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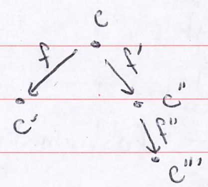
Last time we saw that if C has products, the functor $\times : C^2 \longrightarrow C$, $(c, c') \longmapsto c \times c'$
is a right adjoint to the diagonal functor $\Delta : C \longrightarrow C^2$, $c \longmapsto (c, c)$

& similarly $+ : C^2 \longrightarrow C$, if C has coproducts, is a left adjoint to Δ.

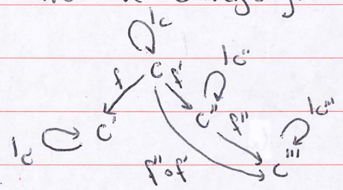
(Thus $\oplus : \mathrm{Vect}_k^2 \longrightarrow \mathrm{Vect}_k$ is both left & right adjoint to $\Delta : \mathrm{Vect}_k \longrightarrow \mathrm{Vect}_k^2$.)

In fact if a category has limits, these limits give a right adjoint to some functor: "limits are right adjoints". Similarly "colimits are left adjoints".

We often think about the limit of a diagram in a category C. What's a "diagram in C", really?

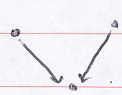


Namely, it's a collection of objects & morphisms between them.
We can make it into a category:

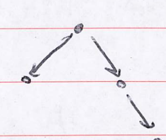


Now it's a subcategory of C.

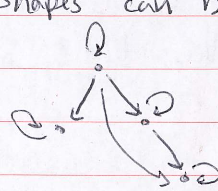
We're often interested in diagram of some shape, like pullbacks



or

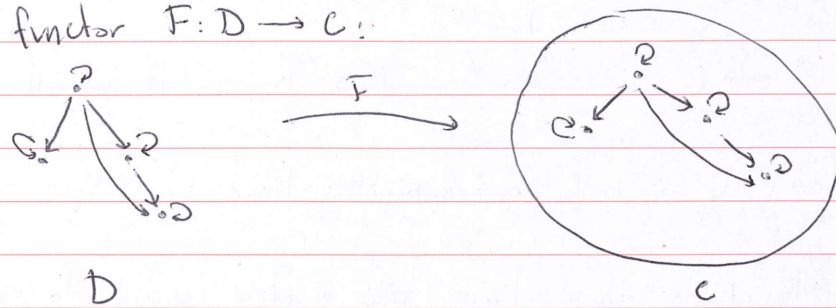


These "shapes" can be interpreted as categories:



Let D be any category: we'll take this as our "diagram shape". What is a D-shaped diagram in some category C?

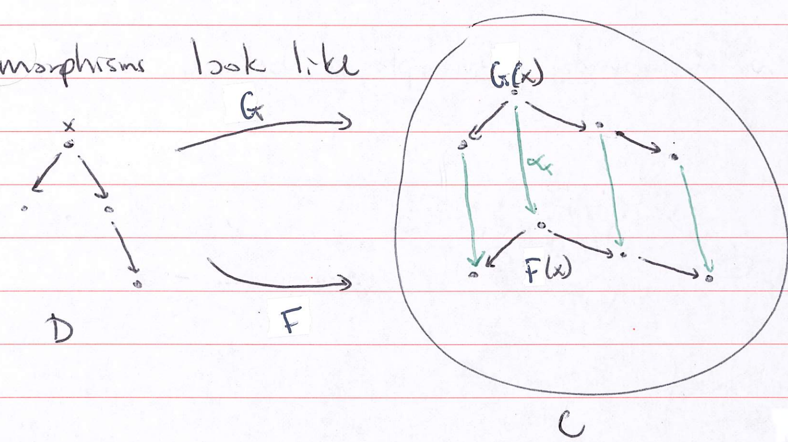
It's a functor $F: D \to C$:



When we take the limit of this diagram, we get an object (defined up to isomorphism) $\lim F \in C$.
What's the process that takes us from $F: D \to C$ to $\lim F \in C$?
The key: there's a category C^D with:
- objects being functors $F: D \to C$
- morphisms being natural transformations

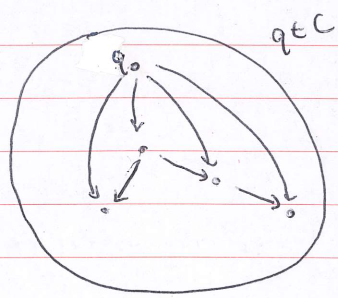
$$\begin{tikzcd} D \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

These morphisms look like

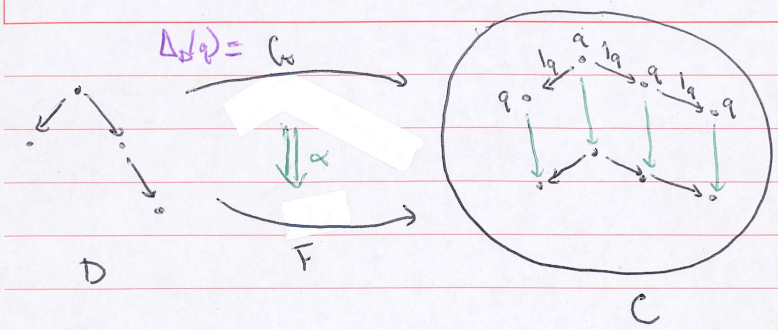


where all the squares commute

When we take a limit of $F: C \to D$, we study cones over F:



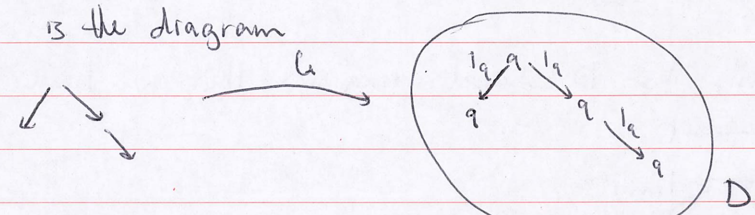
A cone over F is a natural transformation $\alpha: G \Rightarrow F$ where G sends every object of D to some object C & G sends every morphism of D to the identity morphism of that object.



Here $G: D \to C$ was determined by the object $q \in C$, via the above recipe. It turns an object $q \in C$ into an object $G \in C^D$.

So this recipe should be a functor $\Delta_D : C \to C^D$

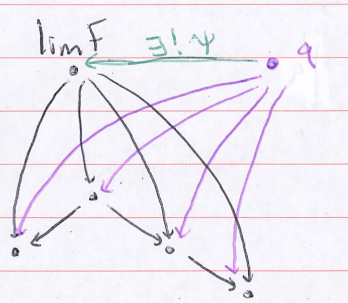
$\Delta_D(q)$ is the diagram



Here $G = \Delta_D(q)$

So: a cone over F with apex $q \in C$ is a natural transformation $\alpha : \Delta_D(q) \Rightarrow F$

What's the limit of a diagram? If $F \in C^D$



It's a universal cone over that diagram.

Remember U is the right adjoint of F if:

$\hom(Fx, y) \cong \hom(x, Uy)$

So adjoint functors are about converting one kind of morphism into another in a bijective way, & that's what we're doing when we're stating the universal property:

- morphisms $\psi : q \to \lim F$ in C
- cones over F with apex q, i.e. natural transformations $\alpha : \Delta_D(q) \Rightarrow F$ morphisms α from $\Delta_D(q)$ to F in C^D.

So: $\hom(\Delta_D(q), F) \cong \hom(q, \lim F)$

So it looks like we have

$$\lim : C^D \longrightarrow C$$

which is right adjoint to

$$\Delta_D : C \longrightarrow C^D$$

This is true – you need to check that

$$\hom(\Delta_D(q), F) \cong \hom(q, \lim F)$$

is a <u>natural</u> bijection to finish the proof of:

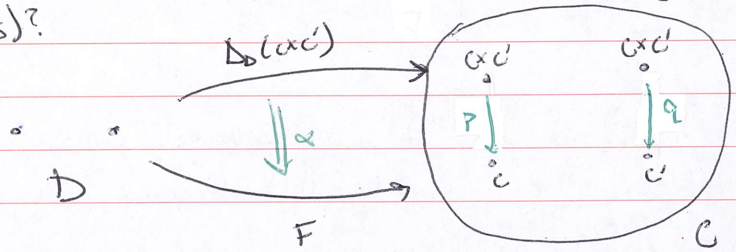
$\boxed{\text{Thm}}$ If C has all limits for D-shaped diagrams, then we have a functor

$$\lim : C^D \longrightarrow C$$
$$F \longmapsto \lim F$$

which is right adjoint to $\Delta_D : C \longrightarrow C^D$.

The converse is true too: if $\Delta_D : C \longrightarrow C^D$ has a right adjoint, then this gives limits of D-shaped diagrams in C.

What choice of D gives the case of binary products (a special case of limits)?



Here D has 2 objects & only identity morphisms, so we could call it 2, so $C^D = C^2$ & $\times : C^2 \longrightarrow C$ is right adjoint to $\Delta_2 = \Delta : C \longrightarrow C^2$.

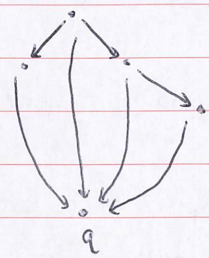
Similarly,

$\boxed{\text{Thm}}$ If a category C has colimits of all D-shaped diagrams, there's a functor $\operatorname{colim} : C^D \rightarrow C$ left adjoint to $\Delta_D : C \longrightarrow C^D$ & conversely.

So $\hom(\operatorname{colim} F, q) \cong \hom(F, \Delta_D q)$.

Note

$\alpha \in \hom(F, \Delta q)$ is a cocone:



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Thm Left adjoints preserve colimits; right adjoints preserve limits.

Pf: (sketch)

Let's show that if $F: C \to D$ is a left adjoint to $U: D \to C$, then F preserves colimits.

For concreteness, let's show F preserves pushouts – general case is analogous.

So suppose we have a pushout in C:

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] \arrow[d] & \\ b \arrow[r] & x & c \arrow[l] \end{tikzcd}$$

Here x is the apex of a cocone on the diagram we're taking a colimit of, & the universal property holds.

The claim is that applying F to this universal cocone gives a universal cocone in D:

$$\begin{tikzcd} & F(a) \arrow[dl] \arrow[dr] \arrow[d] \arrow[drr] & & \\ F(b) \arrow[r] \arrow[rr, bend right] & F(x) \arrow[r, dashed, "\text{want } \exists! \psi"'] & Q & F(c) \arrow[ll] \arrow[l] \end{tikzcd}$$

Choose a competitor cocone with apex Q. Need to show $\exists!\ \psi: F(x) \to Q$ making the newly formed triangle commute.

We can look at $U(Q) \in C$

Note $\hom(F(x), Q) \cong \hom(x, U(Q))$

So to get $\psi: F(x) \to Q$, let's find $\varphi: x \to U(Q)$.

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] \arrow[d] \arrow[drr] & & \\ b \arrow[r] \arrow[rrr, bend right] & x & c \arrow[l] \arrow[r] & U(Q) \end{tikzcd}$$

$U(Q)$ becomes a competitor due to the adjointness of F & U, e.g. $\hom(F(a), Q) \cong \hom(a, U(Q))$

For same reason, the triangles involving $U(Q)$ commute since those involving Q commute.

So $U(Q)$ is a competitor.

Thus $\exists!\ \varphi: x \to U(Q)$ making the newly formed triangles commute.

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] \arrow[d] \arrow[drr] & & \\ b \arrow[r] \arrow[rrr, bend right] & x \arrow[rr, dashed, bend right=40, "\exists! \varphi"'] & c \arrow[l] \arrow[r] & U(Q) \end{tikzcd}$$

This gives us $\psi: F(x) \to Q$, check it makes its newly formed triangle commute & is unique (since φ is).

Ex $F: \mathrm{Set} \longrightarrow \mathrm{Grp}$ preserves colimits, e.g. coproducts, so
$F(S+T) \cong F(S) + F(T)$
Here $S+T$ is the disjoint S & T, $F(S+T)$ is the free group with elements of $S+T$ as generators, and $F(S) + F(T) = F(S) * F(T)$ is the "free product" of $F(S)$ & $F(T)$.

Ex $U: \mathrm{Grp} \longrightarrow \mathrm{Set}$ preserves limits, e.g. products:
$U(G \times H) \cong U(G) \times U(H)$
where $G \times H$ is the usual product of groups G & H.

Thm The composite of left adjoints is a left adjoint. The composite of right adjoints is a right adjoint.

Pf:

Suppose we have functors $C \xrightarrow{F} D \xrightarrow{F'} E$ and F & F' are left adjoints of functors U, U'. $C \xleftarrow{U} D \xleftarrow{U'} E$

We'll show that $F' \circ F : C \longrightarrow E$ is the left adjoint of $U \circ U' : E \longrightarrow C$.

Want a natural isomorphism $\hom(F' \circ F(c), e) \cong \hom(c, U \circ U'(e))$

Here's how we get it:

$$\hom(F' \circ F(c), e) \cong \hom(F(c), U'(e)) \quad \text{, since } F' \text{ is left adjoint to } U'$$
$$\cong \hom(c, U \circ U'(e)) \quad \text{, since } F \text{ is left adjoint to } U \qquad \blacksquare$$

Ex

$$\begin{tikzcd}
\mathrm{Ring} \arrow[d, bend left, "U'"] \arrow[dd, bend left=80, "U \circ U'"] \\
\mathrm{Ab\ Grp} \arrow[u, bend left, "F'"] \arrow[d, bend left, "U"] \\
\mathrm{Set} \arrow[u, bend left, "F"] \arrow[uu, bend left=80, "F' \circ F"]
\end{tikzcd}$$

$F' \circ F$ is left adjoint to the forgetful functor $U \circ U'$ from Ring to Set.

Starting from $\emptyset$ (the initial set) we get $F(\emptyset) = \{0\}$ (the trivial abelian group, which is the initial abelian group) & then $F'(F(\emptyset)) = \mathbb{Z}$ (the ring of integers, which is the initial ring).

Starting from a one-element set $\{x\}$, we get
$F(\{x\}) = \{\ldots, -x, 0, x, x+x, \ldots\} \cong \mathbb{Z}$
& then $F'(F(x)) = \mathbb{Z}[x]$ the ring of polynomials in x with integer coefficients.

Units and counits of adjunctions (= pair of adjoint functors)

Suppose we have $F \curvearrowright \substack{D \\ C} \curvearrowleft U$ with F left adjoint to U.

$\hom(Fc, d) \cong \hom(c, Ud)$ $\quad \forall\ c \in C,\ \forall d \in D$

We can apply this bijection to an identity morphism & get something interesting. We can do this if $d = Fc$.

$$\begin{array}{ccc} \hom(Fc, Fc) & \xrightarrow[\sim]{\varphi} & \hom(c, UFc) \\ \Psi & & \Psi \\ 1_{Fc} & & \varphi(1_{Fc}) \end{array}$$

$\varphi(1_{Fc})$ is called the unit, η_c: $\quad \eta_c : c \longrightarrow UFc$

We can also apply φ^{-1} to an identity if $c = Ud$.

$$\begin{array}{ccc} \hom(FUd, d) & \xleftarrow[\sim]{\varphi^{-1}} & \hom(Ud, Ud) \\ \Psi & & \Psi \\ \varphi^{-1}(1_{Ud}) & & 1_{Ud} \end{array}$$

$\varphi^{-1}(1_{Ud})$ is called the counit, ε_d: $\quad \varepsilon_d : FUd \longrightarrow d$

These give various famous morphisms.

$\boxed{\text{Ex}}$ $F: \text{Set} \longrightarrow \text{Grp}$

$U: \text{Grp} \longrightarrow \text{Set}$

Given any set S, we get a unit: $\eta_S : S \longrightarrow UFS$

This is the "inclusion of the generators": elements of S are generators of FS.

Given a group G, get: $\varepsilon_G : FUG \longrightarrow G$

$$g_1^{\pm 1} * g_2^{\pm 1} * \cdots * g_n^{\pm 1} \longmapsto g_1^{\pm 1} \cdots g_n^{\pm 1} \qquad g_i \in G$$

"formal product" in FUG $\qquad$ "actual product" in G.

The counits "convert formal expressions into actual ones."