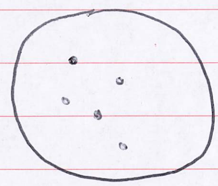
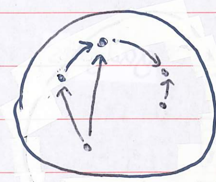
Category Theory Course 1/5/16

Category Theory:
- unifies mathematics
- studies the mathematics of mathematics
- moves toward higher-dimensional algebra ("homotopifying" mathematics)



set theory
0-dimensional



category theory
1-dimensional

Def A category C consists of:
- a class $Ob(C)$ of objects
  (If $x \in Ob(C)$ we write simply $x \in C$)
- Given $x, y \in C$ there's a set $hom(x,y)$, called a homset, whose elements are called morphisms or arrows from x to y. If $f \in hom(x,y)$ we write $f : x \longrightarrow y$.
- Given $f : x \longrightarrow y$ and $g : y \longrightarrow z$ there is a morphism called their composite $g \circ f : x \longrightarrow z$

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & z \end{tikzcd}$$

- Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$ if either side is well-defined.

$$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[dr, "h \circ g"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[u, "f"] \arrow[ur, "g \circ f"] \arrow[r] & \cdot \end{tikzcd}$$

$h \circ (g \circ f) = (h \circ g) \circ f$

- For any $x \in C$, there is an identity morphism $1_x : x \longrightarrow x$

$$\begin{tikzcd} x \arrow[loop above, "1_x"] \end{tikzcd}$$

- We have the left and right unit laws:
  $1_x \circ f = f$ for any $f : x' \longrightarrow x$
  $f \circ 1_x = f$ for any $f : x \longrightarrow x'$

Examples of categories

Categories of mathematical objects
For any kind of mathematical object, there's a category with objects of that kind & morphisms being the structure-preserving maps between objects of that kind.

- **Set** is the category with sets as objects & functions as morphisms.
- **Grp** is the category of groups and homomorphisms
- For k any field, Vect_k of vector spaces over k and linear maps.
- **Ring** is the category of rings & ring homomorphisms

These are categories of "algebraic" objects, namely:

- a set] stuff
- with operations] structure
- obeying equations] properties

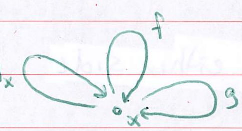
with morphisms being functions that preserve the operations.
All this is formalized in "universal algebra", using "algebraic theories".

There are also categories of non-algebraic gadgets:

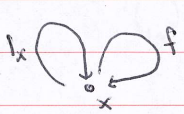
- **Top**, the category of topological spaces & continuous maps.
- **Met**, the category of metric spaces & continuous maps.
- **Meas**, the category of measurable spaces & measurable maps

Categories as mathematical objects
There are lots of small, manageable categories:

Def A <u>monoid</u> is a category with one object. (Then $\hom(x,x)$ for this object x is a set with associative product & unit.)



Ex



with $1_x \circ f = f$
$f \circ 1_x = f$
$f \circ f = 1_x$ is usually called $\mathbb{Z}/2$

Or we could take $f \circ f = f$, and this gives another famous monoid:

1_x = TRUE, f = FALSE, $\circ$ = AND   or 1_x = FALSE, f = TRUE, $\circ$ = OR

Def A morphism $f: x \to y$ is an <u>isomorphism</u> if it has an <u>inverse</u> $g: y \to x$, i.e. a morphism with

$$g \circ f = 1_x$$
$$f \circ g = 1_y$$

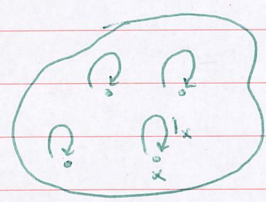
If there exists an isomorphism between 2 objects $x, y \in C$, we say they're <u>isomorphic</u>.

Def A category where all morphisms are isomorphisms is called a <u>groupoid</u>.

Ex "the groupoid of finite sets" is obtained by taking FinSet, with finite sets as objects and functions as morphisms, and then throwing out all morphisms except isomorphisms (i.e. bijections), getting a groupoid.

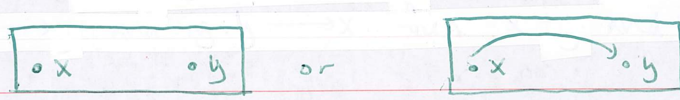
Def A monoid that is a groupoid is called a <u>group</u>.
(The usual "elements" of a group are now the morphisms.)

Def A category with only identity morphisms is a <u>discrete</u> category.



So any set is the set of objects of some discrete category in a unique way.
So a discrete category is "essentially the same" as a set.

Def A <u>preorder</u> is a category with at most one morphism in each hom set.



is hom(x,y)

If there is a morphism $f: x \to y$ in a preorder we say "$x \le y$"; if not we say "$x \not\le y$".
For a preorder, the category axioms just say
- composition: $x \le y$ & $y \le z \Rightarrow x \le z$
- associativity is automatic
- identities: $x \le x$ always
- left & right unit laws are automatic

We're not getting antisymmetry:
$$x \le y \ \& \ y \le x \Rightarrow x = y$$

1/7/16

Categories as Mathematical Object, cont.

$\boxed{\text{Def}}$ A preorder is a category C where for all $x, y \in C$ there is at most one morphism $f: x \to y$.
We write "$x \le y$" iff $\exists$ $f: x \to y$.

We know what C is if we know this relation on objects (if C is a preorder), & then the category axioms simply say:
- there's a class of objects
- there's a relation $\le$ on objects
- $x \le y$ & $y \le z \Rightarrow x \le z$ (composition) $\forall x, y, z \in C$
- $x \le x$ (identities) $\forall x \in C$

$\boxed{\text{Def}}$ An equivalence relation is a preorder that's also a groupoid.

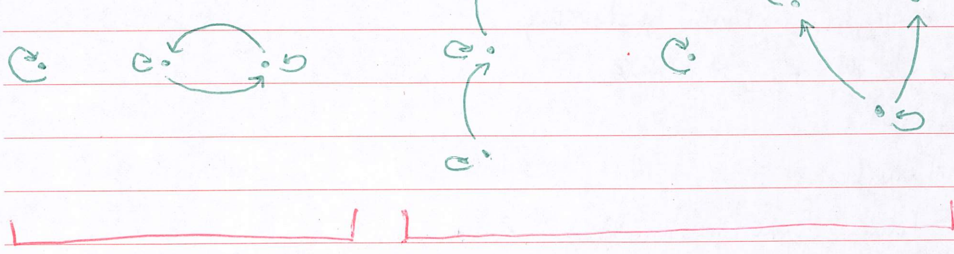
$\boxed{\text{Prop}}$ A preorder C is a groupoid iff this extra law holds:
$x \le y \Rightarrow y \le x$ $\forall x, y \in C$.

Here we have transitivity, reflexivity, & symmetry of "$\le$" so we usually call this relation $\sim$.

$\boxed{\text{Prop}}$ A preorder is skeletal, i.e. isomorphic objects are equal, iff this extra law holds:
$x \le y$ & $y \le x \Rightarrow x = y$ $\forall x, y \in C$

In this case we say C is a poset.

Preorder:



this part is a groupoid but not a poset

this part is a poset but not a groupoid

Since categories can be seen as mathematical objects, we should define maps between them:

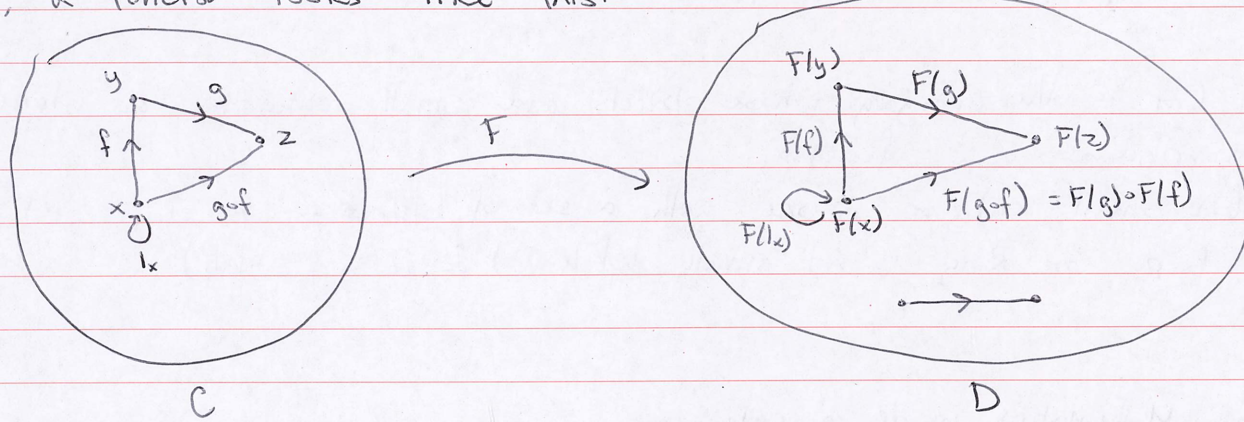
$\boxed{\text{Def}}$ Given categories C & D, a <u>functor</u> $F: C \longrightarrow D$ consists of:

- a function called F from $\mathrm{Ob}(C)$ to $\mathrm{Ob}(D)$: if $x \in C$ then $F(x) \in D$
- functions called F from $\hom(x,y)$ $(\forall x,y \in C)$ to $\hom(F(x),F(y))$: if $f: x \longrightarrow y$ then $F(f): F(x) \longrightarrow F(y)$

such that:

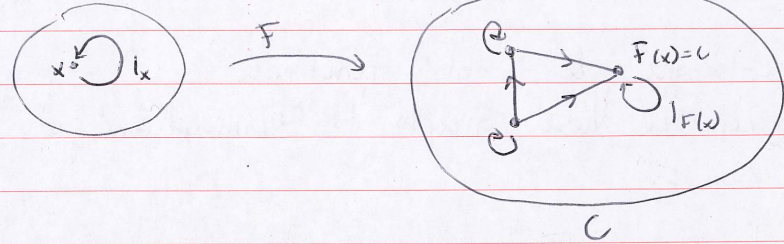
- $F(g \circ f) = F(g) \circ F(f)$ whenever either side is well-defined.
- $F(1_x) = 1_{F(x)}$ $\forall$ $x \in C$

So, a functor looks like this:



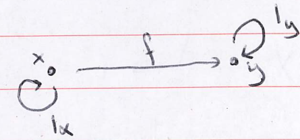
$\boxed{\text{Ex}}$ There's a category called "1". It looks like this: $x \circlearrowleft 1_x$

What is a functor $F: 1 \longrightarrow C$, where C is any category?



The answer is: "an object in C", since for any $c \in C$ $\exists!$ $F: 1 \longrightarrow C$ s.t. $F(x) = c$.

$\boxed{\text{Ex}}$ There's a category called "2". (a poset)



What is a functor $F: 2 \to C$?

It's just a morphism or arrow in C!

For any morphism $g: c \to c'$ in C, $\exists!$ functor $F: 2 \to C$ s.t. $F(f) = g$.

Prop If $F: C \to D$ & $G: D \to E$ are functors then you can define a functor $G \circ F: C \to E$, and $(H \circ G) \circ F = H \circ (G \circ F)$.

Also, for any category C there's an <u>identity functor</u> $1_C: C \to C$ with

$$1_C(x) = x \quad \forall x \in C$$
$$1_C(f) = f \quad \forall f: x \to y \text{ in } C$$

and

$$F \circ 1_C = F \quad \forall F: C \to D$$
$$1_C \circ H = H \quad \forall H: D \to C$$

Def Cat is the category whose objects are "small" categories & whose morphisms are functors.

(A "small" category is one with a set of objects --- so e.g. Set or Grp or Ring is not small, while 1 & 2 are small.)

Doing Mathematics inside a category.

A lot of math is done in Set, the category of sets & functions. Let's try to generalize all that stuff to other categories: replace Set by a general category C.

In Set we have "one-to-one" & "onto" functions.

In a category C we generalize these concepts to "epimorphisms" or "epis" & "monomorphisms" or "monos".

Def A morphism $f: X \to Y$ is a <u>mono</u> if $\forall g, h: Q \to X$ we have

$$f \circ g = f \circ h \Rightarrow g = h$$

$$\begin{tikzcd} Q \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$\boxed{\text{Prop}}$ In Set, a morphism is monic iff it's a one-to-one function.

Turning around the arrows in the definition of mono, we get:

$\boxed{\text{Def}}$ A morphism $f: Y \longrightarrow X$ is an <u>epi</u> if $\forall$ $g, h: X \longrightarrow Q$ we have $g \circ f = h \circ f \Rightarrow g = h$.

$$Y \xrightarrow{f} X \underset{h}{\overset{g}{\rightrightarrows}} Q$$

$\boxed{\text{Prop}}$ In Set, a morphism is epic iff it's an onto function.