Set}$

Given any set S, we get a unit: $\quad \eta_S : S \longrightarrow UFS$

This is the "inclusion of the generators": elements of S are generators of FS.

Given a group G, get: $\varepsilon_G : FUG \longrightarrow G$

$$g_1^{\pm 1} * g_2^{\pm 1} * \cdots * g_n^{\pm 1} \longmapsto g_1^{\pm 1} \cdots g_n^{\pm 1} \qquad g_i \in G$$

"formal product" in FUG — "actual product" in G.

The counits "convert formal expressions into actual ones."

2/16/16

An adjunction is a pair of categories $\mathcal{C}$ & $\mathcal{D}$, a pair of functors

$$\begin{tikzcd} D \arrow[d, shift left, "U"] \\ C \arrow[u, shift left, "F"] \end{tikzcd}$$

& a natural isomorphism $\alpha_{c,d} : \hom(Fc, d) \xrightarrow{\sim} \hom(c, Ud)$

So F is left adjoint to U & U is right adjoint to F.

What does the naturality of α actually say?

We're taking $(c,d) \in C^{op} \times D$ & applying two functors to it, getting objects $\hom(Fc, d)$ & $\hom(c, Ud) \in Set$.

α is a natural isomorphism between these functors, so we get a commuting square for each morphism $(f, g) : (c,d) \longrightarrow (c', d')$ in $C^{op} \times D$

i.e. $f : c' \longrightarrow c$ in C

$g : d \longrightarrow d'$ in D

The commuting square is:

$$\begin{tikzcd} Fc \arrow[r, "h"] & d \arrow[d, "g"] \\ Fc' \arrow[u, "Ff"] \arrow[r] & d' \end{tikzcd}$$

$$\begin{tikzcd} h \arrow[d, mapsto] & \hom(Fc, d) \arrow[r, "\alpha_{c,d}"] \arrow[d, "\hom(Ff, g)"'] & \hom(c, Ud) \arrow[d, "\hom(f, Ug)"] & k \arrow[d, mapsto] \\ g \circ h \circ Ff & \hom(Fc', d') \arrow[r, "\alpha_{c',d'}"] & \hom(c', Ud') & Ug \circ k \circ f \end{tikzcd}$$

Here $\hom(Ff, g)$ is just a name for the map

$$\hom(Fc, d) \longrightarrow \hom(Fc', d')$$
$$h \longmapsto g \circ h \circ Ff$$

& similarly

$$\hom(f, Ug) : \hom(c, Ud) \longrightarrow \hom(c', Ud')$$
$$k \longmapsto Ug \circ k \circ f$$

So saying that α is natural says the square commutes, i.e.

$$\alpha_{c',d'}(g \circ h \circ Ff) = Ug \circ \alpha_{c,d}(h) \circ f$$

Let's use naturality to prove something about the unit & counit of an adjunction.

Recall: $\alpha_{c,Fc} : \hom(Fc, Fc) \xrightarrow{\sim} \hom(c, UFc)$

$$1_{Fc} \longmapsto \eta_c$$

where $\eta_c = \alpha_{c,Fc}(1_{Fc})$ is the unit of the adjunction: $\eta_c : c \longrightarrow UFc$

Also

$$\hom(FUd, d) \xleftarrow[\sim]{\alpha^{-1}_{Ud,d}} \hom(Ud, Ud)$$

$$\varepsilon_d \longleftarrow\!\shortmid \; 1_{Ud}$$

where $\varepsilon_d = \alpha^{-1}_{Ud,d}(1_{Ud})$ is the <u>counit</u>: $\varepsilon_d : FUd \longrightarrow d$.

$\boxed{\text{Ex}}$ if

$$\begin{tikzcd} \mathrm{Vect} \arrow[d, bend left, "U"] \\ \mathrm{Set} \arrow[u, bend left, "F"] \end{tikzcd}$$

is our adjunction

$\eta_S : S \longrightarrow UFS \qquad S \in \mathrm{Set}$

is the "inclusion of the basis elements into the vector space they're a basis of"

$\varepsilon_V : FUV \longrightarrow V \qquad V \in \mathrm{Vect}$

$\underbrace{c_1[v_1] + \cdots + c_n[v_n]}_{\text{formal lin. comb.}} \longmapsto c_1 v_1 + \cdots + c_n v_n.$

is "evaluation of formal linear combinations"

$\boxed{\text{Thm}}$ Given an adjunction, we can recover the natural isomorphism $\alpha_{c,d} : \hom(Fc, d) \longrightarrow \hom(c, Ud)$ and its inverse from the unit & counit.

i.e. there's a formula for α in terms of η & α^{-1} in terms of ε.

Pf:

Use naturality of α:

$$\begin{tikzcd} & 1_{Fc} \arrow[r, mapsto] & \eta_c & \\ h \arrow[d, mapsto] & \hom(Fc, Fc) \arrow[r] \arrow[d, "\hom(F1_c, f)"] & \hom(c, UFc) \arrow[d, "\hom(1_c, Uf)"] & k \arrow[d, mapsto] \\ f \circ h \circ F1_c & \hom(Fc, d) \arrow[r, "\alpha_{c,d}"] & \hom(c, Ud) & Uf \circ k \circ 1_c \\ & f \arrow[r, mapsto] & \alpha_{c,d}(f) & \end{tikzcd}$$

So we get

$$\begin{tikzcd} 1_{Fc} \arrow[r, mapsto] \arrow[d, mapsto] & \eta_c \arrow[d, mapsto] \\ f \arrow[r, mapsto] & \alpha_{c,d}(f) = Uf \circ \eta_c \end{tikzcd}$$

So we can recover α from the unit.

Pf: (continued)

Similarly

$$\begin{array}{ccccccc}
 & & \varepsilon_d & \longmapsfrom & 1_{Ud} & & \\
h & & \hom(FUd,d) & \xleftarrow{\alpha^{-1}_{Ud,d}} & \hom(Ud,Ud) & & k \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
1_d \circ h \circ Fg & & \hom(Fc,d) & \xleftarrow{\alpha^{-1}_{c,d}} & \hom(c,Ud) & & 1_{Ud}\circ k \circ g \\
 & & \alpha^{-1}_{c,d}(g) & \longmapsfrom & g & &
\end{array}$$

So

$$\begin{array}{ccc}
\varepsilon_d & \longmapsfrom & 1_{Ud} \\
\big\downarrow & & \big\downarrow \\
\alpha^{-1}_{c,d}(g) = \varepsilon_d \circ Fg & \longmapsfrom & g
\end{array}$$

So:

$$\alpha^{-1}_{c,d}(g) = \varepsilon_d \circ Fg$$ ▨

This hints that there's another way to think about adjoint functors that says $F: C \to D$ & $U: D \to C$ are adjoint if we have natural transformations

$\eta : 1_C \Rightarrow UF$ (giving $\eta_c : c \to UFc$)

$\varepsilon : FU \Rightarrow 1_D$ (giving $\varepsilon_d : FUd \to d$)

obeying 2 equations.

We can draw η & ε as:

& the equations say:

These are called the zigzag equations

These pictures make sense as part of 2-dim. diagramatic notation for:

- categories
- functors
- natural transformations

} an example of a 2-category

Toward Topos Theory

A topos is a category that's enough like Set that you can do "all mathematics" in it.

In brief: a topos is a cartesian closed category with finite limits & a subobject classifier.

Recall a category is cartesian if it has binary products $\times$ & a terminal object 1, i.e. it has finite products.
A category has finite limits if you can take the limit of any finite-sized diagram.
In fact, a category with finite limits is the same as a cartesian category with equalizers.

Roughly, a category is cartesian closed if it has objects like X^Y (in Set these are sets of functions).

In Set, subsets $S \subseteq T$ are in 1-1 correspondence with characteristic functions $\chi : T \longrightarrow \{0,1\}$

$$\{0,1\} \cong \{F, T\}$$

$\{F, T\}$ is the subobject classifier in Set.
Any topos has its own subobject classifier Ω.

2/18/16

Cartesian Closed Categories or <u>CCC's</u>

We've studied addition:

$X+Y$ - coproduct: left adjoint to Δ

& multiplication

$X \times Y$ - product: right adjoint to Δ

You can show that $+$ & $\times$ are assoc. & comm. up to isomorphism, we saw there's a canonical morphism $X \times Z + Y \times Z \longrightarrow (X+Y) \times Z$ but it's not always an iso.

We say a category with $+, \times$ is <u>distributive</u> if this is an iso.

Now let's do exponentiation!

In Set, $X^Y = \{f : Y \to X\}$ and $|X^Y| = |X|^{|Y|}$

Can we define X^Y in some other categories?

In any category C we have $\hom(Y,X) = \{f : Y \to X\} \in \text{Set}$ & in Set this is the same as X^Y, but we'd like to define $X^Y \in C$ whenever possible: not the <u>set</u> of morphisms from Y to X but the <u>object</u> of morphisms from Y to X, or <u>hom-object</u> (also <u>exponential</u>, or <u>internal hom</u>). One goal is to free category theory from the domination of Set, & work with $X^Y \in C$ as a substitute for $\hom(Y,X) \in \text{Set}$.

$\boxed{\text{Prop}}$ In Set, the functor $- \times Y : \text{Set} \longrightarrow \text{Set}$ has a right adjoint

$$X \longmapsto X \times Y$$

$$-^Y : \text{Set} \longrightarrow \text{Set}$$

$$X \longmapsto X^Y$$

So $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$.

Sketch of Pf:

What's this isomorphism?

$$\hom(X \times Y, Z) \longrightarrow \hom(X, Z^Y)$$

$$f \longmapsto \underline{f}$$

(continued)

Sketch of Pf: (continued)

Given $f: X \times Y \to Z$, we need $\underline{f}: X \to Z^Y$

Computer scientists call $f \mapsto \underline{f}$ "currying", after Haskell Curry.

$\underline{f}(x)(y) = f(x,y)$ $\quad x \in X, \ \underline{f}(x) \in Z^Y, \ \underline{f}(x)(y) \in Z$

Next, need to explain what $- \times Y$ & $-^Y$ do to morphisms; check they're functors, & show that this iso. $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$ is natural. $\blacksquare$

Def A cartesian closed category or CCC is a cartesian category C such that for every $Y \in C$ the functor $- \times Y: C \to C$ has a right adjoint $-^Y: C \to C$. So we have a natural isomorphism: $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$.

Ex

1) Set is CCC

2) FinSet (the category of finite sets & functions between them) is also CCC

3) G-Set (the category of sets with an action of the group G & functions preserving this action: $f(gx) = gf(x)$.)

4) Graphs (the category where an object is a graph meaning a pair of sets E, V and functions $s, t: E \to V$

$$\underset{s(e)}{\bullet} \longrightarrow \underset{t(e)}{\bullet}$$

"source" "target"

A morphism from $s, t: E \to V$ to $s', t': E' \to V'$ consists of functions $\phi_0: V \to V'$ and $\phi_1: E \to E'$ s.t.

$$\begin{array}{ccc} E & \xrightarrow{\phi_1} & E' \\ s \downarrow & & \downarrow s' \\ V & \xrightarrow{\phi_0} & V' \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{\phi_1} & E' \\ t \downarrow & & \downarrow t' \\ V & \xrightarrow{\phi_0} & V' \end{array}$$

$$\begin{array}{l} \bullet \xrightarrow{e} \bullet \ t(e) \\ \quad \downarrow \phi_1 \\ \bullet \xrightarrow[\phi_1(e)]{} \bullet \end{array} \qquad t'(\phi_1(e)) = \phi_0(t(e))$$

Given graphs X & Y there's a graph X^Y whose vertices are maps of graphs $\phi: Y \to X$ but where there are also edges, corresponding to transformations

$$\begin{tikzcd} x \in Y \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\phi'"'] \arrow[r, phantom, "\Downarrow\alpha"] & {} \end{tikzcd} \qquad \begin{array}{c} \phi(x) \\ \downarrow \alpha_x \\ \phi'(x) \end{array}$$

5) Cat — a category is a graph with the ability to compose edges:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

& a functor is a map of graphs that preserve composition.

Cat is also cartesian closed: given $X, Y \in$ Cat we get a category X^Y with:
- functors $F: Y \to X$ as its objects
- natural transformation

$$\begin{tikzcd} Y \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & X \end{tikzcd}$$

as morphisms from F to G.

Puzzle: show $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$

$$f \longmapsto \underline{f}$$

where $\underline{f}(x)(y) = f(x,y)$ $\qquad x \in X,\ y \in Y$

What does $\underline{f}$ do to morphisms?

Show $\cong$ is a natural iso.

b) Grp, Vect$_k$, Ring, ... are all cartesian categories that are not cartesian closed.
If $G, H \in$ Grp, there's no group G^H with one-to-one correspondence between homos $K \times H \to G$ and $K \to G^H$.

$\boxed{\text{Prop}}$ Any cartesian closed category with coproducts is distributive.

Sketch the Proof:

Being distributive means the canonical morphism $X \times Z + Y \times Z \to (X+Y) \times Z$ is an iso.

I'll just show $(X+Y) \times Z \cong X \times Z + Y \times Z$

This would follow if we knew the functor $- \times Z : C \to C$ preserves coproducts

We know left adjoints preserve colimits, e.g. coproducts.

Since C is cartesian closed, $- \times Z$ is the left adjoint of $-^Z$:

$$\hom(X \times Z, Y) \cong \hom(X, Y^Z)$$

$\boxed{\text{Prop}}$

In a cartesian closed category, $(X \times Y)^Z \cong X^Z \times Y^Z$ & $1^Z \cong 1$.

Pf:

$(X \times Y)^Z \cong X^Z \times Y^Z$ says $-^Z : C \to C$ preserves products. We know right adjoints preserve limits, e.g. products, and $-^Z$ is right adjoint to $- \times Z$.

The terminal object 1 is the limit of the empty diagram, so right adjoints also preserve 1, so $1^Z \cong 1$. $\blacksquare$

Puzzle: We know $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$

Is the analogue true when we use "hom-objects" instead of hom-sets?

Is $Z^{X \times Y} \cong (Z^Y)^X$?

Yes this is true in any ccc. But why?

Next time: evaluation morphism $\mathrm{ev}: X^Y \times Y \longrightarrow X$ in Set,

$$(f, y) \longmapsto f(y)$$

can be generalized to any ccc.

So can composition: $\circ: Z^Y \times Y^X \longrightarrow Z^X$

$$(f, g) \longmapsto f \circ g$$

2/23/16

Cartesian Closed Categories

Any category has a <u>set</u> $\hom(X,Y)$ of morphisms from one object X to another object Y, but in a cartesian closed category (or <u>ccc</u>) you also have an object Y^X of morphisms from X to Y.

$\boxed{\text{Ex}}$ if $C = \mathrm{Cat}$, $\hom(X,Y)$ is the set of functors $F: X \to Y$, while Y^X is the category of functors $F: X \to Y$ & natural transformations between them.
In general, you can get $\hom(X,Y)$ from Y^X but not vice versa.
We call $\hom(X,Y)$ the homset or <u>external</u> hom (it lives outside of C, in Set), & Y^X the exponential or <u>internal hom</u> (since it lives inside C).

<u>Internalization</u> is the process of taking math that lives in Set & moving it into some category C.

E.g. in Set you can define a group to be an object $G \in \mathrm{Set}$ with morphisms:

$m: G \times G \longrightarrow G$ multiplication

$inv: G \longrightarrow G$ inverses

$i: 1 \longrightarrow G$ the identity-assigning map
it maps the one element of 1 to the identity element in G

s.t.

$$\begin{tikzcd}
& G \times G \times G \arrow[dl, "m \times 1_G"'] \arrow[dr, "1_G \times m"] & \\
G \times G \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\
& G &
\end{tikzcd}$$

commutes (associative law)

Left and right unit laws:

$$\begin{tikzcd}
1 \times G \arrow[d, "i \times 1_G"'] & G \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "1_G"] & G \times 1 \arrow[d, "1_G \times i"] \\
G \arrow[r, "m"] & G & G \times G \arrow[l, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(*,g) & g \arrow[l, mapsto] \arrow[r, mapsto] \arrow[d] & (g,*) \arrow[d] \\
(1,g) \arrow[r, mapsto] & 1g = g = g1 & (g,1) \arrow[l, mapsto]
\end{tikzcd}$$

inverse laws

$$\begin{tikzcd}
G \times G \arrow[d, "inv \times 1_G"'] & G \arrow[l, "\Delta"'] \arrow[r] \arrow[d, "\exists!"] & G \times G \arrow[d, "1_G \times inv"] \\
& 1 \arrow[d, "i"] & \\
G \times G \arrow[r, "m"] & G & G \times G \arrow[l, "m"']
\end{tikzcd}$$

(continued)

All these diagrams make sense in any cartesian category (= category with finite products = category with binary products & terminal object). So we can define a <u>group internal to C</u> or <u>group object in C</u> or <u>group in C</u> using these axioms whenever C is cartesian.

E.g.:

- if $C = \mathrm{Top}$, a group in C is called a <u>topological group</u>
- if $C = \mathrm{Diff}$, a group in C is called a Lie group
- if C = algebraic varieties, a group in C is called an <u>algebraic group</u>.

Puzzle: if $C = \mathrm{Grp}$, a group in C is a very famous thing. What is it?

Recall a cartesian category C is a <u>ccc</u> if for any $Y \in C$, $- \times C$ has a right adjoint: $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$

Any adjunction
$$\begin{tikzcd} D \arrow[d, bend left, "U"] \\ C \arrow[u, bend left, "F"] \end{tikzcd}$$
has a unit & counit:

$\eta_X : X \longrightarrow UFX \qquad X \in C$

$\varepsilon_Y : FUY \longrightarrow Y \qquad Y \in D$

Now we have an adjunction
$$\begin{tikzcd} C \arrow[d, bend left, "-^Y"] \\ C \arrow[u, bend left, "- \times Y"] \end{tikzcd}$$

$\eta_X : X \longrightarrow (X \times Y)^Y \qquad X \in C$

$\varepsilon_X : X^Y \times Y \longrightarrow X$

The second one is called <u>evaluation</u>: in Set

$$\begin{aligned} \varepsilon_X : X^Y \times Y &\longrightarrow X \\ (f, y) &\longmapsto f(y) \end{aligned}$$

The first one is called <u>coevaluation</u>: in Set

$\eta_X : X \longrightarrow (X \times Y)^Y$ has $\eta_X(x)(y) = (x, y) \qquad x \in X,\ \eta_X(x) : Y \longrightarrow X \times Y$

So we have analogues of these in any ccc.

Next: in any category we have composition:

$$\circ : \hom(Y,Z) \times \hom(X,Y) \longrightarrow \hom(X,Z)$$
$$(f,g) \longmapsto f\circ g$$

In a ccc, can we internalize this & define "internal composition":

$\circ_{X,Y,Z} = \circ : Z^Y \times Y^X \longrightarrow Z^X$?

$$\circ \in \hom(Z^Y \times Y^X, Z^X) \cong \hom(Z^Y, (Z^X)^{(Y^X)})$$
or
$$\cong \hom(Z^Y \times Y^X \times X, Z) \qquad \leftarrow \text{useful!}$$

So we get $\circ$ from a morphism

$$\tilde{\circ} : Z^Y \times Y^X \times X \longrightarrow Z$$

which we indeed have in any ccc

$$Z^Y \times Y^X \times X \xrightarrow{1_{Z^Y} \times \varepsilon} Z^Y \times Y \xrightarrow{\varepsilon} Z \quad \text{where } \varepsilon \text{ is evaluation.}$$

This is just an internalized way of saying the old def. of composition:

$$(f\circ g)(x) = \underbrace{f(g(x))}_{\text{two evaluations}}$$

[Emily Riehl, Categories in Context, Dover Pub.
free on the web]

Elements

Sets have elements, but what about objects in other categories?
Elements of a set X are in 1-1 correspondence with functions $f: 1 \longrightarrow X$
where 1 is a terminal object in Set (1 = a one element set)

So:

$\boxed{\text{Def}}$ If C is a category with a terminal object, an element of an object $X \in C$ to be a morphism $f: 1 \to X$. We define the set $\mathrm{elt}(X)$ to be $\hom(1,X)$.

$\boxed{\text{Ex}}$ If $C = \text{Top}$, $\mathrm{elt}(X) = \{\text{continuous maps } f:\{*\} \to X\}$, where $\{*\}$, the one-point space, is the terminal object of Top. In fact $\mathrm{elt}(X)$ is in 1-1 correspondence with the underlying set of X: given $x \in X$,

$$f: \{*\} \longrightarrow X$$
$$* \longmapsto x$$

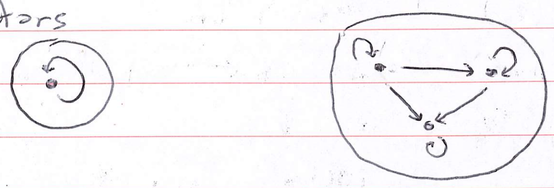
& conversely any such $f(*) \in X$.

Ex If $C = \mathrm{Grp}$, $\mathrm{elt}(G) = \{\text{homomorphisms } f: 1 \to G\}$ where 1 is the trivial group, the terminal object in Grp. So $\mathrm{elt}(G)$ has just one element: there's just one homomorphism $f: 1 \to G$, since 1 is also initial!

Ex If $C = \mathrm{Cat}$

$\mathrm{elt}(D) = \{\text{functors } f: 1 \to D\}$ where 1 is the terminal category: $\bullet\circlearrowright 1$.

functors



$f: 1 \longrightarrow D$

are in one-to-one correspondence with the objects of D.

So $\mathrm{elt}(D) \cong \{\text{objects in } D\}$

Here, as in the previous example, elt forgets a lot of information:

$$\mathrm{elt}(\bullet \longrightarrow \bullet) \cong \mathrm{elt}(\bullet \qquad \bullet)$$

Symmetric monoidal categories

"A category theorist is sort of like a sociologist. He looks at mathematical objects — he doesn't pry it open and see how it works — but sees how it behaves in relation to all other things."

— Chris Heunen

Def A monoid is a nonempty set G together with a binary operation on G which is

(i) associative: $(xy)z = x(yz) \quad \forall\, x,y,z \in G$

(ii) and contains a (two-sided) identity element $e \in G$ such that $xe = ex = x \quad \forall\, x \in G$

[i.e. take the definition of a group and drop the requirement of inverses]

Def A <u>monoidal category</u> is a category $\mathcal{C}$ which is equipped with

1. a tensor product functor $\otimes : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ where the image of a pair of objects (x,y) is denoted by $x\otimes y$
2. a <u>unit object</u> I
3. for every $x,y,z \in ob(\mathcal{C})$, an associativity isomorphism
$$a_{x,y,z} : (x\otimes y)\otimes z \longrightarrow x\otimes(y\otimes z),$$
natural in the objects x, y, and z,
4. for every $x \in ob(\mathcal{C})$, a left unit isomorphism $l_x : I\otimes x \longrightarrow x$ and a right unit isomorphism $r_x : x\otimes I \longrightarrow x$, both natural in x.

We further assume the following diagrams commute for any objects w, x, y, and z:

$$\begin{tikzcd}
 & ((w\otimes x)\otimes y)\otimes z \arrow[dl, "a_{w,x,y}\otimes id_z"'] \arrow[dr, "a_{w\otimes x,y,z}"] & \\
(w\otimes(x\otimes y))\otimes z \arrow[d, "a_{w,x\otimes y,z}"'] & & (w\otimes x)\otimes(y\otimes z) \arrow[d, "a_{w,x,y\otimes z}"] \\
w\otimes((x\otimes y)\otimes z) \arrow[rr, "id_w\otimes a_{x,y,z}"] & & w\otimes(x\otimes(y\otimes z))
\end{tikzcd}$$

$$\begin{tikzcd}
(x\otimes I)\otimes y \arrow[rr, "a_{x,I,y}"] \arrow[dr, "r_x\otimes id_y"'] & & x\otimes(I\otimes y) \arrow[dl, "id_x\otimes l_y"] \\
 & x\otimes y &
\end{tikzcd}$$

When we want to emphasize the tensor product and unit, we denote a monoidal category by $(\mathcal{C},\otimes,I)$.

Ex
- $(\mathrm{Set}, \times, \{\bullet\})$ } it's structure!
- $(\mathrm{Set}, \amalg, \emptyset)$
- $(\mathrm{Grp}, \times, \{e\})$
- $(\mathrm{Hilb}, \otimes, \mathbb{C})$
  - objects: Hilbert spaces
  - morphisms: short linear maps (linear maps of norm at most 1)

Why is

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

an isomorphism, and not an equality?

Let's consider the example $(\mathrm{Set}, \times, \{\circ\})$:

$$(X \times Y) \times Z = \{ (w, z) \mid w \in X \times Y,\ z \in Z \}$$
$$= \{ ((x,y), z) \mid x \in X, y \in Y, z \in Z \}$$

$$X \times (Y \times Z) = \{ (x, w) \mid x \in X,\ w \in Y \times Z \}$$
$$= \{ (x, (y,z)) \mid x \in X,\ y \in Y,\ z \in Z \}$$

These sets are not equal – but we can easily construct an isomorphism.

Ex How can we take a monoid G and construct a monoidal category?

First, we need a category $\mathcal{C}$:

- objects: elements of G
- morphisms: identity morphisms

We get a monoidal category $(\mathcal{C}, \circ, e)$ where $\circ$ is the binary product of G and e is the identity element of G.

Note In general,

- if $\mathcal{C}$ has products, we get a monoidal category $(\mathcal{C}, \times, 1)$
- if $\mathcal{C}$ has coproducts, we get a monoidal category $(\mathcal{C}, +, 0)$

Def A monoidal category $(\mathcal{C}, \otimes; I)$ is <u>symmetric</u> if it additionally is equipped with an isomorphism $s_{x,y} : x \otimes y \longrightarrow y \otimes x$ for any objects x and y of $\mathcal{C}$, natural in x and y, such that the following diagrams commute for all objects x, y, and z:

$$\begin{tikzcd}
 & (x \otimes y) \otimes z \arrow[r, "s_{x,y} \otimes id_z"] \arrow[dl, "a_{x,y,z}"'] & (y \otimes x) \otimes z \arrow[dr, "a_{y,x,z}"] & \\
x \otimes (y \otimes z) \arrow[dr, "s_{x,y\otimes z}"'] & & & y \otimes (x \otimes z) \arrow[dl, "id_y \otimes s_{x,z}"] \\
 & (y \otimes z) \otimes x \arrow[r, "a_{y,z,x}"] & y \otimes (z \otimes x) &
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes I \arrow[rr, "s_{x,I}"] \arrow[dr, "r_x"'] & & I \otimes x \arrow[dl, "l_x"] \\
 & x &
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes y \arrow[r, "s_{x,y}"] \arrow[dr, "id_{x \otimes y}"'] & y \otimes x \arrow[d, "s_{y,x}"] \\
 & x \otimes y
\end{tikzcd}$$

Most of the examples of monoidal categories we have talked about are symmetric.
What's an example of a monoidal category that is not symmetric?

Let R be a non-commutative ring

The category of R-R-bimodules, with $_R\otimes_R$ as the tensor and R as the unit, is such an example.

Note Let $(\mathcal{C}, \circ, e)$ be the monoidal category given by the monoid G. If G is an abelian group, then $(\mathcal{C}, \circ, e)$ is symmetric.

Going back to the definition of a symmetric monoidal category...

Q: Why is the hexagon commuting diagram sufficient?

- There are 6 different ways to order 3 elements
- There are 2 ways of associating 3 elements

$\Rightarrow$ 12 possibilities

(and we would expect all of these to be isomorphic)

A: repeat!

$$\begin{tikzcd}
 & (x\otimes y)\otimes z \arrow[r] \arrow[dl] & (y\otimes x)\otimes z \arrow[dr] & \\
x\otimes(y\otimes z) \arrow[dr] & & & y\otimes(x\otimes z) \arrow[dl] \\
 & (y\otimes z)\otimes x \arrow[r, "a_{y,z,x}"] & y\otimes(z\otimes x) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (y\otimes z)\otimes x \arrow[r] \arrow[dl] & (z\otimes y)\otimes x \arrow[dr] & \\
y\otimes(z\otimes x) \arrow[dr] & & & z\otimes(y\otimes x) \arrow[dl] \\
 & (z\otimes x)\otimes y \arrow[r] & z\otimes(x\otimes y) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (z\otimes x)\otimes y \arrow[r] \arrow[dl] & (x\otimes z)\otimes y \arrow[dr] & \\
z\otimes(x\otimes y) \arrow[dr] & & & x\otimes(z\otimes y) \arrow[dl] \\
 & (x\otimes y)\otimes z \arrow[r] & x\otimes(y\otimes z) &
\end{tikzcd}$$

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Prop Suppose $\mathcal{C}$ is a category with a terminal object $1 \in \mathcal{C}$. Then there's a functor $\mathrm{elt}: \mathcal{C} \longrightarrow \mathrm{Set}$ with

$$\mathrm{elt}(X) = \{f: 1 \longrightarrow X\} \qquad \forall X \in \mathcal{C}$$

and given any morphism $g: X \longrightarrow Y$ in $\mathcal{C}$, $\mathrm{elt}(g): \mathrm{elt}(X) \longrightarrow \mathrm{elt}(Y)$ is defined as follows:

$$\begin{tikzcd} 1 \arrow[r, "f"] \arrow[dr, "\mathrm{elt}(g)f = g \circ f"'] & X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

Pf:

elt preserves composition: given $X \xrightarrow{g} Y \xrightarrow{h} Z$ we need $\mathrm{elt}(h \circ g) = \mathrm{elt}(h) \circ \mathrm{elt}(g)$.

$$\begin{tikzcd} 1 \arrow[r, "f"] & X \arrow[d, "g"] \\ & Y \arrow[d, "h"] \\ & Z \end{tikzcd}$$

Given $f \in \mathrm{elt}(X)$ we have

$$\begin{aligned} \mathrm{elt}(h \circ g)f &= (h \circ g) \circ f \\ &= h \circ (g \circ f) \\ &= h \circ (\mathrm{elt}(g)f) \\ &= \mathrm{elt}(h)(\mathrm{elt}(g)f). \end{aligned}$$

Similarly

$$\begin{aligned} \mathrm{elt}(1_X)f &= 1_X \circ f \qquad \forall f \in \mathrm{elt}(X) \\ &= f \end{aligned}$$

So $\mathrm{elt}(1_X) = 1_{\mathrm{elt}(X)}$ ∎

Ex $\mathrm{elt}: \mathcal{C} \longrightarrow \mathrm{Set}$ may not be faithful, i.e. we can have two different morphisms $g, g': X \longrightarrow Y$ in $\mathcal{C}$ with $\mathrm{elt}(g) = \mathrm{elt}(g')$.

If $\mathcal{C} = \mathrm{Grp}$, we saw $\mathrm{elt}(G) = 1 \in \mathrm{Set}$ for all G, so any homomorphism $h: G \longrightarrow G'$ will get sent to a function $\mathrm{elt}(h): 1 \longrightarrow 1$, but there's only one of these.

$\boxed{\text{Prop}}$ If C is a cartesian category, $\mathrm{elt}: C \longrightarrow \mathrm{Set}$ preserves finite products.

Pf:

It's easy to show elt preserves the terminal object:

if $1 \in C$ then $\mathrm{elt}(1) = \{f : 1 \longrightarrow 1\}$ is a one-element set, so it's terminal in Set.

Why does elt preserve binary products?

Suppose $X, Y \in C$; then their product is a universal cone:

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

To show elt preserves products, we need to show this cone is universal in Set:

$$\begin{tikzcd} & \mathrm{elt}(X \times Y) \arrow[dl, "\mathrm{elt}(p)"'] \arrow[dr, "\mathrm{elt}(q)"] & \\ \mathrm{elt}(X) & & \mathrm{elt}(Y) \end{tikzcd}$$

Choose a competitor:

$$\begin{tikzcd} & Q \arrow[d, dashed, "\Psi"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & \mathrm{elt}(X \times Y) \arrow[dl, "\mathrm{elt}(p)"'] \arrow[dr, "\mathrm{elt}(q)"] & \\ \mathrm{elt}(X) & & \mathrm{elt}(Y) \end{tikzcd}$$

$Q \in \mathrm{Set}$

$(\star)$

Want $\exists!\ \Psi : Q \longrightarrow \mathrm{elt}(X \times Y)$ making the newly formed triangles commute.

$f : Q \longrightarrow \mathrm{elt}(X)$ sends any point $a \in Q$ to a point $f(a) \in \mathrm{elt}(X) = \{h : 1 \longrightarrow X\}$,

so $f(a) : 1 \longrightarrow X$.

Similarly $g(a) : 1 \longrightarrow Y$.

We want to define $\Psi : Q \longrightarrow \mathrm{elt}(X \times Y)$; this will send any $a \in Q$ to $\Psi(a) : 1 \longrightarrow X \times Y$.

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\exists!\,\Psi(a)"] \arrow[ddl, bend right, "f(a)"'] \arrow[ddr, bend left, "g(a)"] & \\ & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

By the universal property of $X \times Y$, for each $a \in Q$ $\exists!\ \Psi(a) : 1 \longrightarrow X \times Y$ s.t. this commutes.

(continued)

Pf: (continued)

Define Ψ this way, check that (*) commutes, and moreover (*) commuting forces us to choose this Ψ, so Ψ is unique. ∎

What if C is a ccc?

Then
$$\begin{aligned} \hom(X,Y) &\cong \hom(1\times X, Y) \\ &\cong \hom(1, Y^X) \\ &= \mathrm{elt}(Y^X) \end{aligned}$$

Since $1\times X \cong X$ so:

$$\begin{tikzcd} 1\times X \arrow[r, "\alpha"] \arrow[dr, "f\circ\alpha"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha^{-1}"] \arrow[dr, "g\circ\alpha^{-1}"'] & 1\times X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

gives us a bijection

$$\begin{aligned} \hom(X,Y) &\cong \hom(1\times X, Y) \\ f &\longmapsto f\circ\alpha \\ g\circ\alpha^{-1} &\longleftarrow\!\shortmid\ g \end{aligned}$$

The moral: we can convert the hom-object $Y^X \in C$ into the hom-set $\hom(X,Y) \in \mathrm{Set}$ by taking elements.

Given $f: X \to Y$ in $\hom(X,Y)$, we can convert it into an element of Y^X, called the <u>name</u> of f: $\ulcorner f \urcorner : 1 \to Y^X$

Conversely, any element of Y^X is the name of a unique morphism $f: X \to Y$.

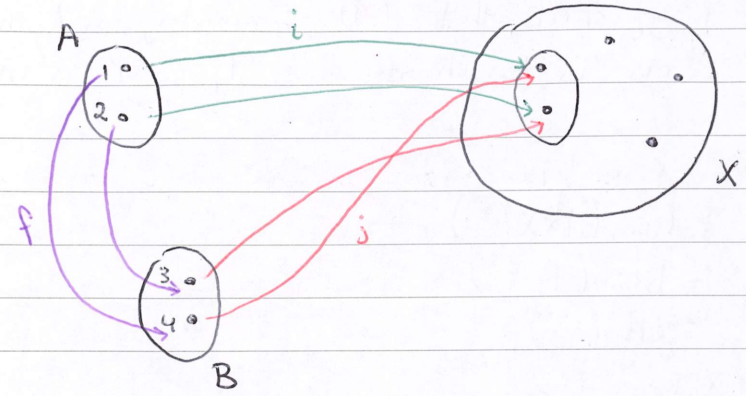
In functional programming, objects are data types, morphisms are programs, & any program $f: X \to Y$ has a "name" $\ulcorner f \urcorner \in \mathrm{elt}(Y^X)$.

<u>Subobjects</u>

$\boxed{\text{Def}}$ In a category C, a <u>subobject</u> of an object $X \in C$ is an equivalence class of monomorphisms $i: A \to X$ where monos $i: A \to X$, $j: B \to X$ are equivalent if there's an isomorphism $f: A \to B$ s.t. this commutes:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

Ex If $C = \mathrm{Set}$, subobjects of $X \in \mathrm{Set}$ correspond to subsets of X.



Given a mono $i : A \to X$ we get a subset $\mathrm{im}(i) \subseteq X$. Any subset $S \subseteq X$ arises in this way via the inclusion:

$$i : S \longrightarrow X$$
$$s \longmapsto s \in X$$

This has $\mathrm{Im}(i) = S$.

Finally, given monos $i : A \to X$ & $j : B \to X$ that define the same subset: $\mathrm{im}(i) = \mathrm{im}(j)$

then there exists a bijection $f : A \to B$ s.t.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

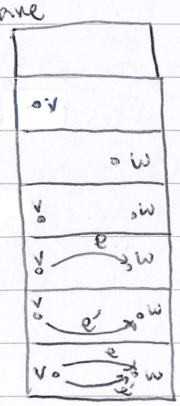
commutes

namely $f = (j|_{\mathrm{im}\, j})^{-1} \circ i$.

Ex In Graph, how many subobjects does this graph:

$$\begin{tikzcd} v \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] & w \end{tikzcd}$$

Here they are



the initial graph

graphs w/o edges

any object gives a subobject of itself.

$1_X : X \to X$ is a mono.

(A graph is a pair of fns $E \underset{t}{\overset{s}{\rightrightarrows}} V$.)

Prop In Set, subobjects of $S \in Set$ are in 1-1 correspondence with functions $\chi : S \longrightarrow 2$ where $2 = \{F, T\}$.

Pf:

Subobjects of S are just subsets $A \subseteq S$.

Any such subset has a <u>characteristic function</u> $\chi : S \longrightarrow 2$ given by

$$\chi(s) = \begin{cases} F & s \notin A \\ T & s \in A \end{cases}$$

Conversely, given $\chi : S \longrightarrow 2$, let

$$\begin{aligned} A &= \chi^{-1}(T) \\ &= \{ s \in S : \chi(s) = T \}. \end{aligned}$$

∎

Roughly, a "subobject classifier" in a category $\mathcal{C}$ is an object $\Omega \in \mathcal{C}$ that plays the role of $2 = \{F, T\}$, in that subobjects of any object $S \in \mathcal{C}$ are going to be in 1-1 correspondence with morphisms $\chi : S \longrightarrow \Omega$.

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Set has the "subobject classifier" $2 = \{F, T\}$.
What does this really mean?
First, there's a function called true:
$$t : 1 \longrightarrow 2$$
from $1 = \{*\}$ to 2 given by $t(*) = T \in 2$

For any set A there's a unique function
$$!_A : A \longrightarrow 1$$
since 1 is terminal.

I claim that for any monomorphism $i : A \to X$ (that is a 1-1 function), there exists a unique function
$$\chi_i : X \longrightarrow 2$$
called the <u>characteristic function</u> of i, such that:

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "i"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi_i"] & 2 \end{tikzcd}$$

is a pullback.
χ_i, in more familiar terms, will be the characteristic function of the subset $\mathrm{im}(i) \subseteq X$, but we call it the characteristic function of the mono, i.
First let's show that this χ_i:
$$\chi_i(x) = \begin{cases} T & x \in \mathrm{im}\, i \\ F & x \notin \mathrm{im}\, i \end{cases}$$
Let Q be a competitor

$$\begin{tikzcd} Q \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "!_Q"] \arrow[ddr, bend right, "f"'] & & \\ & A \arrow[r, "!_A"] \arrow[d, "i"] & 1 \arrow[d, "t"] \\ & X \arrow[r, "\chi_i"] & 2 \end{tikzcd}$$

Then show $\exists!\ \psi : Q \longrightarrow A$ making the newly formed triangles commute.
Since Q is a competitor:
$$\chi_i(f(q)) = t(!_Q(q)) \qquad q \in Q$$
(continued)

$$\chi_i(f(q)) = t(!_Q(q)) \qquad q \in Q$$
$$= t(*)$$
$$= T$$

$\Rightarrow$ (using the def. of χ_i) $f(q) \in \operatorname{im} i$

So since i is 1-1, for each $q \in Q$, $\exists!\ a \in A$ with $f(q) = i(a)$

So define $\Psi : Q \longrightarrow A$ by $\Psi(q) = a$.

This makes $f = i \circ \Psi$ and it's the unique $\Psi : Q \longrightarrow A$ that does so (since i is 1-1).

The other newly formed triangle automatically commutes:

$$\begin{tikzcd} Q \arrow[d, "\Psi"'] \arrow[dr, bend left, "!_Q"] & \\ A \arrow[r, "!_A"'] & 1 \end{tikzcd}$$

You can also check that $\chi_i : X \longrightarrow 2$ is the unique morphism from X to 2 that makes the square a pullback.

So generalizing:

Def Given a category C with a terminal object, a <u>subobject classifier</u> is an object $\Omega \in C$ with a morphism $t : 1 \longrightarrow \Omega$ such that:
for any mono $i : A \longrightarrow X$ there exists a unique $\chi_i : X \longrightarrow \Omega$ such that this square is a pullback:

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "i"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi_i"] & \Omega \end{tikzcd}$$

Def A <u>(elementary)</u> <u>topos</u> is a cartesian closed category with finite limits (limits of finite sized diagrams) and a subobject classifier.

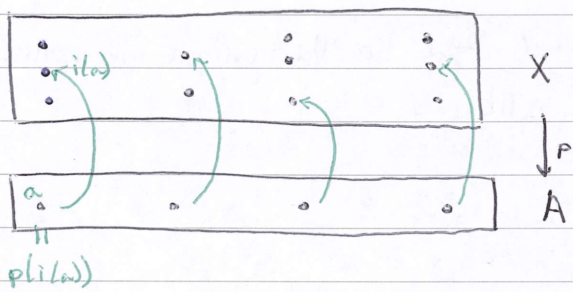
Grothendieck in the 1960's introduced a concept of topos, now <u>Grothendieck topos</u>, which is a special case of an elementary topos, as part of proving the Weil hypotheses in number theory. Later, in the late 60's & early 70's, Lawvere & Tierney simplified & generalized the concept of topos to define an "elementary topos".

Examples of elementary topoi:

1) Set: category of sets & functions
2) FinSet: category of finite sets & functions
   - this doesn't have all limits, only finite limits.
   So topos theory includes finitist mathematics
3) Set′: category of sets & functions as defined using ZF - Zermelo-Fraenkel axioms without axiom of choice.

The axiom of choice is equivalent to:

for every epimorphism $p: X \to A$ there exists a mono $i: A \to X$ s.t. $p \circ i = 1_A$.

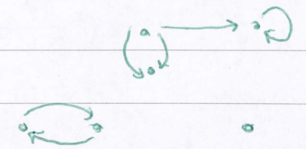


If this is true we say the epimorphism <u>splits</u>.

In a general topos, not every epi splits so the axiom of choice need not hold.

4) Graphs: the category of graphs:

$$E \overset{s}{\underset{t}{\rightrightarrows}} V$$



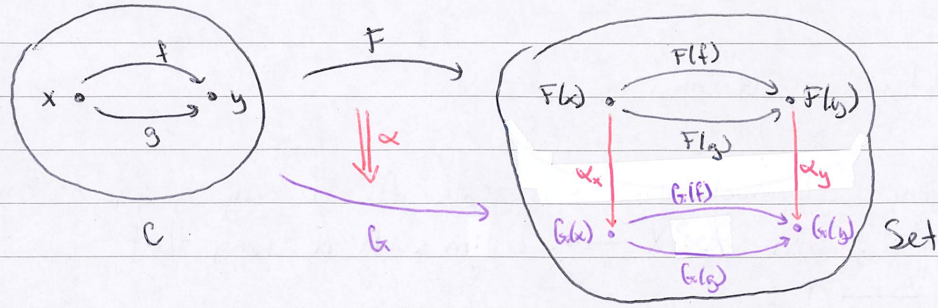
5) Previous example is a special case of a category Set^C, where C is any category. These are called <u>presheaf categories</u> when we write them as $Set^{(D^{op})}$ (e.g. $D = C^{op}$ so $D^{op} = C$)

E.g. if

$$C = \begin{tikzcd} x \arrow[loop left, "1_x"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & y \arrow[loop right, "1_y"] \end{tikzcd}$$

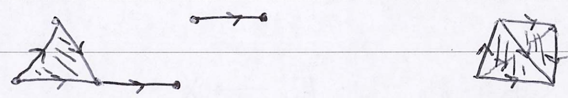
then $Set^C \simeq Graph$

(continued)



A functor $F: C \longrightarrow Set$ is a graph with $E = F(x)$, $V = F(y)$, $s = F(f)$, $t = F(g)$.
So a graph is an object in Set^C.
Similarly, a morphism in Set^C is a morphism between graphs.

6) Another example of a presheaf category is the category of simplicial sets:



These are fundamental to algebraic topology.

7) Presheaf categories are closely connected to categories of sheaves, which are also topoi.
Sheaves are fundamental to algebraic geometry.

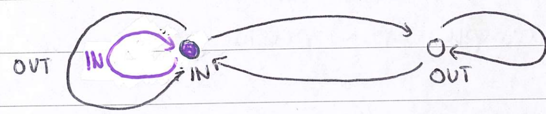
The subobject classifier in Graph

This is some graph Ω such that subgraphs A of any graph X correspond to morphisms of graphs $\chi : X \longrightarrow \Omega$ in such a way that

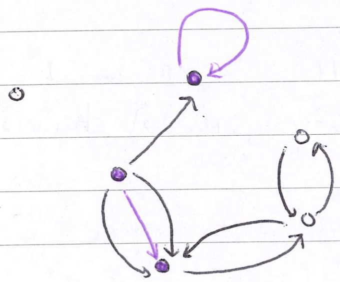
$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "i \text{ mono}"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi"] & \Omega \end{tikzcd}$$

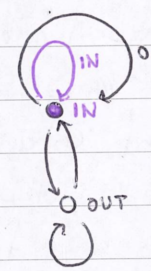
is a pullback.

Ω looks like this:



Here's a graph X with a subgraph $i: A \longrightarrow X$





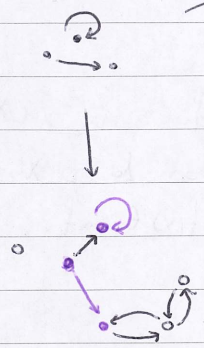
$$X \xrightarrow{\quad \chi \quad} \Omega$$

χ sends purple vertices/edges to purple ones.

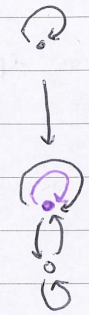
Conversely any morphism of graphs $\chi : X \longrightarrow \Omega$ determines a subgraph of X, consisting of vertices & edges that are purple in Ω.

The terminal graph, 1, looks like this: 

The purple subgraph of Ω is a copy of 1 (it's isomorphic to 1).
We get this from the morphism $t : 1 \longrightarrow \Omega$ which you have in any topos.
A vertex or edge of X will be mapped to this subgraph of Ω iff it's true that the vertex or edge is in A.



$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$



i.e., this diagram commutes. Moreover it's a pullback, allowing you to reconstruct A knowing χ.

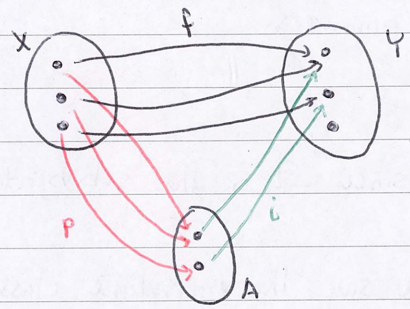
The most important basic properties of topoi:

Prop A topos has finite colimits, meaning it has colimits of finite-sized diagrams.

Prop Any morphism $f: X \to Y$ in a topos has an <u>epi-mono factorization</u>, i.e. there exists an epi $p: X \to A$ & mono $i: A \to Y$ making this triangle commute:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & A \arrow[ur, "i"'] & \end{tikzcd}$$

Ex in Set



every function is a composite $f = i \circ p$ with p onto & i one-to-one.

Prop In a topos, the epi-mono factorization of any morphism $f: X \to Y$ is unique up to a unique isomorphism:
given two epi-mono factorization:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"] \arrow[ddr, bend right, "p'"'] & & Y \\ & A \arrow[ur, "i"'] \arrow[d, dashed] & \\ & A' \arrow[uur, bend right, "i'"'] & \end{tikzcd}$$

there exists a unique isomorphism $g: A \to A'$ making the resulting diagram commute.

Ex In Set, we have an epi-mono factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & \mathrm{im} f \arrow[ur, "i"'] & \end{tikzcd}$$

where $\mathrm{im} f = \{y \in Y : y = f(x) \text{ for some } x \in X\}$
$i : \mathrm{im} f \to Y$ is the inclusion & $p : X \to \mathrm{im} f$
is the obvious function $p(x) = f(x) \in \mathrm{im} f$.

So:

Def Given an epi-mono factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & A \arrow[ur, "i"'] & \end{tikzcd}$$

we call A "the" image of f (it's unique up to isomorphism) & denote it as $\mathrm{im} f$.

Generalize $\subseteq, \cap, \cup,$ to any topos

Henceforth suppose C is a topos.

Def Given $X \in C$, define $\mathrm{Sub}(X)$ to be the set of all subobjects of X: equivalence classes of monos $i : A \to X$, where $i : A \to X$ and $j : B \to X$ are equivalent if there exists an iso $g : A \to B$ s.t.

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "g"', "\wr"] & & X \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

commutes.

Note $\mathrm{Sub}(X) \cong \hom(X, \Omega)$ since Ω is the subobject classifier.

Prop $\mathrm{Sub}(X)$ is a poset where we say the equivalence class of $i : A \to X$ is contained in (or $\subseteq$) the equivalence class of $j : B \to X$ if there exists $f : A \to B$ making this commute:

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "f"'] & & X \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

(Note: f must be a mono, and it's unique.)
Let's say $[i] \subseteq [j]$ in this case.

(Pf on next page)

Pf:

Need to check:

1) $[i] \subseteq [j]$, $[j] \subseteq [k] \Rightarrow [i] \subseteq [k]$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"] \arrow[r, "g"'] & C \arrow[u, "k"'] \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g\circ f"'] & X \\ C \arrow[ur, "k"'] & \end{tikzcd}$$

2) $[i] \subseteq [i]$ – easy

3) $[i] \subseteq [j]$ and $[j] \subseteq [i] \Rightarrow [i] = [j]$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd} \quad \begin{tikzcd} A \arrow[r, "i"] & X \\ B \arrow[u, "g"] \arrow[ur, "j"'] & \end{tikzcd} \quad \text{commute.}$$

To show $[i] = [j]$, it suffices to show:

g is the inverse of f (so f is an isomorphism).

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "i"] & \\ B \arrow[r, "j"] \arrow[d, "g"'] & X \\ A \arrow[ur, "i"'] & \end{tikzcd} \quad \begin{tikzcd} B \arrow[d, "g"'] \arrow[dr, "j"] & \\ A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

commute, so $i \circ g \circ f = i \circ 1_A$ and $j \circ f \circ g = j \circ 1_B$

& since i & j are monic, they're left cancellable:

$g \circ f = 1_A$ & $f \circ g = 1_B$. ∎

Next time we'll define $\cup$ for subobjects, & this makes $\mathrm{Sub}(X)$, which is a poset hence a category, into a category with coproducts: $\cup$ is the coproduct in $\mathrm{Sub}(X)$.

Similarly $\cap$ is the product in the category $\mathrm{Sub}(X)$.

3/10/16

Set Theory, Topos, & Logic

In Set, every subset of $X \in \mathbf{Set}$ corresponds to a predicate on elements of X:

$$\chi : X \longrightarrow \{T, F\}$$

i.e. a characteristic function.

χ determines a subset $A \subseteq X$ via:

$$A = \{x \in X : \chi(x) = T\}$$

& conversely any subset $A \subseteq X$ determines $\chi : X \to \{T, F\}$ via

$$\chi(x) = \begin{cases} T & x \in A \\ F & x \notin A \end{cases}$$

In a topos we get a similar bijection between $\mathrm{Sub}(X)$ & $\hom(X, \Omega)$.

The concepts of $\cup$ & $\cap$ for subsets correspond to the operations of $\vee$ (or) & $\wedge$ (and) on predicates.

$$\{x \in X : \chi(x) = T\} \cup \{x \in X : \varphi(x) = T\} = \{x \in X : (\chi \vee \varphi)(x) = T\}$$

& similarly for $\cap$ & $\wedge$.

$\boxed{\text{Prop}}$ In Set, $\mathrm{Sub}(X)$ for $X \in \mathbf{Set}$ is a poset via $\subseteq$, and thus a category where there exists a unique morphism from A to B iff $A \subseteq B$ $(A, B \subseteq X)$.

In this category $A \cap B$ is the product of A and B, and $A \cup B$ is the coproduct.

Sketch of Pf:

We have

$$\begin{tikzcd} & A \cap B \arrow[dl, "\subseteq"'] \arrow[dr, "\subseteq"] & \\ A & & B \end{tikzcd}$$

& this cone is universal:

$$\begin{tikzcd} & Q \arrow[ddl, bend right] \arrow[ddr, bend left] \arrow[d, dashed, "\exists! \psi"] & \\ & A \cap B \arrow[dl, "\subseteq"'] \arrow[dr, "\subseteq"] & \\ A & & B \end{tikzcd}$$

which is true since $Q \subseteq A$, $Q \subseteq B \Rightarrow Q \subseteq A \cap B$.

The proof for $\cup$ is the same but with all $\subseteq$'s turned around.

In fact, in Set, Sub(X) has all finite limits and all finite colimits!

A category has all finite limits iff it has:
- binary products
- terminal object

  $\Rightarrow$ all finite products exist (Cartesian)
- equalizers

Sub(X) has binary products ($\cap$), terminal object (X, since $A \subseteq X$ for all $A \in \mathrm{Sub}(X)$), and equalizers:

$$B \overset{f}{\underset{g}{\rightrightarrows}} C \quad \text{in any poset is really} \quad B \overset{f}{\underset{g}{\rightrightarrows}} C$$

& the equalizer is:

$$B \xrightarrow{1} B \overset{f}{\underset{f}{\rightrightarrows}} C$$

so equalizers exist in any poset.

Similarly, in Set, Sub(X) has all finite colimits because it has:
- binary coproducts
- initial object
- coequalizers

The binary product of A & B is $A \cup B$, the initial object is $\emptyset$ (since $\emptyset \subseteq A$ for all $A \in \mathrm{Sub}(X)$), and coequalizers (which exist in any poset: just turn arrows around in argument for equalizers).

Def A lattice is a poset with all finite limits & colimits.
(This is equivalent to other more popular definitions, though some evil people don't demand the initial and terminal object.)

In fact we have:

	SET THEORY	LOGIC	CATEGORY THEORY
FINITE LIMITS	$\cap$	$\wedge$	binary product
	X (the whole set)	$\top$	terminal object
FINITE COLIMITS	$\cup$	$\vee$	binary coproduct
	$\emptyset$	F	initial object
	$\subseteq$	$\Rightarrow$	$\longrightarrow$
CARTESIAN CLOSEDNESS	$B \cup A^c$	$Q \vee \neg P$ or "P implies Q"	exponentiation

Note $X = \{x \in X : T = T\}$
$\phi = \{x \in X : F = T\}$

In fact the poset $Sub(X)$ is cartesian closed.
In general this means
$$\hom(B \times C, D) \cong \hom(B, D^C)$$
but for $Sub(X)$, being a poset, these sets either have 0 elements or 1 element. Also, the product is the intersection.
So this says
$$B \cap C \subseteq D \quad \text{iff} \quad B \subseteq D \cup C^c$$
or in terms of logic
$$P \wedge Q \Rightarrow R \quad \text{iff} \quad P \Rightarrow \underbrace{R \vee \neg Q}_{\text{or "Q implies R"}}$$

Thm In any topos, for any object X the poset $Sub(X)$ is a Heyting algebra: it's a poset that has finite limits, finite colimits, & is Cartesian closed. I.e. it's a Cartesian closed lattice.

Sketch of sketch of pf:

Given two subobjects of X, $[i], [j]$, we want to form $[i] \cap [j]$ and $[i] \cup [j]$.

Taking the pullback gives us the intersection
$$\begin{tikzcd} A \cap B \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, "i \circ f = j \circ g"] & B \arrow[d, "j"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$
Since this is a pullback & i, j are monic $\Rightarrow f, g$ are monic.
$\Rightarrow i \circ f = j \circ g$ is also a monic, so we get a new subobject of X, which is $[i] \wedge [j]$.

For unions, we start with the coproduct
$$\begin{tikzcd} A + B \arrow[dr, "\exists ! \psi"] & B \arrow[l, "g"'] \arrow[d, "j"] \\ A \arrow[u, "f"] \arrow[r, "i"] & X \end{tikzcd}$$
where we get ψ from the universal property of the coproduct.
But ψ need not be monic, so do the epi-mono factorization:
(continued)

(continued)

$$\begin{tikzcd} A+B \arrow[rr, "\psi"] \arrow[dr, "p"'] & & X \\ & \operatorname{im}\psi \arrow[ur, "k"'] & \end{tikzcd}$$

where p is epic and k is monic.

k gives a new subobject of X, which is $[i] \cup [j]$.

Where does topos theory go from here?

Many directions ... e.g.:

- Using the "Mitchell-Benabou language", we can reason inside any topos: we can write things like:

  $$\{x \in A \cap B : \forall y \in Y \ \exists z \in Z \ f(x,z) = y\}$$

  & prove things about them, using the logic internal to the topos, & "generalized elements".

- There are also maps between topoi:

  $$C \rightleftarrows D$$

  consisting of certain nice adjunctions.

  These maps are called "geometric morphisms".

  There's a topos called $\mathrm{Th}(\mathrm{Grp})$ — "the theory of a group", and then a geometric morphism from some other topos C to $\mathrm{Th}(\mathrm{Grp})$ is the same as a group object in C.

  This idea works for lots of concepts, not just the concept of a group.