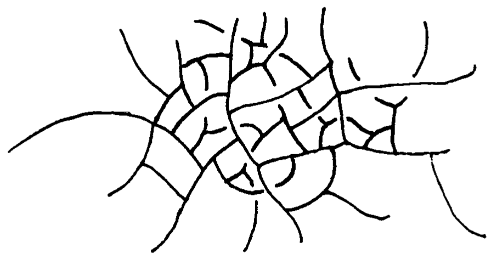
SPIN FOAM MODELS

We've seen a bit of how <u>spin</u> <u>networks</u> can describe the quantum geometry of <u>space</u>:



edges contribute area

vertices contribute volume

How can we understand the quantum geometry of <u>spacetime</u>? The problem of time makes this tough in the canonical (Hamiltonian) approach. It's natural to try the path-integral (Lagrangian) approach.

This leads to <u>spin</u> <u>foams</u>.

It's easiest to define "closed" spin networks & spin foams. Given a group G,

A closed spin network $\Psi = (\gamma, \rho, \iota)$ is a graph γ with:

1) edges e labelled by unitary irreps ρ_e of G.

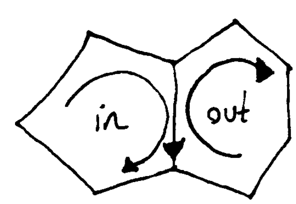
2) vertices v labelled by intertwiners

$$\iota_v : \underbrace{\rho_{e_1} \otimes \cdots \otimes \rho_{e_n}}_{\text{incoming}} \to \underbrace{\rho_{e_1'} \otimes \cdots \otimes \rho_{e_m'}}_{\text{outgoing}}$$

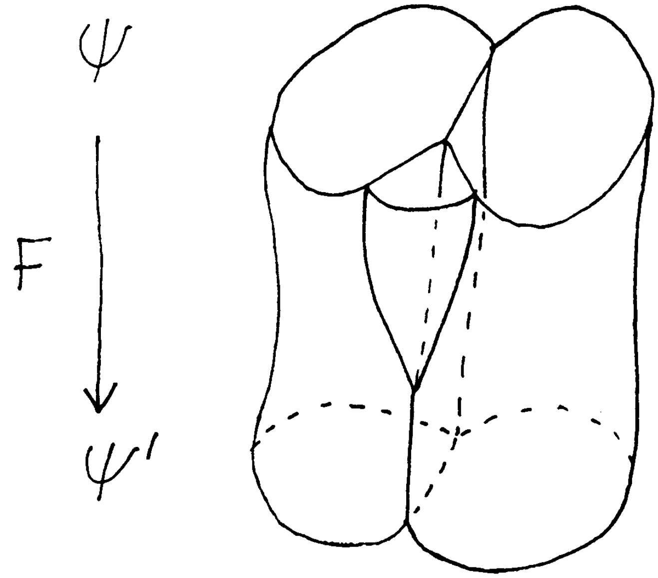
A closed spin foam $F = (\kappa, \rho, \iota)$ is an oriented 2d PLCW complex κ with:

1) faces f labelled by unitary irreps ρ_f of G.

2) edges e labelled by intertwiners

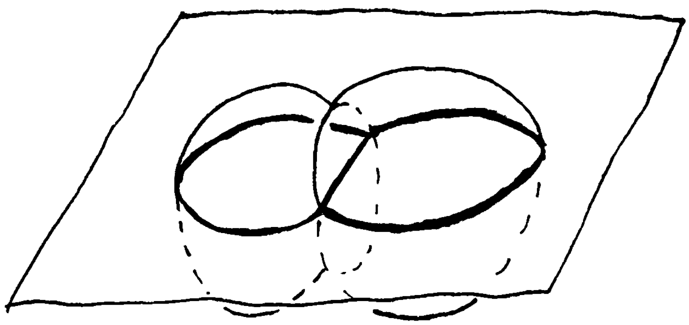


$$\iota_e : \underbrace{\rho_{f_1} \otimes \cdots \otimes \rho_{f_n}}_{\text{incoming}} \to \underbrace{\rho_{f_1'} \otimes \cdots \otimes \rho_{f_m'}}_{\text{outgoing}}$$



A spin foam should be something going from one spin network to another.

Or: a generic "slice" of a spin foam should be a spin network:

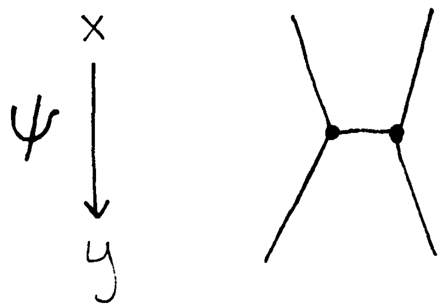


spin foam faces $\longrightarrow$ spin network edges

spin foam edges $\longrightarrow$ spin network vertices

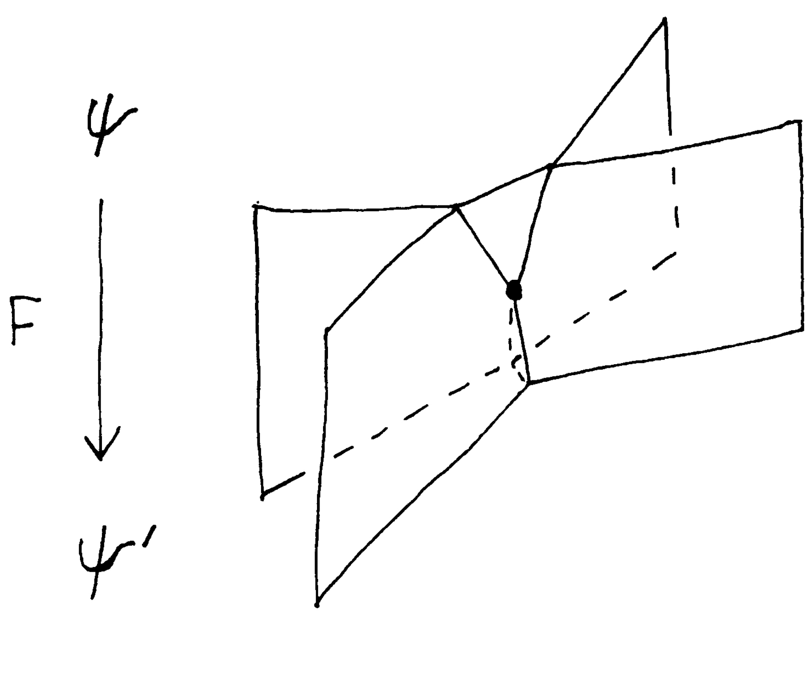
Spin foam vertices correspond to "interactions" where a spin network changes topology.

We can also define "open" spin networks. These are the morphisms in a category:



When G is the Poincaré group + a compact Lie group, these are just Feynman diagrams!

Similarly, "open spin foams" are the 2-morphisms in a certain 2-category:

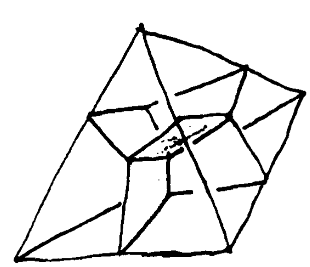


A *spin foam model* assigns an amplitude $Z(F) \in \mathbb{C}$ to each spin foam F, computed as a product of:

- vertex amplitudes] "interactions"
- edge amplitudes] "propagators"
- face amplitudes]

(A categorified analogue of Feynman diagrams.)

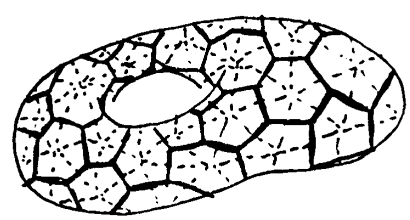
One can study spin foams in a real-analytic manifold, or *not embedded*, but lets consider them in a triangulated manifold, living in the 2-skeleton of the dual complex:



Given a triangulated $(n-1)$-manifold S representing "space", define the kinematical Hilbert space $Z(S)$ by

$$Z(S) = L^2(\mathcal{A}_\gamma / \mathcal{G}_\gamma)$$

where the graph γ is the dual 1-skeleton:



Given a triangulated cobordism $M: S \to S'$ representing "spacetime", let

$$Z(M): Z(S) \to Z(S')$$

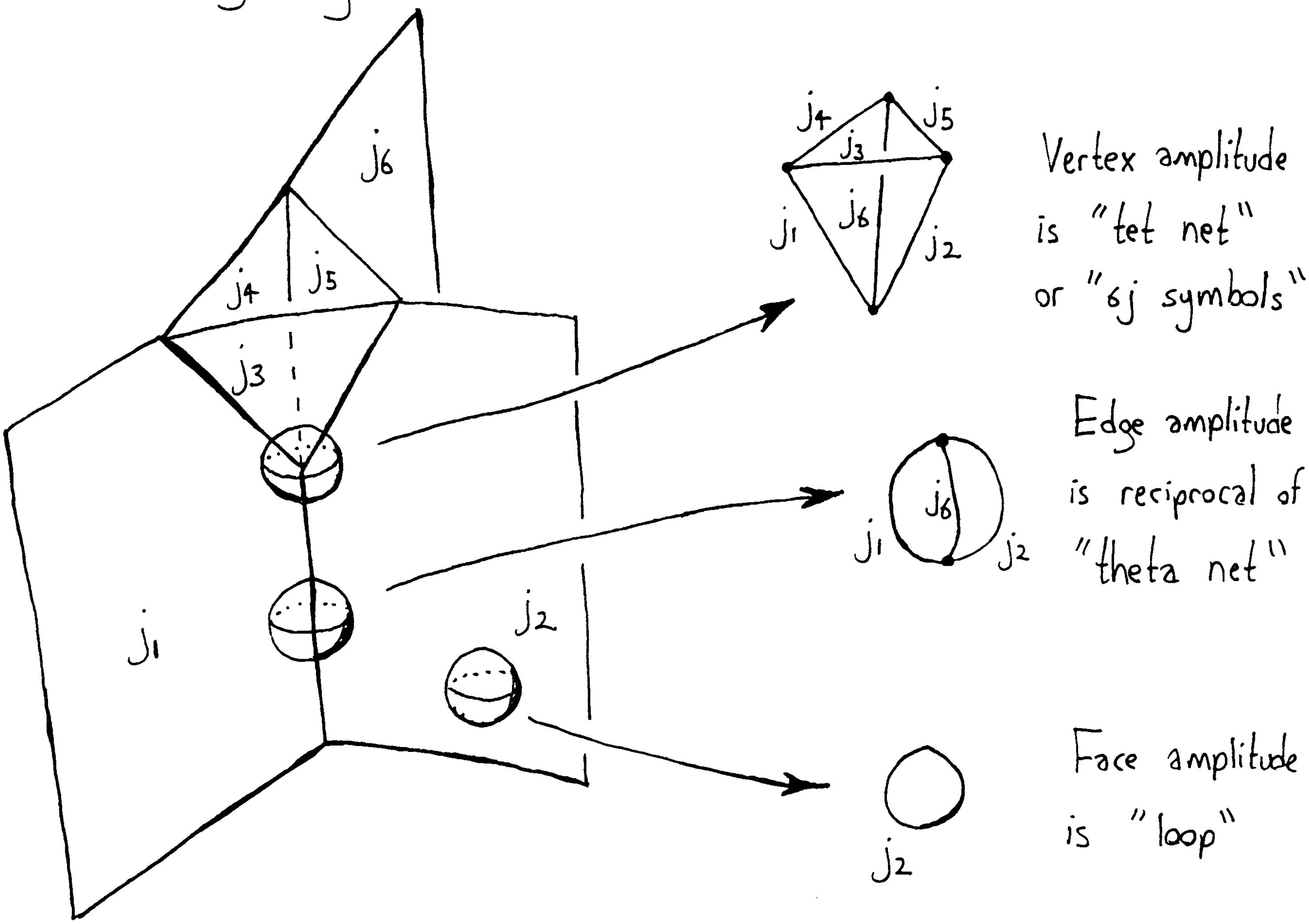
be given by

$$\langle \psi', Z(M)\psi \rangle = \sum_{\substack{F: \psi \to \psi' \\ \text{in dual 2-skeleton of } M}} Z(F)$$

If convergent & triangulation-independent, we can often promote Z to a TQFT.

PONZANO-REGGE MODEL
~1968

This is a theory of <u>Riemannian</u> quantum gravity in 3d spacetime, with $G = SU(2)$.



Problem: sum over spin foams diverges, corresponding to <u>infrared</u> <u>divergence</u> in path integral.

TURAEV-VIRO MODEL
~1992

The divergences in the Ponzano-Regge model were cured by replacing $SU(2)$ by the corresponding <u>quantum group</u>, with q a suitable root of unity. Now there are finitely many "spins" to sum over & we get a TQFT.

This corresponds to introducing a <u>cosmological</u> <u>constant</u> Λ in Riemannian general relativity: $q = e^{\frac{2\pi i}{k+2}}$, $k = \frac{4\pi}{\sqrt{\Lambda}}$.

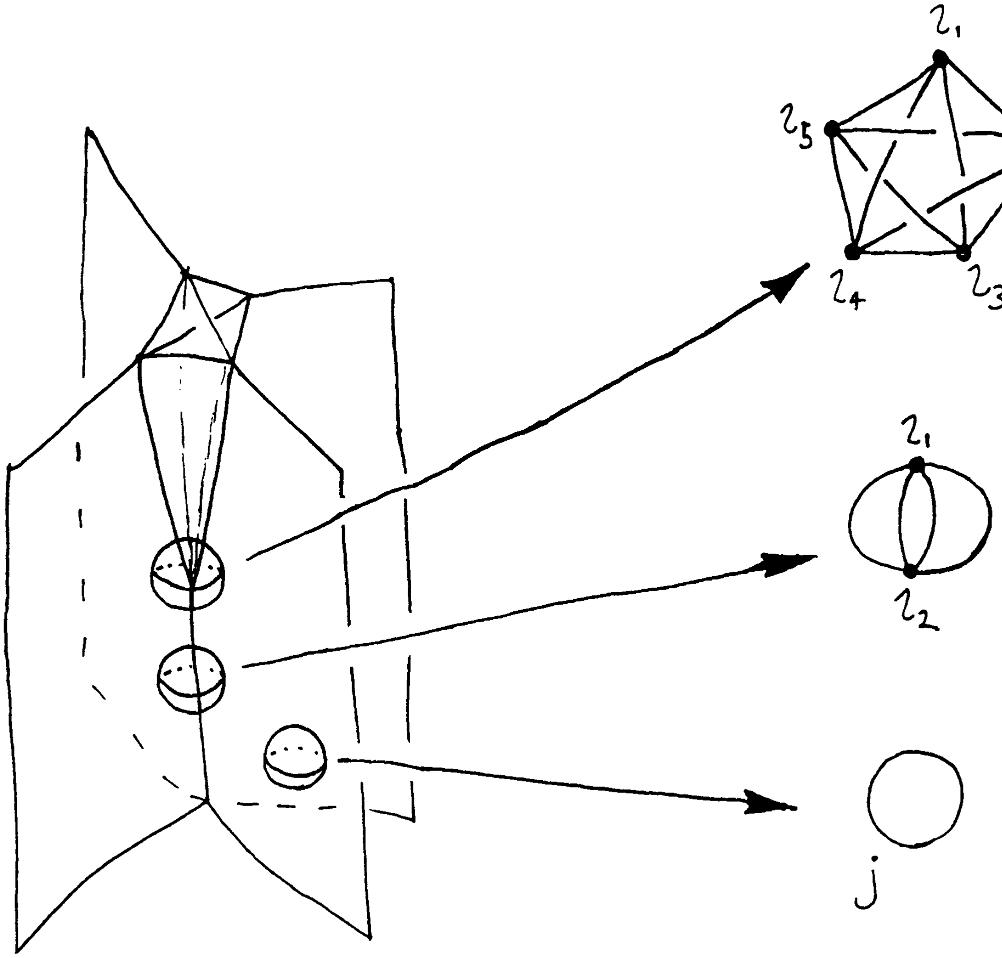
Positive curvature eliminates infrared divergences!

Same idea works for other quantum groups.

OOGURI MODEL

~1992

$G = SU(2)$, but now in 4d spacetime:



Vertex amplitude is "15j symbols"

Edge amplitude is reciprocal of this net

Face amplitude is "loop"

Corresponds to "topological gravity" with

$$S = \int_M tr(E \wedge F)$$

F ← curvature of $SU(2)$ connection

E ← $Ad(P)$-valued 2-form

Again suffers from divergences.

TURAEV / CRANE-YETTER MODEL

~1992

Replacing SU(2) by the corresponding quantum group in the Ooguri model, the sum over spins becomes finite & we get a 4d TQFT.

We <u>believe</u> this is the quantization of the field theory with

$$S = \int_M \mathrm{tr}\left(E \wedge F + \frac{\Lambda}{12} E \wedge E\right),$$

but this needs more work!

We get similar TQFTs from other quantum groups.

RIEMANNIAN BARRETT-CRANE MODEL
~1997

4d Riemannian general relativity can be expressed as a theory with

$$G = \widetilde{SO}(4) \cong SU(2) \times SU(2)$$

and

$$S = \int_M \mathrm{tr}\,(E \wedge F)$$

but with <u>extra constraints</u> saying

$$\boxed{E = e \wedge e}$$

$e : TM \to \mathcal{T} \cong TM$

with metric, so

$e \wedge e \in \Omega^2(M, \Lambda^2 \mathcal{T}) \cong \Omega^2(M, \mathrm{Ad}\,P)$

Imposing a version of this in corresponding Ooguri-like spin foam model, we get a <u>convergent</u> theory w/o needing quantum group technology!

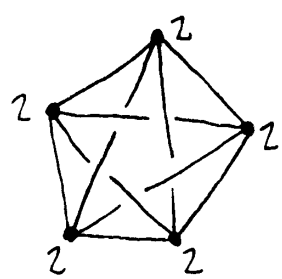
gr-qc/0011058

LORENTZIAN BARRETT-CRANE MODEL 1999

A similar model works in the Lorentzian case, where

$$G = \widetilde{SO}(3,1) \cong SL(2,\mathbb{C})$$

Here the relevant unitary irreps are <u>infinite</u>-<u>dimensional</u> so the vertex amplitude



is not obviously convergent... but it <u>does</u> converge (gr/qc-0101107)... and even better, the integral over labellings converges (gr-qc/0104057)! Stay tuned....