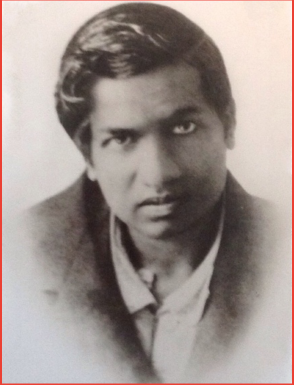
RAMANUJAN'S EASIEST FORMULA



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π is a bit bigger than 3; e is a bit less. You should be curious about their average.

Is there anything interesting about their geometric mean?

$$\sqrt{\pi e} = 2.92228236532\ldots$$

In 1914, Ramanujan posed this puzzle in *The Journal of the Indian Mathematical Society*:

Prove that

$$\left(\frac{1}{1}+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\cdots\right)+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{\ddots}}}}}}=\sqrt{\frac{\pi e}{2}}$$

This is an infinite series:

$$\frac{1}{1} + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 3 \cdot 5} + \cdots$$

This is a continued fraction:

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{3}{1 + \cfrac{4}{1 + \cfrac{5}{\ddots}}}}}}$$

Infinite series are easier, so let's start with that!

If we knew which function had this Taylor series:

$$f(x) = \frac{x}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots$$

then we'd know

$$f(1) = \frac{1}{1} + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 3 \cdot 5} + \cdots$$

We could try to *guess* $f(x)$, but let's figure out a differential equation it satisfies, and solve that.

If we differentiate

$$f(x) = \frac{x}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots$$

we get

$$f'(x) = 1 + \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} + \cdots$$

This looks a lot like

$$xf(x) = \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} + \cdots$$

Indeed, we have

$$f'(x) = xf(x) + 1$$

So, let's solve $f'(x) = xf(x) + 1$ or

$$f'(x) - xf(x) = 1$$

This is a first-order linear ODE. The trick is to use an "integrating factor". Multiply both sides by $e^{-\int x\,dx} = e^{-x^2/2}$:

$$e^{-x^2/2} f'(x) - xe^{-x^2/2} f(x) = e^{-x^2/2}$$

$$\frac{d}{dx}\left(e^{-x^2/2} f(x)\right) = e^{-x^2/2}$$

$$e^{-x^2/2} f(x) = \int e^{-x^2/2}\,dx$$

$$f(x) = e^{x^2/2} \int e^{-x^2/2}\,dx$$

So we're getting

$$\frac{x}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots = e^{x^2/2} \int e^{-x^2/2}\, dx$$

But there's a constant of integration! Really we have

$$\frac{x}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots = e^{x^2/2} \left(\int_0^x e^{-x^2/2}\, dx + C \right)$$

But the left side is zero when $x = 0$, so $C = 0$:

$$\frac{x}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots = e^{x^2/2} \int_0^x e^{-t^2/2}\, dt$$

Remember, we really care about this when $x = 1$:

$$\mathbf{1 + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 3 \cdot 5} + \cdots = \sqrt{e} \int_0^1 e^{-t^2/2}\, dt}$$

We wanted to show

$$\left(\frac{1}{1}+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\cdots\right)+\frac{1}{1+\frac{1}{1+\frac{2}{1+\frac{3}{1+\frac{4}{1+\frac{5}{\ddots}}}}}}=\sqrt{\frac{\pi e}{2}}$$

Now we know

$$1+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\cdots=\sqrt{e}\int_0^1 e^{-x^2/2}\,dx$$

so we “just” need to show

$$\frac{1}{1+\frac{1}{1+\frac{2}{1+\frac{3}{1+\frac{4}{1+\frac{5}{\ddots}}}}}}=\sqrt{\frac{\pi e}{2}}-\sqrt{e}\int_0^1 e^{-t^2/2}\,dt$$

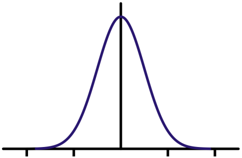
This still looks hard:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{\ddots}}}}}} = \sqrt{\frac{\pi e}{2}} - \sqrt{e}\int_0^1 e^{-t^2/2}\,dt$$

This famous integral is a clue:

$$\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$$

It's the area under a Gaussian:



Take *half* the area under the Gaussian:

$$\int_0^\infty e^{-t^2/2}\,dt = \sqrt{\frac{\pi}{2}}$$

This is starting to look a bit like what we want to prove:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{\ddots}}}}}} = \sqrt{\frac{\pi e}{2}} - \sqrt{e}\int_0^1 e^{-t^2/2}\,dt$$

so multiply it by $\sqrt{e}$:

$$\sqrt{e}\int_0^\infty e^{-t^2/2}\,dt = \sqrt{\frac{\pi e}{2}}$$

and plug this into the equation we want to prove!

We get

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{\ddots}}}}}} = \sqrt{e}\int_0^\infty e^{-t^2/2}\,dt - \sqrt{e}\int_0^1 e^{-t^2/2}\,dt$$

or just:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{\ddots}}}}}} = \sqrt{e}\int_1^\infty e^{-t^2/2}\,dt$$

If we can prove this, we're done! This is Ramanujan's "puzzle within a puzzle".

In the first part of his puzzle, he was secretly asking us to show

$$f(x) = \frac{x}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots$$

obeys the differential equation $f'(x) = xf(x) + 1$,and then solve this getting

$$\boldsymbol{\frac{x}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots = e^{x^2/2} \int_0^x e^{-t^2/2}\, dt}$$

In his “puzzle within a puzzle”, he’s secretly asking us to show

$$g(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}}$$

obeys the differential equation $g'(x) = xg(x) - 1$, and then solve this getting

$$\cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}} = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt$$

This part is harder.

Adding them together we get

$$\left(\frac{x}{1}+\frac{x^3}{1\cdot 3}+\frac{x^5}{1\cdot 3\cdot 5}+\cdots\right)+\cfrac{1}{x+\cfrac{1}{x+\cfrac{2}{x+\cfrac{3}{x+\cfrac{4}{\ddots}}}}}=$$

$$e^{x^2/2}\int_0^x e^{-x^2/2}\,dx+e^{x^2/2}\int_x^\infty e^{-t^2/2}\,dt=$$

$$e^{x^2/2}\int_0^\infty e^{-t^2/2}\,dt=$$

$$\sqrt{\frac{\pi}{2}}e^{x^2/2}$$

and setting $x=1$ we're done:

$$\left(\frac{1}{1}+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\cdots\right)+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{\ddots}}}}}}=\sqrt{\frac{\pi e}{2}}$$

In 1913, Ramanujan wrote a letter to Hardy introducing himself. It contained many formulas he'd proved. His formula (1.8) was equivalent to this:

$$\cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}} = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt$$

Later Hardy wrote:

> *The first question was whether I could recognise anything. I had proved things rather like (1.7) myself, and seemed vaguely familiar with (1.8). Actually (1.8) is classical; it is a formula of Laplace first proved properly by Jacobi. [....] On the whole the integral formulae seemed the least impressive.*

Laplace's derivation of the formula

$$\cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}} = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt$$

made no sense to me.

Jacobi wrote a 2-page paper about it in 1834. Even though it was in Latin and *full of errors*, it was pretty easy to understand, so I blogged about it.

On Twitter my friend Leo Stein found a simpler proof, which you can see here:

- John Baez, Chasing the Tail of the Gaussian (Part 2), *The n-Category Café*.

Let's write:

$$g(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt$$

as a continued fraction. If we differentiate g we can see that

$$g'(x) = xg(x) - 1$$

But *keep on* differentiating g, and see what happens:

$$\begin{aligned} g''(x) &= xg'(x) + g(x) \\ g'''(x) &= xg''(x) + 2g'(x) \\ g^{(4)}(x) &= xg'''(x) + 3g''(x) \end{aligned}$$

We notice a pattern which we can prove inductively:

$$g^{(n+2)} = xg^{(n+1)} + (n+1)g^{(n)} \qquad \text{for } n \geq 0$$

To get a continued fraction, let's take our formula

$$g^{(n+2)} = xg^{(n+1)} + (n+1)g^{(n)} \qquad \text{for } n \geq 0$$

and divide it by $g^{(n+1)}$:

$$\frac{g^{(n+2)}}{g^{(n+1)}} = x + (n+1)\frac{g^{(n)}}{g^{(n+1)}} \qquad \text{for } n \geq 0$$

This looks simpler in terms of the ratios $r_n = g^{(n+1)}/g^{(n)}$:

$$r_{n+1} = x + \frac{(n+1)}{r_n}$$

or solving for r_n:

$$r_n = \frac{n+1}{-x + r_{n+1}} \qquad \text{for } n \geq 0$$

♠

$$r_n = \frac{n+1}{-x + r_{n+1}} \qquad \text{for } n \geq 0 \qquad\qquad \spadesuit$$

In terms of $r_0 = g'/g$, our original equation $g' = xg - 1$ gets a similar look:

$$g = \frac{1}{x - r_0}$$

Starting from here, and repeatedly using ♠, we get

$$g = \frac{1}{x - r_0} = \frac{1}{x - \frac{1}{-x+r_1}} = \frac{1}{x - \frac{1}{-x+\frac{2}{-x+r_2}}} = \frac{1}{x - \frac{1}{-x+\frac{2}{-x+\frac{3}{-x+r_3}}}} = \cdots$$

and so on.

If we go on forever, we get

$$g = \cfrac{1}{x - \cfrac{1}{-x + \cfrac{2}{-x + \cfrac{3}{-x + \cfrac{4}{\ddots}}}}}$$

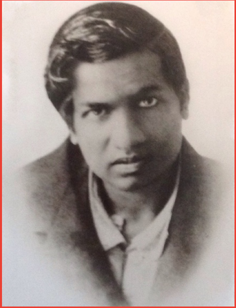
A bit of algebra gives

$$g = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}}$$

or in other words:

$$e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}}$$

So we're done!



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