1/19/15

Equalizer

$\boxed{\text{Def}}$ A limit of this diagram:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is called an <u>equalizer</u>.

$\boxed{\text{Prop}}$ In Set, the equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[d, bend left, "q"] \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd} \qquad q = f\circ p = g \circ p$$

with

$$Z = \{a \in A : f(a) = g(a)\}$$

where $p : Z \to A$ has $p(a) = a$ for all $a \in Z$. (It's an inclusion)

q is forced to be $f \circ p = g \circ p$.

$\boxed{\text{Note}}$ Since q is determined by p, we usually don't draw it, & write an equalizer like $Z \xrightarrow{p} A \underset{g}{\overset{f}{\rightrightarrows}} B$

Similarly for lots of other limits and colimits.

Pf:

We need to check that this cone is universal, so take a competitor:

$$\begin{tikzcd} Z \arrow[dr, "p"'] & Q \arrow[d, "p'"] \\ & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

and we want to show $\exists !\ \Psi : Q \to Z$ making everything commute: $p \circ \Psi = p'$

$(p \circ \overbrace{\Psi)(q)}^{\in Z} = \Psi(q) \quad \forall q \in Q$ since $p(a) = a \quad \forall a \in Z$.

Thus $\Psi \circ p = p'$ simply says $\Psi(q) = p'(q) \quad \forall q \in Q$

Thus $\exists !\ \Psi$ making everything commute, namely $\Psi = p'$.

$\boxed{\text{Prop}}$ In Grp, AbGrp, or Vect_k, the equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is $\ker(f-g)$.

$\boxed{\text{Note}}$ $\ker(f-g) = \{a \in A : f(a) = g(a)\}$

Pf: the same as before

Prop If $Z \xrightarrow{i} A \overset{f}{\underset{g}{\rightrightarrows}} B$ is an equalizer then i is monic.

Moral monics and limits get along well;
epics and colimits do too.

Pf:

Assume we have an equalizer

To check that i is monic we consider $Y \overset{h}{\underset{k}{\rightrightarrows}} Z \xrightarrow{i} A \overset{f}{\underset{g}{\rightrightarrows}} B$

and show $i \circ h = i \circ k \Rightarrow h = k$

Y is a competitor to Z.

Since Z is universal, $\exists !\, \Psi : Y \to Z$ making everything commute, so

$\Psi = h = k$ ∎

Coequalizers

Def A coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is a universal cocone over this diagram,

i.e.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[d, "i"] \\ & Z \end{tikzcd}$$

commutes s.t. if we have a competitor

$$\begin{tikzcd} & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[dll, "i'"'] \arrow[d, "i"] \\ Q & & Z \end{tikzcd}$$

$\exists !\, \Psi : Z \to Q$ making everything commute.

Prop In Set, the coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{i} Z$

where $Z = B/\sim$ where $\sim$ is the finest equivalence relation s.t. $f(a) \sim g(a)\ \forall a \in A$

and $i : B \to Z$

$b \mapsto [b]$ ← its equivalence class

Pf:

$i \circ f = i \circ g$ with this definition, so this is a cocone. Why is it universal?

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[dl, "i'"'] \arrow[d, "i"] \\ Q & Z \arrow[l, "\Psi"] \end{tikzcd}$$

Why $\exists !\, \Psi : Z \to Q$ making this commute?

To commute, we need

$\Psi \circ i = i'$

$\Psi(i(b)) = i'(b) \quad \forall b \in B$

$\Psi([b]) = i'(b)$

This shows Ψ is unique if it exists; to show it exists need to check it's well-defined:

(continued)

Pf: (continued)

$$[b] = [b'] \overset{\text{need}}{\Longrightarrow} i'(b) = i'(b')$$
$$\Updownarrow$$
$$b \sim b'$$
$$\Updownarrow$$

??? Either $b, b' \in \mathrm{im}(f) \cap \mathrm{im}(g)$ and $b = b'$

or $b, b' \in \mathrm{im}(f) \cap \mathrm{im}(g)$ and ~~~

Find the good proof!

$\boxed{\text{Prop}}$ In AbGp or Vect_k, the coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is $\mathrm{coker}(f-g) = B/\mathrm{im}(f-g)$

$\boxed{\text{Prop}}$ If $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{p} Z$ is a coequalizer, p is epic.

Pf:

Same as proof of the "dual" proposition for equalizers.

Pullbacks

$\boxed{\text{Def}}$ The limit of this diagram
$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$
is called a pullback,

& denoted
$$\begin{tikzcd} A \times_C B \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$
The object here "A times B over C", or the fibred product, and we only need to draw its morphisms to A & B called projections.

We write
$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$
when Z is a pullback.

$\boxed{\text{Prop}}$ In Set, the pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is
$A \times_C B = \{(a,b) \in A \times B : f(a) = g(b)\}$ with
$$p : A \times_C B \longrightarrow A, \quad (a,b) \longmapsto a \qquad q : A \times_C B \longrightarrow B, \quad (a,b) \longmapsto b$$

Pf:

This is clearly a cone; to show it's universal use the next Prop. ∎

$\boxed{\text{Prop}}$ Given

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

if the product $A \times B$ exists and if this equalizer exists:

$$\begin{tikzcd} Z \arrow[dr, "i"] & & \\ & A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

where $i : Z \longrightarrow A \times B$ is the equalizer of $A \times B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} C$

then this is a pullback:

$$\begin{tikzcd} Z \arrow[r, "\pi_2 \circ i"] \arrow[d, "\pi_1 \circ i"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

1/21/16

$\boxed{\text{Prop}}$ In Set, a coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is the set $Z = B/\sim$ where $\sim$ is the finest equiv. relation on B s.t. $f(a) \sim g(a)$ $\forall a \in A$. with map $i: A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{i} Z$ s.t. $i(b) = [b]$. Note: $i \circ f = i \circ g$

Pf:

Consider a competitor

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "i"] \arrow[dr, "i'"'] & Z \\ & & Q \end{tikzcd}$$

with $i' \circ f = i' \circ g$

Want: $\exists!\ \Psi: Z \to Q$

Try $\Psi([b]) = i'(b)$ $\forall b \in B$

If Ψ is well-defined, then the diagram commutes since $\Psi(i(b)) = i'(b)$, and Ψ is unique since this formula specifies it.

Why is Ψ well-defined?

Assume $b \sim b'$, need to show $i'(b) = i'(b')$.

If $b \sim b'$, then $b = b_1 \sim b_2 \sim b_3 \sim \cdots \sim b_n = b'$ where for each i either $b_i = f(a)$ & $b_{i+1} = g(a)$ OR $b_i = g(a)$ & $b_{i+1} = f(a)$ for some a (depending on i).

Need to check $i'(b_i) = i'(b_{i+1})$ for each $i = 1, \ldots, n-1$.

Either $b_i = f(a)$ & $b_{i+1} = g(a)$ — in which case $i'(b_i) = i'(f(a)) = i'(g(a)) = i'(b_{i+1})$

OR $b_i = g(a)$ & $b_{i+1} = f(a)$, which works similarly. ∎

Pullbacks & Pushouts

$\boxed{\text{Prop}}$ To compute a pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ it suffices to take a product of A & B:

$$\begin{tikzcd} A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

and then form the equalizer of: $Z \xrightarrow{i} A \times B \overset{f \circ \pi_1}{\underset{g \circ \pi_2}{\rightrightarrows}} C$

giving the desired pullback:

$$\begin{tikzcd} Z \arrow[r, "\pi_2 \circ i"] \arrow[d, "\pi_1 \circ i"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

(Pf on next page)

Pf.

Note the last square commutes since $f\circ\pi_1\circ i = g\circ\pi_2\circ i$, so it's a candidate for being the pullback.

To show it's universal, consider a competitor:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!\psi"] \arrow[drrr, bend left, "q"] \arrow[ddrr, bend right, "p"'] & & & \\
& Z \arrow[dr, "i"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\
& & A \arrow[r, "f"] & C
\end{tikzcd}$$

only little square does not commute

How do we show $\exists!\ \psi: Q\to Z$ making the newly formed triangle commute?

By the universal property of the product, we get

$$\begin{tikzcd}
Q \arrow[dr, "\varphi"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \\
& A &
\end{tikzcd}$$

making this commute

Why is Q a competitor?

Need: $f\circ\pi_1\circ\varphi = g\circ\pi_2\circ\varphi$

$$\begin{aligned} f\circ\pi_1\circ\varphi &= f\circ p \\ &= g\circ q \\ &= g\circ\pi_2\circ\varphi \end{aligned}$$

(by various comm. diagrams)

By the universal property of the equalizer, $\exists!\ \psi: Q\to Z$ making this diagram commute:

$$\begin{tikzcd}
Z \arrow[r, "i"] & A\times B \arrow[r, shift left, "f\circ\pi_1"] \arrow[r, shift right, "g\circ\pi_2"'] & C \\
Q \arrow[u, "\psi"] \arrow[ur, "\varphi"'] & &
\end{tikzcd}$$

In particular, $\varphi = i\circ\psi$.

Why does this imply:

(1) $\pi_1\circ i\circ\psi = p$

(2) $\pi_2\circ i\circ\psi = q$

(3) a unique ψ making (1) & (2) true.

(continued)

Pf: (continued)

For (1) & (2), suffices to show $\pi_1 \circ \varphi = p$ and $\pi_2 \circ \varphi = q$, but we already had this by the universal property of the product.

Exercise: check (3) ∎

"Category theory makes trivial things trivially trivial." —Michael Barr

"I'm content to let them be trivial." – Timothy Gowers

$\boxed{\text{Prop}}$ In Set, a pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is
$Z = \{(a,b) \in A \times B : f(a) = g(b)\}$ with obvious maps to A & B.

Pf:

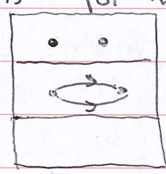
Previous Prop. combined with our description of products & equalizers in Set. ∎

In fact, a category has limits for all finite diagrams iff it has:

products

equalizers

terminal object 1



$\boxed{\text{Prop}}$ If this is a pullback:

$$\begin{tikzcd} A \times_C B \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

and g is mono, then p is a mono:

Pf:

Assume g is a mono. Show p is a mono:

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & A \times_C B \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

Need: $p \circ h = p \circ k \Rightarrow h = k$.

$$\begin{aligned} p \circ h = p \circ k &\Rightarrow f \circ p \circ h = f \circ p \circ k \\ &\Rightarrow g \circ q \circ h = g \circ q \circ k \\ &\Rightarrow q \circ h = q \circ k \qquad \text{since } g \text{ is mono} \end{aligned}$$

(continued)

Pf: (continued)

Note X is a competitor to the pullback

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists! \psi"] \arrow[drr, bend left, "q \circ h = q \circ k"] \arrow[ddr, bend right, "p \circ h = p \circ k"'] & & \\
& A \times_C B \arrow[r, "q"] \arrow[d, "p"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"] & C
\end{tikzcd}$$

$$\begin{aligned} f \circ p \circ h &= g \circ q \circ h \\ &= g \circ q \circ k \end{aligned}$$

So $\exists ! \psi : X \longrightarrow A \times_C B$ making this commute.

Both h & k do make it commute.

So $h = k$ $\blacksquare$

$\boxed{\text{Prop}}$ Given

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathcal{A}"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathcal{B}"] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

1) If $\mathcal{A}$ & $\mathcal{B}$ are pullbacks, so is the combined square $\mathcal{AB}$.

2) If $\mathcal{B}$ & $\mathcal{AB}$ are pullbacks, so is $\mathcal{A}$.