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Prop Suppose C is a category with a terminal object $1 \in C$. Then there's a functor $\mathrm{elt}: C \longrightarrow \mathrm{Set}$ with

$$\mathrm{elt}(X) = \{f: 1 \longrightarrow X\} \quad \forall X \in C$$

and given any morphism $g: X \longrightarrow Y$ in C, $\mathrm{elt}(g): \mathrm{elt}(X) \longrightarrow \mathrm{elt}(Y)$ is defined as follows:

$$\begin{tikzcd} 1 \arrow[r, "f"] \arrow[dr, "{\mathrm{elt}(g)f = g \circ f}"'] & X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

Pf:

elt preserves composition: given $X \xrightarrow{g} Y \xrightarrow{h} Z$ we need $\mathrm{elt}(h \circ g) = \mathrm{elt}(h) \circ \mathrm{elt}(g)$.

$$\begin{tikzcd} 1 \arrow[r, "f"] & X \arrow[d, "g"] \\ & Y \arrow[d, "h"] \\ & Z \end{tikzcd}$$

Given $f \in \mathrm{elt}(X)$ we have

$$\begin{aligned} \mathrm{elt}(h \circ g)f &= (h \circ g) \circ f \\ &= h \circ (g \circ f) \\ &= h \circ (\mathrm{elt}(g)f) \\ &= \mathrm{elt}(h)(\mathrm{elt}(g)f). \end{aligned}$$

Similarly

$$\begin{aligned} \mathrm{elt}(1_X)f &= 1_X \circ f \qquad \forall f \in \mathrm{elt}(X) \\ &= f \end{aligned}$$

So $\mathrm{elt}(1_X) = 1_{\mathrm{elt}(X)}$ ∎

Ex $\mathrm{elt}: C \longrightarrow \mathrm{Set}$ may not be faithful, i.e. we can have two different morphisms $g, g': X \longrightarrow Y$ in C with $\mathrm{elt}(g) = \mathrm{elt}(g')$.

If $C = \mathrm{Grp}$, we saw $\mathrm{elt}(G) = 1 \in \mathrm{Set}$ for all G, so any homomorphism $h: G \longrightarrow G'$ will get sent to a function $\mathrm{elt}(h): 1 \longrightarrow 1$, but there's only one of these.

$\boxed{\text{Prop}}$ If C is a cartesian category, $\mathrm{elt}: C \longrightarrow \mathrm{Set}$ preserves finite products.

Pf:

It's easy to show elt preserves the terminal object:

if $1 \in C$ then $\mathrm{elt}(1) = \{f: 1 \longrightarrow 1\}$ is a one-element set, so it's terminal in Set.

Why does elt preserve binary products?

Suppose $X, Y \in C$; then their product is a universal cone:

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

To show elt preserves products, we need to show this cone is universal in Set:

$$\begin{tikzcd} & \mathrm{elt}(X \times Y) \arrow[dl, "\mathrm{elt}(p)"'] \arrow[dr, "\mathrm{elt}(q)"] & \\ \mathrm{elt}(X) & & \mathrm{elt}(Y) \end{tikzcd}$$

Choose a competitor:

$$\begin{tikzcd} & Q \arrow[d, dashed, "\psi"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & \mathrm{elt}(X \times Y) \arrow[dl, "\mathrm{elt}(p)"'] \arrow[dr, "\mathrm{elt}(q)"] & \\ \mathrm{elt}(X) & & \mathrm{elt}(Y) \end{tikzcd}$$

$Q \in \mathrm{Set}$

$(\star)$

Want $\exists!\ \psi: Q \longrightarrow \mathrm{elt}(X \times Y)$ making the newly formed triangles commute.

$f: Q \longrightarrow \mathrm{elt}(X)$ sends any point $a \in Q$ to a point $f(a) \in \mathrm{elt}(X) = \{h: 1 \longrightarrow X\}$,

so $f(a): 1 \longrightarrow X$.

Similarly $g(a): 1 \longrightarrow Y$.

We want to define $\psi: Q \longrightarrow \mathrm{elt}(X \times Y)$; this will send any $a \in Q$ to $\psi(a): 1 \longrightarrow X \times Y$.

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\exists!\, \psi(a)"] \arrow[ddl, bend right, "f(a)"'] \arrow[ddr, bend left, "g(a)"] & \\ & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

By the universal property of $X \times Y$, for each $a \in Q$ $\exists!\ \psi(a): 1 \longrightarrow X \times Y$ s.t. this commutes.

(continued)

Pf: (continued)

Define Ψ this way, check that $(\star)$ commutes, and moreover $(\star)$ commuting forces us to choose this Ψ, so Ψ is unique. $\blacksquare$

What if C is a ccc?

Then
$$\begin{aligned} \hom(X,Y) &\cong \hom(1\times X, Y) \\ &\cong \hom(1, Y^X) \\ &= \mathrm{elt}(Y^X) \end{aligned}$$

since $1\times X \cong X$ so:

$$\begin{tikzcd} 1\times X \arrow[r, "\alpha"] \arrow[dr, "f\circ\alpha"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha^{-1}"] \arrow[dr, "g\circ\alpha^{-1}"'] & 1\times X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

gives us a bijection

$$\begin{aligned} \hom(X,Y) &\cong \hom(1\times X, Y) \\ f &\longmapsto f\circ\alpha \\ g\circ\alpha^{-1} &\longleftarrow\!\shortmid\ g \end{aligned}$$

The moral: we can convert the hom-object $Y^X \in C$ into the hom-set $\hom(X,Y) \in \mathrm{Set}$ by taking elements.

Given $f: X \to Y$ in $\hom(X,Y)$, we can convert it into an element of Y^X, called the <u>name</u> of f: $\ulcorner f \urcorner : 1 \to Y^X$

Conversely, any element of Y^X is the name of a unique morphism $f: X \to Y$.

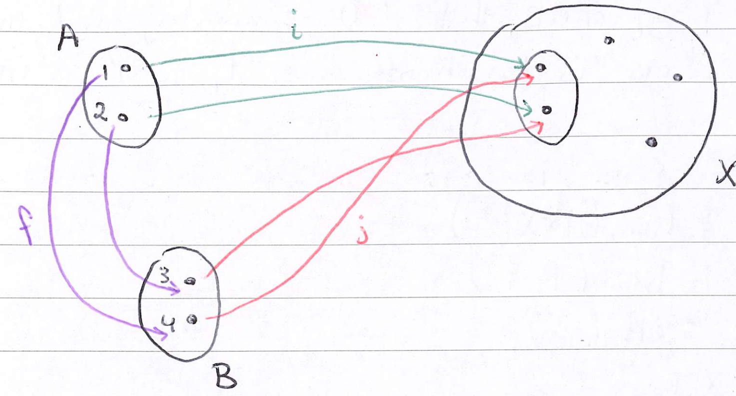
In functional programming, objects are data types, morphisms are programs, & any program $f: X \to Y$ has a "name" $\ulcorner f \urcorner \in \mathrm{elt}(Y^X)$.

<u>Subobjects</u>

$\boxed{\text{Def}}$ In a category C, a <u>subobject</u> of an object $X \in C$ is an equivalence class of monomorphisms $i: A \to X$ where monos $i: A \to X$, $j: B \to X$ are equivalent if there's an isomorphism $f: A \to B$ s.t. this commutes:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

$\boxed{\text{Ex}}$ If $C = \text{Set}$, subobjects of $X \in \text{Set}$ correspond to subsets of X.



Given a mono $i: A \to X$ we get a subset $\text{im}(i) \subseteq X$. Any subset $S \subseteq X$ arises in this way via the inclusion:

$$i: S \longrightarrow X$$
$$s \longmapsto s \in X$$

This has $\text{Im}(i) = S$.

Finally, given monos $i: A \to X$ & $j: B \to X$ that define the same subset: $\text{im}(i) = \text{im}(j)$

then there exists a bijection $f: A \to B$ s.t.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

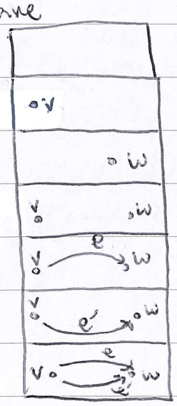
commutes

namely $f = (j|_{\text{im} j})^{-1} \circ i$.

$\boxed{\text{Ex}}$ In Graph, how many subobjects does this graph:

$$\begin{tikzcd} v \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] & w \end{tikzcd}$$

Here they are



the initial graph

graphs w/o edges

any object gives a subobject of itself.

$1_X: X \to X$ is a mono.

(A graph is a pair of fns $E \underset{t}{\overset{s}{\rightrightarrows}} V$.)

Prop In Set, subobjects of $S \in Set$ are in 1-1 correspondence with functions $\chi : S \longrightarrow 2$ where $2 = \{F, T\}$.

Pf:

Subobjects of S are just subsets $A \subseteq S$.

Any such subset has a <u>characteristic function</u> $\chi : S \longrightarrow 2$ given by

$$\chi(s) = \begin{cases} F & s \notin A \\ T & s \in A \end{cases}$$

Conversely, given $\chi : S \longrightarrow 2$, let

$$\begin{aligned} A &= \chi^{-1}(T) \\ &= \{ s \in S : \chi(s) = T \}. \end{aligned}$$

∎

Roughly, a "subobject classifier" in a category C is an object $\Omega \in C$ that plays the role of $2 = \{F, T\}$, in that subobjects of any object $S \in C$ are going to be in 1-1 correspondence with morphisms $\chi : S \longrightarrow \Omega$.

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Set has the "subobject classifier" $2 = \{F, T\}$.
What does this really mean?
First, there's a function called true:

$$t : 1 \longrightarrow 2$$

from $1 = \{*\}$ to 2 given by $t(*) = T \in 2$

For any set A there's a unique function

$$!_A : A \longrightarrow 1$$

since 1 is terminal.

I claim that for any monomorphism $i : A \to X$ (that is a 1-1 function), there exists a unique function

$$\chi_i : X \longrightarrow 2$$

called the <u>characteristic function</u> of i, such that:

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "i"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi_i"] & 2 \end{tikzcd}$$

is a pullback.
χ_i, in more familiar terms, will be the characteristic function of the subset $\mathrm{im}(i) \subseteq X$, but we call it the characteristic function of the mono, i.
First let's show that this χ_i:

$$\chi_i(x) = \begin{cases} T & x \in \mathrm{im}\, i \\ F & x \notin \mathrm{im}\, i \end{cases}$$

Let Q be a competitor

$$\begin{tikzcd} Q \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "!_Q"] \arrow[ddr, bend right, "f"'] & & \\ & A \arrow[r, "!_A"] \arrow[d, "i"] & 1 \arrow[d, "t"] \\ & X \arrow[r, "\chi_i"] & 2 \end{tikzcd}$$

Then show $\exists!\ \psi : Q \longrightarrow A$ making the newly formed triangles commute.
Since Q is a competitor:

$$\chi_i(f(q)) = t(!_Q(q)) \qquad q \in Q$$

(continued)

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$$\begin{aligned}\chi_i(f(q)) &= t(!_Q(q)) \qquad q \in Q \\ &= t(*) \\ &= T\end{aligned}$$

$\Rightarrow$ (using the def. of χ_i) $f(q) \in \text{im } i$

So since i is 1-1, for each $q \in Q$, $\exists !\ a \in A$ with $f(q) = i(a)$

So define $\Psi : Q \longrightarrow A$ by $\Psi(q) = a$.

This makes $f = i \circ \Psi$ and it's the unique $\Psi : Q \longrightarrow A$ that does so (since i is 1-1).

The other newly formed triangle automatically commutes:

$$\begin{tikzcd} Q \arrow[d, "\Psi"'] \arrow[dr, bend left, "!_Q"] & \\ A \arrow[r, "!_A"'] & 1 \end{tikzcd}$$

You can also check that $\chi_i : X \longrightarrow 2$ is the unique morphism from X to 2 that makes the square a pullback.

So generalizing:

Def Given a category C with a terminal object, a <u>subobject classifier</u> is an object $\Omega \in C$ with a morphism $t : 1 \longrightarrow \Omega$ such that: for any mono $i : A \longrightarrow X$ there exists a unique $\chi_i : X \longrightarrow \Omega$ such that this square is a pullback:

$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "i"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi_i"'] & \Omega \end{tikzcd}$$

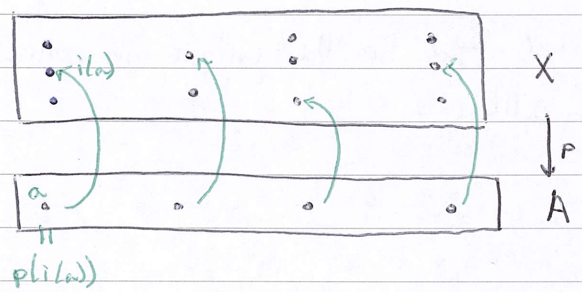
Def A <u>(elementary) topos</u> is a cartesian closed category with finite limits (limits of finite sized diagrams) and a subobject classifier.

Grothendieck in the 1960's introduced a concept of topos, now <u>Grothendieck topos</u>, which is a special case of an elementary topos, as part of proving the Weil hypotheses in number theory. Later, in the late 60's & early 70's, Lawvere & Tierney simplified & generalized the concept of topos to define an "elementary topos".

Examples of elementary topoi:

1) Set: category of sets & functions
2) FinSet: category of finite sets & functions
   - this doesn't have all limits, only finite limits.
   So topos theory includes finitist mathematics
3) Set′: category of sets & functions as defined using ZF - Zermelo-Fraenkel axioms without axiom of choice.

The axiom of choice is equivalent to:
for every epimorphism $p: X \to A$ there exists a mono $i: A \to X$ s.t. $p \circ i = 1_A$.

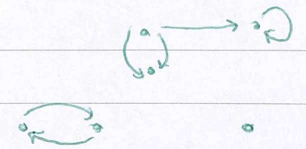


If this is true we say the epimorphism <u>splits</u>.

In a general topos, not every epi splits so the axiom of choice need not hold.

4) Graphs: the category of graphs:

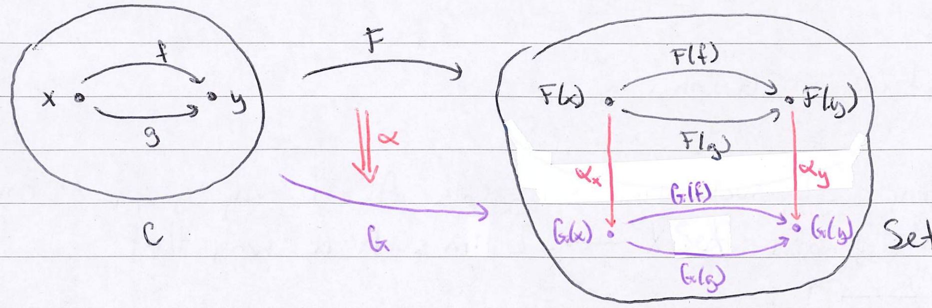
$$\begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \end{tikzcd}$$



5) Previous example is a special case of a category Set^C, where C is any category. These are called <u>presheaf categories</u> when we write them as $Set^{(D^{op})}$ (e.g. $D = C^{op}$ so $D^{op} = C$).
E.g. if

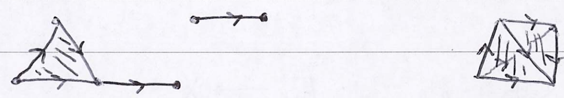
$$C = \begin{tikzcd} x \arrow[loop left, "1_x"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & y \arrow[loop right, "1_y"] \end{tikzcd}$$

then $Set^C \cong Graph$
(continued)



A functor $F: C \longrightarrow Set$ is a graph with $E = F(x)$, $V = F(y)$, $s = F(f)$, $t = F(g)$.
So a graph is an object in Set^C.
Similarly, a morphism in Set^C is a morphism between graphs.

6) Another example of a presheaf category is the category of simplicial sets:



These are fundamental to algebraic topology.

7) Presheaf categories are closely connected to categories of sheaves, which are also topoi.
Sheaves are fundamental to algebraic geometry.