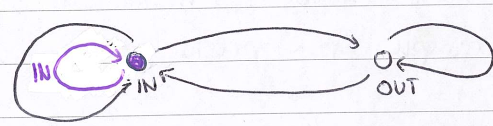
The subobject classifier in Graph

This is some graph Ω such that subgraphs A of any graph X correspond to morphisms of graphs $\chi : X \to \Omega$ in such a way that

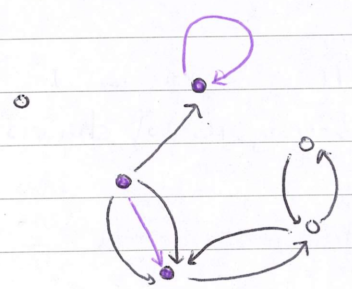
$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d, "i \text{ mono}"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi"] & \Omega \end{tikzcd}$$

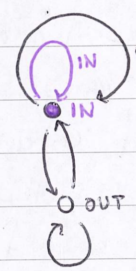
is a pullback.

Ω looks like this:



Here's a graph X with a subgraph $i: A \to X$





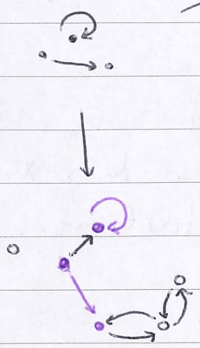
$$X \xrightarrow{\qquad\chi\qquad} \Omega$$

χ sends purple vertices/edges to purple ones.

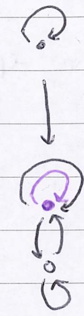
Conversely any morphism of graphs $\chi : X \to \Omega$ determines a subgraph of X, consisting of vertices & edges that are purple in Ω.

The terminal graph, 1, looks like this: 

The purple subgraph of Ω is a copy of 1 (it's isomorphic to 1). We get this from the morphism $t: 1 \to \Omega$ which you have in any topos. A vertex or edge of X will be mapped to this subgraph of Ω iff it's true that the vertex or edge is in A.



$$\begin{tikzcd} A \arrow[r, "!_A"] \arrow[d] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$



i.e., this diagram commutes. Moreover it's a pullback, allowing you to reconstruct A knowing χ.

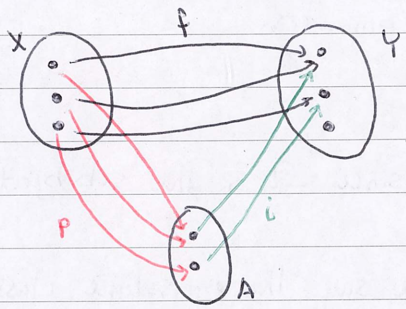
The most important basic properties of topoi:

Prop A topos has finite colimits, meaning it has colimits of finite-sized diagrams.

Prop Any morphism $f: X \to Y$ in a topos has an <u>epi-mono factorization</u>, i.e. there exists an epi $p: X \to A$ & mono $i: A \to Y$ making this triangle commute:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & A \arrow[ur, "i"'] & \end{tikzcd}$$

Ex in Set



every function is a composite $f = i \circ p$ with p onto & i one-to-one.

Prop In a topos, the epi-mono factorization of any morphism $f: X \to Y$ is unique up to a unique isomorphism:
given two epi-mono factorization:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"] \arrow[ddr, bend right, "p'"'] & & Y \\ & A \arrow[ur, "i"'] \arrow[d, dashed] & \\ & A' \arrow[uur, bend right, "i'"'] & \end{tikzcd}$$

there exists a unique isomorphism $g: A \to A'$ making the resulting diagram commute.

Ex In Set, we have an epi-mono factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & \mathrm{im} f \arrow[ur, "i"'] & \end{tikzcd}$$

where $\mathrm{im} f = \{y \in Y : y = f(x) \text{ for some } x \in X\}$
$i : \mathrm{im} f \to Y$ is the inclusion & $p : X \to \mathrm{im} f$
is the obvious function $p(x) = f(x) \in \mathrm{im} f$.

So:

Def Given an epi-mono factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & A \arrow[ur, "i"'] & \end{tikzcd}$$

we call A "the" image of f (it's unique up to isomorphism) & denote it as $\mathrm{im} f$.

Generalize $\subseteq$, $\cap$, $\cup$, to any topos

Henceforth suppose C is a topos.

Def Given $X \in C$, define $\mathrm{Sub}(X)$ to be the set of all subobjects of X: equivalence classes of monos $i : A \to X$, where $i : A \to X$ and $j : B \to X$ are equivalent if there exists an iso $g : A \to B$ s.t.

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "g"', "\wr"] & & X \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

commutes.

Note $\mathrm{Sub}(X) \cong \hom(X, \Omega)$ since Ω is the subobject classifier.

Prop $\mathrm{Sub}(X)$ is a poset where we say the equivalence class of $i : A \to X$ is contained in (or $\subseteq$) the equivalence class of $j : B \to X$ if there exists $f : A \to B$ making this commute:

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "f"'] & & X \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

(Note: f must be a mono, and it's unique.)
Let's say $[i] \subseteq [j]$ in this case.

(Pf on next page)

Pf:

Need to check:

1) $[i] \subseteq [j]$, $[j] \subseteq [k] \Rightarrow [i] \subseteq [k]$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"] \arrow[r, "g"'] & C \arrow[u, "k"'] \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g\circ f"'] & X \\ C \arrow[ur, "k"'] \end{tikzcd}$$

2) $[i] \subseteq [i]$ - easy

3) $[i] \subseteq [j]$ and $[j] \subseteq [i] \Rightarrow [i] = [j]$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i"] & X \\ B \arrow[u, "g"] \arrow[ur, "j"'] \end{tikzcd} \quad \text{commute.}$$

To show $[i] = [j]$, it suffices to show:

g is the inverse of f (so f is an isomorphism).

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "i"] & \\ B \arrow[r, "j"] \arrow[d, "g"'] & X \\ A \arrow[ur, "i"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "g"'] \arrow[dr, "j"] & \\ A \arrow[r, "i"] \arrow[d, "f"'] & X \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

commute, so $i \circ g \circ f = i \circ 1_A$ and $j \circ f \circ g = j \circ 1_B$

& since i & j are monic, they're left cancellable:

$g \circ f = 1_A$ & $f \circ g = 1_B$ ∎

Next time we'll define $\cup$ for subobjects, & this makes $Sub(X)$, which is a poset hence a category, into a category with coproducts: $\cup$ is the coproduct in $Sub(X)$.

Similarly $\cap$ is the product in the category $Sub(X)$.

3/10/16

Set Theory, Topos, & Logic

In Set, every subset of $X \in \mathbf{Set}$ corresponds to a predicate on elements of X:

$$\chi : X \longrightarrow \{T, F\}$$

i.e. a characteristic function.

χ determines a subset $A \subseteq X$ via:

$$A = \{x \in X : \chi(x) = T\}$$

& conversely any subset $A \subseteq X$ determines $\chi : X \to \{T,F\}$ via

$$\chi(x) = \begin{cases} T & x \in A \\ F & x \notin A \end{cases}$$

In a topos we get a similar bijection between $\mathrm{Sub}(X)$ & $\hom(X, \Omega)$.

The concepts of $\cup$ & $\cap$ for subsets correspond to the operations of $\vee$ (or) & $\wedge$ (and) on predicates.

$$\{x \in X : \chi(x) = T\} \cup \{x \in X : \varphi(x) = T\} = \{x \in X : (\chi \vee \varphi)(x) = T\}$$

& similarly for $\cap$ & $\wedge$.

$\boxed{\text{Prop}}$ In Set, $\mathrm{Sub}(X)$ for $X \in \mathbf{Set}$ is a poset via $\subseteq$, and thus a category where there exists a unique morphism from A to B iff $A \subseteq B$ $(A, B \subseteq X)$.

In this category $A \cap B$ is the product of A and B, and $A \cup B$ is the coproduct.

Sketch of Pf:

We have

$$\begin{tikzcd} & A \cap B \arrow[dl, "\subseteq"'] \arrow[dr, "\subseteq"] & \\ A & & B \end{tikzcd}$$

& this cone is universal:

$$\begin{tikzcd} & Q \arrow[ddl, bend right, "\subseteq"'] \arrow[ddr, bend left, "\subseteq"] \arrow[d, dashed, "\exists! \varphi"] & \\ & A \cap B \arrow[dl, "\subseteq"'] \arrow[dr, "\subseteq"] & \\ A & & B \end{tikzcd}$$

which is true since $Q \subseteq A$, $Q \subseteq B \Rightarrow Q \subseteq A \cap B$.

The proof for $\cup$ is the same but with all $\subseteq$'s turned around. $\blacksquare$

In fact, in Set, Sub(X) has all finite limits and all finite colimits!

A category has all finite limits iff it has:

- binary products
- terminal object
- equalizers

($\Rightarrow$ all finite products exist (Cartesian), said of binary products and terminal object)

Sub(X) has binary products ($\cap$), terminal object (X, since $A \subseteq X$ for all $A \in \mathrm{Sub}(X)$), and equalizers:

$$B \overset{f}{\underset{g}{\rightrightarrows}} C \quad \text{in any poset is really} \quad B \overset{f}{\underset{g}{\rightrightarrows}} C$$

& the equalizer is:

$$B \xrightarrow{1} B \overset{f}{\underset{f}{\rightrightarrows}} C$$

so equalizers exist in any poset.

Similarly, in Set, Sub(X) has all finite colimits because it has:

- binary coproducts
- initial object
- coequalizers

The binary product of A & B is $A \cup B$, the initial object is $\emptyset$ (since $\emptyset \subseteq A$ for all $A \in \mathrm{Sub}(X)$), and coequalizers (which exist in any poset: just turn arrows around in argument for equalizers).

Def A lattice is a poset with all finite limits & colimits.
(This is equivalent to other more popular definitions, though some evil people don't demand the initial and terminal object.)

In fact we have:

	SET THEORY	LOGIC	CATEGORY THEORY
FINITE LIMITS	$\cap$	$\wedge$	binary product
	X (the whole set)	$\top$	terminal object
FINITE COLIMITS	$\cup$	$\vee$	binary coproduct
	$\emptyset$	F	initial object
	$\subseteq$	$\Rightarrow$	$\longrightarrow$
CARTESIAN CLOSEDNESS	$B \cup A^c$	$Q \vee \neg P$ or "P implies Q"	exponentiation

$\boxed{\text{Note}}$ $X = \{x \in X : T = T\}$

$\emptyset = \{x \in X : F = T\}$

In fact the poset $\mathrm{Sub}(X)$ is cartesian closed.

In general this means

$$\hom(B \times C, D) \cong \hom(B, D^C)$$

but for $\mathrm{Sub}(X)$, being a poset, these sets either have 0 elements or 1 element. Also, the product is the intersection.

So this says

$$B \cap C \subseteq D \qquad \text{iff} \qquad B \subseteq D \cup C^c$$

or in terms of logic

$$P \wedge Q \Rightarrow R \qquad \text{iff} \qquad P \Rightarrow \underbrace{R \vee \neg Q}_{\text{or "Q implies R"}}$$

$\boxed{\text{Thm}}$ In any topos, for any object X the poset $\mathrm{Sub}(X)$ is a Heyting algebra: it's a poset that has finite limits, finite colimits, & is Cartesian closed. I.e. it's a Cartesian closed lattice.

Sketch of sketch of pf:

Given two subobjects of X, $[i]$, $[j]$, we want to form $[i] \wedge [j]$ and $[i] \vee [j]$.

Taking the pullback gives is the intersection

$$\begin{tikzcd} A \cap B \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, "i \circ f = j \circ g"] & B \arrow[d, "j"] \\ A \arrow[r, "i"'] & X \end{tikzcd}$$

Since this is a pullback & i, j are monic $\Rightarrow f, g$ are monic.

$\Rightarrow i \circ f = j \circ g$ is also a monic, so we get a new subobject of X, which is $[i] \wedge [j]$.

For unions, we start with the coproduct

$$\begin{tikzcd} A + B \arrow[dr, "\exists ! \psi"] & B \arrow[l, "g"'] \arrow[d, "j"] \\ A \arrow[u, "f"] \arrow[r, "i"'] & X \end{tikzcd}$$

where we get ψ from the universal property of the coproduct.

But ψ need not be monic, so do the epi-mono factorization:

(continued)

(continued)

$$\begin{tikzcd} A+B \arrow[rr, "\psi"] \arrow[dr, "p"'] & & X \\ & \operatorname{im}\psi \arrow[ur, "k"'] & \end{tikzcd}$$

where p is epic and k is monic.

k gives a new subobject of X, which is $[i] \cup [j]$.

Where does topos theory go from here?

Many directions ... e.g.:

- Using the "Mitchell-Benabou language", we can reason inside any topos: we can write things like:

  $$\{x \in A \cap B : \forall y \in Y \ \exists z \in Z \ f(x,z) = y\}$$

  & prove things about them, using the logic internal to the topos, & "generalized elements".

- There are also maps between topoi:

  $$C \rightleftarrows D$$

  consisting of certain nice adjunctions.

  These maps are called "geometric morphisms".

  There's a topos called $\mathrm{Th}(\mathrm{Grp})$ — "the theory of a group", and then a geometric morphism from some other topos C to $\mathrm{Th}(\mathrm{Grp})$ is the same as a group object in C.

  This idea works for lots of concepts, not just the concept of a group.

MIQUELRIUS