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Def A morphism $f: X \to Y$ is an <u>iso</u> if $\exists\ f^{-1}: Y \to X$ that's a <u>left inverse</u> $f^{-1} \circ f = 1_X$
& a <u>right inverse</u> $f \circ f^{-1} = 1_Y$

Prop In Set, $f: X \to Y$ is a mono iff it has a left inverse, and an epi iff it has a right inverse (using the Axiom of Choice).
Thus f is an iso iff it's a mono & epi.

assume we have identity

Prop In Ring (rings & ring homomorphisms) $f: \mathbb{Z} \longrightarrow \mathbb{Q}$, $n \longmapsto n$

is a mono and an epi but not an iso; in fact it has neither left nor right inverse.

Pf:

There's no ring homomorphism $g: \mathbb{Q} \longrightarrow \mathbb{Z}$ since it would send $\frac{1}{2}$ to some multiplicative inverse of 2.

Why is f a mono?

Need:

$f \circ g = f \circ h \Rightarrow g = h$

$$R \overset{g}{\underset{h}{\rightrightarrows}} \mathbb{Z} \xrightarrow{f} \mathbb{Q}$$

If $(f \circ g)(r) = (f \circ h)(r)\ \forall r \in R$,

since f is 1-1 $g(r) = h(r)\ \forall r$ (as a function)

$\Rightarrow g = h$

Why is f epi?

Need:

$g \circ f = h \circ f \Rightarrow g = h$

$$\mathbb{Z} \xrightarrow{f} \mathbb{Q} \overset{g}{\underset{h}{\rightrightarrows}} R$$

We know $g(p) = h(p)$ & $g(q) = h(q)$

So $g(1) = g(\frac{q}{q}) = g(q) \cdot g(\frac{1}{q})$, so we can write $g(\frac{1}{q}) = \frac{1}{g(q)}$.

So $g(\frac{p}{q}) = g(p) \cdot g(\frac{1}{q}) = \frac{g(p)}{g(q)}$

So g (& similarly h) is determined by its values on integers; since they agree on $\mathbb{Z}$ they're equal

Puzzle: In Top, find $f: X \to Y$ that is epic & monic but not an isomorphism.

Limits & Colimits

These are ways of building new objects in a category $\mathcal{C}$ from diagrams in $\mathcal{C}$,

e.g.:
$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$$

An example of a limit is:

$\boxed{\text{Def}}$ Given objects $X, Y \in \mathcal{C}$, a <u>product</u> of them is an object Z equipped with morphisms

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

called projections to X & Y,

s.t. for any candidate Q

$$\begin{tikzcd} & Q \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

there $\exists !\ \Psi : Q \to Z$ s.t.

this diagram commutes

$$\begin{tikzcd} Z \arrow[d, "p"'] \arrow[dr, "q" near start] & Q \arrow[l, bend right, "\Psi"'] \arrow[dl, "f"' near start] \arrow[d, "g"] \\ X & Y \end{tikzcd}$$

i.e.
$$f = p \circ \Psi$$
$$g = q \circ \Psi$$

The definition of <u>coproduct</u> is just the same but with all arrows reversed.

$\boxed{\text{Prop}}$ In Set, we get a product of X & Y by taking
$X \times Y = \{ (x,y) : x \in X,\ y \in Y \}$ with $p(x,y) = x$ and $q(x,y) = y$.

Pf:

Given
$$\begin{tikzcd} & Q \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

let $\Psi : Q \longrightarrow X \times Y$ be $\Psi(q) = (f(q), g(q))$

We indeed get $p \circ \Psi = f$, $q \circ \Psi = g$ & Ψ is the unique map obeying these equation.

But we could also take as our product any set S that's isomorphic to $X \times Y$, via some iso. $\alpha : S \to X \times Y$

$$\begin{tikzcd} X \times Y \arrow[d, "p"'] \arrow[drr, "q" near start] & & S \arrow[ll, "\alpha"', "\sim"] \arrow[d, "q \circ \alpha"] \arrow[dll, "p \circ \alpha" near end] \\ X & & Y \end{tikzcd}$$

Use $p \circ \alpha$ & $q \circ \alpha$ as projections; then you can check

$$\begin{tikzcd} & S \arrow[dl, "p \circ \alpha"'] \arrow[dr, "q \circ \alpha"] & \\ X & & Y \end{tikzcd}$$

is also a product of X & Y.

So "any object isomorphic to a product can also be a product."

$\boxed{\text{Prop}}$ Suppose

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Z \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ X & & Y \end{tikzcd} \quad \text{are both a product of } X \text{ \& } Y.$$

Then W & Z are isomorphic

"Products are unique up to isomorphism."

Pf:

Since W is the product

$$(*) \quad \begin{tikzcd} W \arrow[d, "p"'] \arrow[dr, "q" near start] & Z \arrow[l, "\exists! \psi"'] \arrow[d, "q'"] \arrow[dl, "p'" near start] \\ X & Y \end{tikzcd}$$

$\exists! \ \psi : Z \to W$ making this commute

Since Z is the product

$$(**) \quad \begin{tikzcd} W \arrow[r, "\exists!"] \arrow[d, "p"'] \arrow[dr, "q" near start] & Z \arrow[d, "q'"] \arrow[dl, "p'" near start] \\ X & Y \end{tikzcd}$$

$\exists! \ \varphi : W \to Z$ s.t. the diagram commutes

Suffices to show ψ & φ are inverse. Why is $\psi \circ \varphi : W \to W$ the identity?
If we can show this, the same arguement will show $\varphi \circ \psi = 1_Z$.

$$\begin{tikzcd} W \arrow[d, "p"'] \arrow[dr, "q" near start] & W \arrow[l, bend right, "\exists!"'] \arrow[d, "q"] \arrow[dl, "p" near start] \\ X & Y \end{tikzcd}$$

There is a unique arrow making this commute since W is the product.
$1_W : W \to W$ does the job, but also $\psi \circ \varphi$ does the job.

$$\begin{tikzcd}
W \arrow[r, "\varphi"] \arrow[d, "p"'] \arrow[dr, "q"] & Z \arrow[r, "\psi"] \arrow[dl, "p'"'] \arrow[d, "q'"] & W \arrow[dll, "p"] \arrow[dl, "q"] \\
X & Y
\end{tikzcd}$$

This commutes since * & ** do

so this commutes:

$$\begin{tikzcd}
W \arrow[r, "\psi\circ\varphi"] \arrow[d, "p"'] \arrow[dr, "q"] & W \arrow[dl, "p'"'] \arrow[d, "q'"] \\
X & Y
\end{tikzcd}$$

so $\psi \circ \varphi = 1_W$ by uniqueness.

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Whoops:

$\boxed{\text{Prop}}$ If a morphism is an iso, it's both a mono & an epi.
(We've seen the converse is false)

Pf:

If $f: X \to Y$ has a left inverse f^{-1}, it's a mono:

$$f \circ g = f \circ h \Rightarrow f^{-1} \circ f \circ g = f^{-1} \circ f \circ h \Rightarrow g = h \qquad \forall g, h$$

Similarly, if f has a right inverse f^{-1}, it's an epi

$$g \circ f = h \circ f \Rightarrow g \circ f \circ f^{-1} = h \circ f \circ f^{-1} \Rightarrow g = h \qquad \forall g, h$$

$\boxed{\text{Def}}$ A morphism with a left inverse is called a <u>split monomorphism</u>; a morphism with a right inverse is called a <u>split epimorphism</u>.

In Set every mono (or epi) splits, but we saw not in Ring (or Top).

Coproducts

$\boxed{\text{Def}}$ Given objects X & Y, a <u>coproduct</u> of X & Y is an object Z equipped with morphisms

$$\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & Z & \end{tikzcd}$$

(where i & j are called <u>inclusions</u>),

which is <u>universal</u>: for any diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Q & \end{tikzcd}$$

$\exists !\ \Psi: Z \to Q$ making the following diagram commute:

i.e. $f = \Psi \circ i$

$g = \Psi \circ j$

$$\begin{tikzcd} X \arrow[d, "i"'] \arrow[dr, "f"] & Y \arrow[dl, "j"'] \arrow[d, "g"] \\ Z \arrow[r, "\exists ! \Psi"'] & Q \end{tikzcd}$$

$\boxed{\text{Prop}}$ In Set, a coproduct of X & Y is their <u>disjoint union</u>

$X + Y = X \times \{0\} \cup Y \times \{1\}$ with $i: X \to X+Y$, $x \mapsto (x, 0)$; $j: Y \to X+Y$, $y \mapsto (y, 1)$

	PRODUCTS $\times$	COPRODUCTS $+$
Set	cartesian product $S \times T$	disjoint union $S \sqcup T = S + T$
Top	cartesian product $X \times Y$ w/ product topology	disjoint union $X \sqcup Y = X + Y$
Gp	product of groups $G \times H$	free product $G * H$
AbGp	$A \oplus B = A \times B$ product of abelian groups	$A \oplus B$
Vect_k	$V \oplus W = V \times W$ direct sum of vector spaces	$V \oplus W$

abelian categories

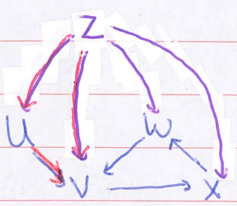
The free product $G * H$ consists of equivalence classes of words
$x_1 x_2 \cdots x_n$ where $x_i \in G \cup H$
where $x_1 \cdots x_{i-1} \cdot 1 \cdot x_{i+1} \cdots x_n \sim x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$ where 1 is the identity in G or H
and $x_1 \cdots x_i x_{i+1} \cdots x_n = x_1 \cdots x_{i-1} y x_{i+2} \cdots x_n$ if $x_i, x_{i+1} \in G$ or $x_i, x_{i+1} \in H$
and $y = x_i x_{i+1}$

General limits & colimits

Given any diagram in a category C:

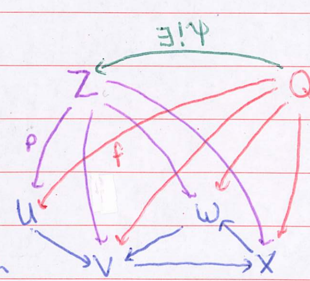
$$\begin{tikzcd} U \arrow[dr, "f"] & & W \arrow[dl, "g"'] & \\ & V \arrow[rr, "i"] & & X \arrow[ul, "h"'] \end{tikzcd}$$

a cone over the diagram is:

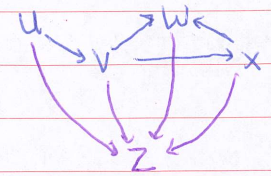


a choice of morphisms from Z to each object in the diagram, such that all newly formed triangles commute.

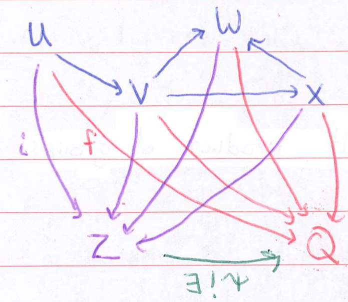
A limit of the diagram is a cone that's universal:
i.e., given any competitor (another candidate), another cone over the same diagram:
$\exists!\, \Psi : Q \to Z$ s.t. all triangles including Ψ commute:
if U is any object in the diagram and $p : Z \to U$ is the morphism in the universal cone, and $f : Q \to U$ is the morphism in the competitor then $f = p \circ \Psi$.



A *cocone* is like a cone, but with morphisms to Z instead of from:



A *colimit* is the 'universal cocone':



$f = \psi \circ i$

Examples of different diagrams

Diagrams	LIMITS	COLIMITS
$\bullet \quad \bullet$	(binary) product	(binary) coproduct
$\bullet \rightrightarrows \bullet$	equalizer	coequalizer
$A \to C \leftarrow B$	pullback	C
$A \leftarrow C \to B$	C	pushout
$\bullet A$	A	A
$A \to B$	A	B
	terminal object 1	initial object 0

What's a limit of the empty diagram:

$$\begin{tikzcd} Z & & Q \arrow[ll, bend right, "\exists! \Psi"'] \end{tikzcd}$$

It's an object Z s.t. for all objects Q, $\exists! \Psi: Q \to Z$.
This is called a terminal object.

In Set, any 1-element set is a terminal object.
In Vect_k, any 0-dim. vector space is a terminal object.

Similarly, an initial object Z is one s.t. for any object Q, $\exists! \Psi: Z \to Q$.

In Set, the empty set is an initial object.
In Vect_k, any 0-dim. vector space is an initial object.

In any abelian category, initial objects are terminal & vice versa.

MIQUELRIUS