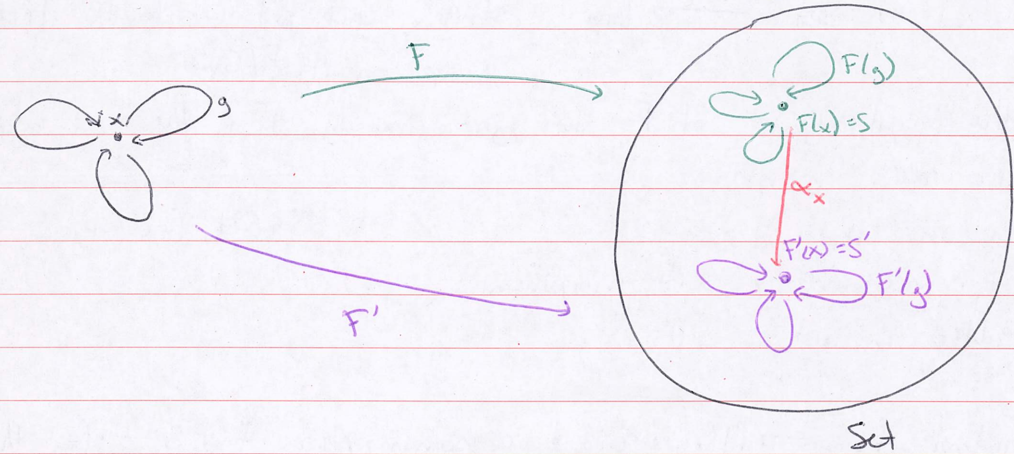
2/2/16

We saw that a 1-object category G with all morphisms invertible is a group.
We saw that a functor $F: G \longrightarrow \text{Set}$ is a <u>G-set</u>: a set with functions $F(g): S \to S$ for each $g \in G$ s.t.

$$F(gg') = F(g) \circ F(g')$$
$$F(1) = 1_S$$

Given 2 functors $F, F': G \longrightarrow \text{Set}$, what's a natural transformation $\alpha: F \Rightarrow F'$?
It's called a <u>map of G-sets</u> or <u>G-equivariant</u> map, but let's draw one:



It's a function $\alpha_x: F(x) \longrightarrow F'(x)$, where $x \in G$ is the one object, $F(x) = S$ is our first G-set, and $F'(x) = S'$ is our second, s.t. all squares like this commute:

$$\begin{tikzcd} F(x) \arrow[r, "F(g)"] \arrow[d, "\alpha_x"'] & F(x) \arrow[d, "\alpha_x"] \\ F'(x) \arrow[r, "F'(g)"] & F'(x) \end{tikzcd}$$

so $F'(g) \circ \alpha_x = \alpha_x \circ F(g)$

Another example of natural transformations

$\boxed{\text{Ex}}$ Two sets are isomorphic if there are functions $F: X \longrightarrow Y$, $G: Y \longrightarrow X$ s.t. $G \circ F = 1_X$ & $F \circ G = 1_Y$.
Given F, when can you find such a G? Iff f is 1-1 & onto.
For categories we say:

$\boxed{\text{Def}}$ An <u>equivalence</u> of categories C & D consists of functors $F: C \to D$, $G: D \longrightarrow C$ and natural isomorphisms $\alpha: G \circ F \overset{\sim}{\Rightarrow} 1_X$, $\beta: F \circ G \overset{\sim}{\Rightarrow} 1_Y$. We say that F & G are <u>weak inverses</u>. We say C & D are equivalent if there exists an equivalence between them.

$\boxed{\text{Thm}}$ Given a functor $F: C \to D$, it's part of an equivalence (F, G, α, β) iff F is faithful, full, & essentially surjective. If such a G exists, it may not be unique, but if G' was another one, it's naturally isomorphic to G.

Another example: adjoint functors

Recall an example: $U: \mathrm{Grp} \longrightarrow \mathrm{Set}$ sending each group G to its underlying set $U(G)$

$F: \mathrm{Set} \longrightarrow \mathrm{Grp}$ sending each set S to the free group on it $F(S)$.

We say U is the "right adjoint" of F, or synonymously, F is the "left adjoint" of U. The basic idea: morphisms

$FS \to G$ in Grp $\qquad S \in \mathrm{Set}, G \in \mathrm{Grp}$

are in 1-1 correspondence with morphisms

$S \to UG$ in Set

Why?

Given a function $f: S \to UG$ we get a homomorphism $\tilde{f}: FS \to G$, the unique one s.t. $\tilde{f}(s) = f(s)$ for $s \in S \subseteq \underbrace{FS}_{\text{the underlying set}}$

Conversely,

given a homomorphism $h: FS \to G$, we get $\underset{\sim}{h}: S \to UG$ by restricting h to $S \subseteq FS$.

The usual picture:

$$\begin{tikzcd}
S \arrow[d, "\text{inclusion function}"'] \arrow[dr, "\text{given a function } f"] & \\
FS \arrow[r, "\exists!\ \text{homomorphism } \tilde{f}"'] & G
\end{tikzcd}$$

mixes up morphisms in Set & Grp in the same diagram.

We prefer to say there's a bijection

$$\underset{\uparrow \text{in Grp}}{\hom(FS, G)} \cong \underset{\uparrow \text{in Set}}{\hom(S, UG)}$$

Note F is on the left of $\hom(F-, -)$,

U is on the right of $\hom(-, U-)$

To define adjoint functors, we need to say that this kind of bijection is "natural".

What functors give $\hom(FS, G)$ & $\hom(S, UG)$?

They must be 2 functors from $\mathrm{Set} \times \mathrm{Grp}$ to Set:

on objects, these do:

$$(S, G) \longmapsto \hom(FS, G) \in \mathrm{Set}$$
$$(S, G) \longmapsto \hom(S, UG) \in \mathrm{Set}$$

What's the "hom" doing here?

$\boxed{\text{Prop}}$ For any category there's a functor called the <u>hom functor</u>.
$$\hom: C^{op} \times C \longrightarrow \mathrm{Set}$$
$$(c, c') \longmapsto \hom(c, c')$$

Here C^{op} is the <u>opposite</u> of C: the category with one morphism $f^{op}: y \to x$ for each $f: x \to y$ in C, and $f^{op} \circ g^{op} = (g \circ f)^{op}$ with same identity morphisms.

Sketch of Pf:

Need to define $\hom: C^{op} \times C \longrightarrow \mathrm{Set}$ on morphisms.

Given a morphism in $C^{op} \times C$:

$$\varphi: (x, y) \longrightarrow (x', y')$$

i.e. a pair of morphisms

$f^{op}: x \longrightarrow x'$ in C^{op} i.e. $f: x' \longrightarrow x$ in C

$g: y \longrightarrow y'$ in C

we need to define a morphism

$$\hom(\varphi): \hom(x, y) \longrightarrow \hom(x', y')$$

in Set, i.e. a function.

Given $h \in \hom(x, y)$, what $\hom(\varphi) h \in \hom(x', y')$?

It's $g \circ h \circ f$

Thus the hom-functor $\hom: C^{op} \times C \to \mathrm{Set}$ will not only describe the hom sets, but also composition in C.

$$\begin{tikzcd} x \arrow[r, "h"] & y \arrow[d, "g"] \\ x' \arrow[u, "f"] \arrow[r, "\underbrace{g \circ h \circ f}_{\hom(\varphi) h}"'] & y' \end{tikzcd}$$

Then: check it's really a functor — e.g. check it preserves composition.

Given functors $F: C \to D$, $U: D \to C$, how can we say the isomorphism
$\hom(Fc, d) \cong \hom(c, Ud)$ is natural?

$$\begin{tikzcd}
 & D^{op} \times D \arrow[dr, "\hom"] \arrow[dd, phantom, "\cong \Downarrow \alpha"] & \\
C^{op} \times D \arrow[ur, "F^{op} \times 1_D"] \arrow[dr, "1 \times U"'] & & Set \\
 & C^{op} \times C \arrow[ur, "\hom"'] &
\end{tikzcd}$$

$(F(c), d) \in D^{op} \times D$

$(c, d) \in C^{op} \times D$

$(c, U(d)) \in C^{op} \times C$

$$\hom(F(c), d) \in Set$$
$$\cong \downarrow \alpha_{c,d}$$
$$\hom(c, U(d)) \in Set$$

Here $F^{op}: C^{op} \to D^{op}$ is just F in disguise $F^{op}(f^{op}) = (F(f))^{op}$
and α is a natural isomorphism.

2/4/16

Given functors
$$\begin{tikzcd} D \arrow[d, shift left, "U"] \\ C \arrow[u, shift left, "F"] \end{tikzcd}$$
we say F is a left adjoint of U or U is a right adjoint of F if there's a natural isomorphism

$$\begin{tikzcd} & D^{op}\times D \arrow[dr, "\hom"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\ C^{op}\times D \arrow[ur, "F^{op}\times 1"] \arrow[dr, "1\times U"'] & & \mathrm{Set} \\ & C^{op}\times C \arrow[ur, "\hom"'] & \end{tikzcd}$$

So we have bijections

$\alpha_{c,d} : \hom(Fc, d) \xrightarrow{\sim} \hom(c, Ud)$ for all $c\in C$, $d\in D$

which are natural, i.e. making certain squares commute.

At first let's downplay the naturality condition & look at examples, focussing on the bijections.

Ex
$$\begin{tikzcd} \mathrm{Grp} \arrow[d, shift left, "U"] \\ \mathrm{Set} \arrow[u, shift left, "F"] \end{tikzcd}$$
where UG is the underlying set of $G\in\mathrm{Grp}$, FS is the free group on $S\in\mathrm{Set}$.

The bijection lets us turn any function $f: S\longrightarrow UG$

into a homomorphism $\tilde{f} = \alpha^{-1}_{S,G}(f) : FS \longrightarrow G$

& conversely: any homomorphism $h: FS\longrightarrow G$

comes from a function $\underset{\sim}{h} = \alpha_{S,G}(h) : S\longrightarrow UG$

Ex Does the forgetful functor
$$\begin{tikzcd} \mathrm{Vect}_k \arrow[d, "U"] \\ \mathrm{Set} \end{tikzcd}$$
have a left adjoint?

Is there some famous functor $F:\mathrm{Set}\longrightarrow\mathrm{Vect}$?

Yes, for any set S there's a vector space FS whose basis is S:

$FS = \{ \sum_{s_i\in S} c_i s_i \ : c_i\in k$, only finitely many nonzero$\}$

where the sums are formal expressions.

What does $F:\mathrm{Set}\longrightarrow\mathrm{Vect}_k$ do to a morphism $f:S\to T$ in Set?

It should give a linear map $Ff: FS\longrightarrow FT$. What is it?

$Ff\left(\sum_{s_i\in S} c_i s_i\right) = \sum_{s_i\in S} c_i f(s_i) \in FT$

Check F is a functor:

$F(g\circ f) = F(g)\circ F(f)$

$F(1_S) = 1_{F(S)}$

(continued)

Why is F left adjoint to U? Need bijections:

$\hom(FS, V) \cong \hom(S, UV) \quad \forall S \in \mathrm{Set} \quad \forall V \in \mathrm{Vect}_k$

(and then check they're natural)

Need: given a function $f : S \longrightarrow UV$

we can define a linear map $\tilde{f} : FS \longrightarrow V$

in some "natural" way. Try

$$\tilde{f}\Big(\sum_{s_i \in S} c_i s_i\Big) = \sum_{s_i \in S} c_i f(s_i)$$

Conversely given a linear map $l : FS \to V$

need a function $\underset{\sim}{l} : S \longrightarrow UV$

Try: $\underset{\sim}{l}(s) = l(s)$

Check these maps are inverses: $\underset{\sim}{(\tilde{f})} = f$ and $\widetilde{(\underset{\sim}{l})} = l$.

So, we have a bijection $\hom(FS, V) \cong \hom(S, UV)$

Sometimes a functor has both a left & right adjoint.

$\boxed{\text{Ex}}$

$$\begin{array}{c} \mathrm{Top} \\ \downarrow U \\ \mathrm{Set} \end{array}$$

To dream up a left adjoint, think of ways to turn a set S into a topological space.

One is the discrete topology: here you give S as many open sets as possible, so every subset is open.

Another is the indiscrete topology: here you give S as few open sets as possible, only $\emptyset$ & S are open.

The left adjoint of $U : \mathrm{Top} \to \mathrm{Set}$, say $L : \mathrm{Set} \longrightarrow \mathrm{Top}$ must have

$\hom(LS, X) \cong \hom(S, UX) \qquad S \in \mathrm{Set}, \quad X \in \mathrm{Top}$

i.e. here continuous maps $\tilde{f} : LS \longrightarrow X$ are "the same" as functions $f : S \to UX$

To make this true, LS should have as many open sets as possible, so LS is S with discrete topology.

The right adjoint of U, say $R : \mathrm{Set} \longrightarrow \mathrm{Top}$, has

$\hom(UX, S) \cong \hom(X, RS)$

i.e. continuous maps $h : X \longrightarrow RS$ are "the same" as functions $\underset{\sim}{h} : UX \longrightarrow S$

To make this true, RS should have as few open sets as possible, i.e. it should be S with indiscrete topology.

Suppose C is any category. There's always a functor $\Delta : C \to C \times C$ called the diagonal with

$$\Delta(c) = (c,c) \qquad c \in C$$

& if $f : c \to c'$

$$\Delta f : \Delta c \to \Delta c' \quad \text{is given by} \quad \Delta f = (f,f) : (c,c) \to (c',c')$$

$\boxed{\text{Prop}}$ If C has binary products then the functor $\times : C \times C \to C$ is the right adjoint of $\Delta : C \to C \times C$.
(In fact, the converse is true: Δ has a right adjoint iff C has binary products, & then it's $\times$.)

Sketch of Pf:

For starter we need bijections

$$\hom(\Delta c, (c', c'')) \cong \hom(c, c' \times c'') \qquad c \in C \qquad (c', c'') \in C \times C$$

The left side:

$$\hom(\Delta c, (c', c'')) = \hom((c,c), (c', c''))$$
$$\cong \hom(c, c') \times \hom(c, c'')$$

since a morphism from (c,c) to (c',c'') is a pair $f : c \to c'$, $g : c \to c''$

So we need:

$$\hom(c, c') \times \hom(c, c'') \cong \hom(c, c' \times c'')$$

Indeed, the universal property of the product says:

$$\begin{tikzcd}
 & c \arrow[d, "\exists!\,\psi"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
 & c' \times c'' \arrow[dl, "p"'] \arrow[dr, "q"] & \\
c' & & c''
\end{tikzcd}$$

So (f,g) gives ψ & conversely ψ gives $f = p \circ \psi$ & $g = q \circ \psi$, so we have a bijection:

$$\hom(c, c') \times \hom(c, c'') \xrightarrow{\sim} \hom(c, c' \times c'')$$
$$(f,g) \longmapsto \psi$$

$\blacksquare$

$\boxed{\text{Prop}}$ If C has binary coproducts, $\Delta : C \to C \times C$ has a left adjoint, $+ : C \times C \to C$ assigning to each pair (c', c'') their coproduct $c' + c''$.
(Conversely, if Δ has a left adjoint, C has binary coproducts & left adjoint is $+$.)

(Sketch of Pf on next page)

Sketch of Pf:

For starters we need a bijection

$$\hom(c'+c'', c) \cong \hom((c', c''), \Delta c).$$

Note: $\hom((c', c''), \Delta c) \cong \hom((c', c''), (c, c))$

$$\cong \hom(c', c) \times \hom(c'', c)$$

So need:

$$\hom(c'+c'', c) \cong \hom(c', c) \times \hom(c'', c)$$

and indeed the definition of coproducts gives

$$\begin{tikzcd} c' \arrow[dr, "i"] \arrow[ddr, bend right, "f"'] & & c'' \arrow[dl, "j"'] \arrow[ddl, bend left, "g"] \\ & c'+c'' \arrow[d, "\exists! \psi"] & \\ & c & \end{tikzcd}$$

So our bijection sends $\psi \mapsto (f, g)$ ∎

A product (an example of a limit) is an example of a right adjoint – it's easy to describe morphisms going into it.

A coproduct (an example of a colimit) is an example of a left adjoint – it's easy to describe morphisms going out of it.