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Cartesian Closed Categories

Any category has a *set* hom(X, Y) of morphisms from one object X to another object Y, but in a cartesian closed category (or ccc) you also have an object Y^X of morphisms from X to Y.

$\boxed{\text{Ex}}$ if $C = \text{Cat}$, $\hom(X,Y)$ is the set of functors $F: X \to Y$, while Y^X is the category of functors $F: X \to Y$ & natural transformations between them.
In general, you can get $\hom(X,Y)$ from Y^X but not vice versa.
We call $\hom(X,Y)$ the homset or external hom (it lives outside of C, in Set), & Y^X the exponential or internal hom (since it lives inside C).

Internalization is the process of taking math that lives in Set & moving it into some category C.

E.g. in Set you can define a group to be an object $G \in \text{Set}$ with morphisms:

$m: G \times G \longrightarrow G$	multiplication
$\text{inv}: G \longrightarrow G$	inverses
$i: 1 \longrightarrow G$	the identity-assigning map it maps the one element of 1 to the identity element in G

s.t.

$$\begin{tikzcd}
 & G\times G\times G \arrow[dl, "m\times 1_G"'] \arrow[dr, "1_G\times m"] & \\
G\times G \arrow[dr, "m"'] & & G\times G \arrow[dl, "m"] \\
 & G &
\end{tikzcd}$$

commutes (associative law)

left and right unit laws:

$$\begin{tikzcd}
1\times G \arrow[d, "i\times 1_G"'] & G \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "1_G"] & G\times 1 \arrow[d, "1_G\times i"] \\
G \arrow[r, "m"] & G & G\times G \arrow[l, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(*,g) & g \arrow[l, mapsto] \arrow[r, mapsto] \arrow[d] & (g,*) \arrow[d] \\
(1,g) \arrow[r, mapsto] & 1g = g = g1 & (g,1) \arrow[l, mapsto]
\end{tikzcd}$$

inverse laws

$$\begin{tikzcd}
G\times G \arrow[d, "\text{inv}\times 1_G"'] & G \arrow[l, "\Delta"'] \arrow[r] \arrow[d, "\exists!"] & G\times G \arrow[dd, "1_G\times \text{inv}"] \\
 & 1 \arrow[d, "i"] & \\
G\times G \arrow[r, "m"'] & G & G\times G \arrow[l, "m"]
\end{tikzcd}$$

(continued)

All these diagrams make sense in any cartesian category (= category with finite products = category with binary products & terminal object). So we can define a <u>group internal to C</u> or <u>group object in C</u> or <u>group in C</u> using these axioms whenever C is cartesian.

E.g.:

- if $C = \mathrm{Top}$, a group in C is called a <u>topological group</u>
- if $C = \mathrm{Diff}$, a group in C is called a Lie group
- if C = algebraic varieties, a group in C is called an <u>algebraic group</u>.

Puzzle: if $C = \mathrm{Grp}$, a group in C is a very famous thing. What is it?

Recall a cartesian category C is a <u>ccc</u> if for any $Y \in C$, $- \times C$ has a right adjoint: $\hom(X \times Y, Z) \cong \hom(X, Z^Y)$

Any adjunction
$$\begin{tikzcd} D \arrow[d, bend left, "U"] \\ C \arrow[u, bend left, "F"] \end{tikzcd}$$
has a unit & counit:

$$\eta_X : X \longrightarrow UFX \qquad X \in C$$
$$\varepsilon_Y : FUY \longrightarrow Y \qquad Y \in D$$

Now we have an adjunction
$$\begin{tikzcd} C \arrow[d, bend left, "-^Y"] \\ C \arrow[u, bend left, "- \times Y"] \end{tikzcd}$$

$$\eta_X : X \longrightarrow (X \times Y)^Y \qquad X \in C$$
$$\varepsilon_X : X^Y \times Y \longrightarrow X$$

The second one is called <u>evaluation</u>: in Set
$$\varepsilon_X : X^Y \times Y \longrightarrow X$$
$$(f, y) \longmapsto f(y)$$

The first one is called <u>coevaluation</u>: in Set
$$\eta_X : X \longrightarrow (X \times Y)^Y \text{ has } \eta_X(x)(y) = (x, y) \qquad x \in X,\ \eta_X(x) : Y \longrightarrow X \times Y$$

So we have analogues of these in any ccc.

Next: in any category we have composition:

$$\circ : \hom(Y,Z) \times \hom(X,Y) \longrightarrow \hom(X,Z)$$
$$(f,g) \longmapsto f\circ g$$

In a CCC, can we internalize this & define "internal composition":

$$\circ_{X,Y,Z} = \bullet : Z^Y \times Y^X \longrightarrow Z^X \quad ?$$

$$\bullet \in \hom(Z^Y \times Y^X, Z^X) \cong \hom(Z^Y, (Z^X)^{(Y^X)})$$
$$\text{or} \quad \cong \hom(Z^Y \times Y^X \times X, Z) \quad \longleftarrow \text{useful!}$$

So we get $\bullet$ from a morphism

$$\tilde{\bullet} : Z^Y \times Y^X \times X \longrightarrow Z$$

which we indeed have in any CCC

$$Z^Y \times Y^X \times X \xrightarrow{1_{Z^Y} \times \mathcal{E}} Z^Y \times Y \xrightarrow{\mathcal{E}} Z \quad \text{where } \mathcal{E} \text{ is evaluation.}$$

This is just an internalized way of saying the old def. of composition:

$$(f\circ g)(x) = \underbrace{f(g(x))}_{\text{two evaluations}}$$

[Emily Riehl, Categories in Context, Dover Pub.
free on the web]

Elements

Sets have elements, but what about objects in other categories?
Elements of a set X are in 1-1 correspondence with functions $f: 1 \longrightarrow X$
where 1 is a terminal object in Set (1 = a one element set)

So:

$\boxed{\text{Def}}$ If C is a category with a terminal object, an element of an object $X \in C$ to be a morphism $f: 1 \to X$. We define the set $elt(X)$ to be $\hom(1,X)$.

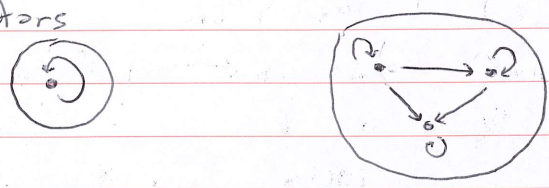
$\boxed{\text{Ex}}$ If $C = \text{Top}$, $elt(X) = \{\text{continuous maps } f: \{*\} \to X\}$, where $\{*\}$, the one-point space, is the terminal object of Top. In fact $elt(X)$ is in 1-1 correspondence with the underlying set of X: given $x \in X$,

$$f: \{*\} \longrightarrow X$$
$$* \longmapsto x$$

& conversely any such $f(*) \in X$.

$\boxed{\text{Ex}}$ If $C = \mathrm{Grp}$, $\mathrm{elt}(G) = \{\text{homomorphisms } f : 1 \to G\}$ where 1 is the trivial group, the terminal object in Grp. So $\mathrm{elt}(G)$ has just one element: there's just one homomorphism $f : 1 \to G$, since 1 is also initial!

$\boxed{\text{Ex}}$ If $C = \mathrm{Cat}$
$\mathrm{elt}(D) = \{\text{functors } f : 1 \to D\}$ where 1 is the terminal category: $\bullet \circlearrowright 1$.
functors



$$f : 1 \longrightarrow D$$

are in one-to-one correspondence with the objects of D.
So $\mathrm{elt}(D) \cong \{\text{objects in } D\}$
Here, as in the previous example, elt forgets a lot of information:

$$\mathrm{elt}(\bullet \longrightarrow \bullet) \cong \mathrm{elt}(\bullet \quad \bullet)$$