Symmetric monoidal categories

"A category theorist is sort of like a sociologist. He looks at mathematical objects — he doesn't pry it open and see how it works — but sees how it behaves in relation to all other things."

— Chris Heunen

$\boxed{\text{Def}}$ A <u>monoid</u> is a nonempty set G together with a binary operation on G which is

(i) associative: $(xy)z = x(yz)$ $\forall\, x,y,z \in G$

(ii) and contains a (two-sided) identity element $e \in G$ such that $xe = ex = x$ $\forall\, x \in G$

[i.e. take the definition of a group and drop the requirement of inverses]

Def A <u>monoidal category</u> is a category $\mathcal{C}$ which is equipped with

1. a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ where the image of a pair of objects (x, y) is denoted by $x \otimes y$
2. a <u>unit object</u> I
3. for every $x, y, z \in ob(\mathcal{C})$, an associativity isomorphism
$$a_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z),$$
natural in the objects $x, y,$ and z,
4. for every $x \in ob(\mathcal{C})$, a left unit isomorphism $l_x : I \otimes x \longrightarrow x$ and a right unit isomorphism $r_x : x \otimes I \longrightarrow x$, both natural in x.

We further assume the following diagrams commute for any objects $w, x, y,$ and z:

$$\begin{tikzcd}
 & ((w \otimes x) \otimes y) \otimes z \arrow[dl, "a_{w,x,y} \otimes id_z"'] \arrow[dr, "a_{w \otimes x, y, z}"] & \\
(w \otimes (x \otimes y)) \otimes z \arrow[d, "a_{w, x \otimes y, z}"'] & & (w \otimes x) \otimes (y \otimes z) \arrow[d, "a_{w,x,y \otimes z}"] \\
w \otimes ((x \otimes y) \otimes z) \arrow[rr, "id_w \otimes a_{x,y,z}"] & & w \otimes (x \otimes (y \otimes z))
\end{tikzcd}$$

$$\begin{tikzcd}
(x \otimes I) \otimes y \arrow[rr, "a_{x,I,y}"] \arrow[dr, "r_x \otimes id_y"'] & & x \otimes (I \otimes y) \arrow[dl, "id_x \otimes l_y"] \\
 & x \otimes y &
\end{tikzcd}$$

When we want to emphasize the tensor product and unit, we denote a monoidal category by $(\mathcal{C}, \otimes, I)$.

Ex
- $(Set, \times, \{\circ\})$
- $(Set, \amalg, \emptyset)$ } it's structure!
- $(Grp, \times, \{e\})$
- $(Hilb, \otimes, \mathbb{C})$
  objects: Hilbert spaces
  morphisms: short linear maps (linear maps of norm at most 1)

Why is

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

an isomorphism, and not an equality?

Let's consider the example $(\mathrm{Set}, \times, \{\circ\})$:

$$(X \times Y) \times Z = \{(w, z) \mid w \in X \times Y,\ z \in Z\}$$
$$= \{((x, y), z) \mid x \in X, y \in Y, z \in Z\}$$

$$X \times (Y \times Z) = \{(x, w) \mid x \in X,\ w \in Y \times Z\}$$
$$= \{(x, (y, z)) \mid x \in X,\ y \in Y,\ z \in Z\}$$

These sets are <u>not</u> equal — but we can easily construct an isomorphism.

Ex How can we take a monoid G and construct a monoidal category?

First, we need a category $\mathcal{C}$:

- objects: elements of G
- morphisms: identity morphisms

We get a monoidal category $(\mathcal{C}, \circ, e)$ where $\circ$ is the binary product of G and e is the identity element of G.

Note In general,

- if $\mathcal{C}$ has products, we get a monoidal category $(\mathcal{C}, \times, 1)$
- if $\mathcal{C}$ has coproducts, we get a monoidal category $(\mathcal{C}, +, 0)$

$\boxed{\text{Def}}$ A monoidal category $(\mathcal{C}, \otimes; I)$ is symmetric if it additionally is equipped with an isomorphism $s_{x,y}: x \otimes y \longrightarrow y \otimes x$ for any objects x and y of $\mathcal{C}$, natural in x and y, such that the following diagrams commute for all objects x, y, and z:

$$\begin{tikzcd}
 & (x \otimes y) \otimes z \arrow[dl, "a_{x,y,z}"'] \arrow[rr, "s_{x,y} \otimes id_z"] & & (y \otimes x) \otimes z \arrow[dr, "a_{y,x,z}"] & \\
x \otimes (y \otimes z) \arrow[dr, "s_{x,y\otimes z}"'] & & & & y \otimes (x \otimes z) \arrow[dl, "id_y \otimes s_{x,z}"] \\
 & (y \otimes z) \otimes x \arrow[rr, "a_{y,z,x}"] & & y \otimes (z \otimes x) &
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes I \arrow[rr, "s_{x,I}"] \arrow[dr, "r_x"'] & & I \otimes x \arrow[dl, "l_x"] \\
 & x &
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes y \arrow[r, "s_{x,y}"] \arrow[dr, "id_{x \otimes y}"'] & y \otimes x \arrow[d, "s_{y,x}"] \\
 & x \otimes y
\end{tikzcd}$$

Most of the examples of monoidal categories we have talked about are symmetric. What's an example of a monoidal category that is not symmetric?

Let R be a non-commutative ring

The category of R-R-bimodules, with ${}_R\otimes_R$ as the tensor and R as the unit, is such an example.

$\boxed{\text{Note}}$ Let $(\mathcal{C}, \circ, e)$ be the monoidal category given by the monoid G. If G is an abelian group, then $(\mathcal{C}, \circ, e)$ is symmetric.

Going back to the definition of a symmetric monoidal category...

Q: Why is the hexagon commuting diagram sufficient?

- There are 6 different ways to order 3 elements
- There are 2 ways of associating 3 elements

$\Rightarrow$ 12 possibilities

(and we would expect all of these to be isomorphic)

A: repeat!

$$\begin{tikzcd}
 & (x\otimes y)\otimes z \arrow[r] \arrow[dl] & (y\otimes x)\otimes z \arrow[dr] & \\
x\otimes (y\otimes z) \arrow[dr] & & & y\otimes (x\otimes z) \arrow[dl] \\
 & (y\otimes z)\otimes x \arrow[r, "a_{y,z,x}"] & y\otimes (z\otimes x) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (y\otimes z)\otimes x \arrow[r] \arrow[dl] & (z\otimes y)\otimes x \arrow[dr] & \\
y\otimes (z\otimes x) \arrow[dr] & & & z\otimes (y\otimes x) \arrow[dl] \\
 & (z\otimes x)\otimes y \arrow[r] & z\otimes (x\otimes y) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (z\otimes x)\otimes y \arrow[r] \arrow[dl] & (x\otimes z)\otimes y \arrow[dr] & \\
z\otimes (x\otimes y) \arrow[dr] & & & x\otimes (z\otimes y) \arrow[dl] \\
 & (x\otimes y)\otimes z \arrow[r] & x\otimes (y\otimes z) &
\end{tikzcd}$$