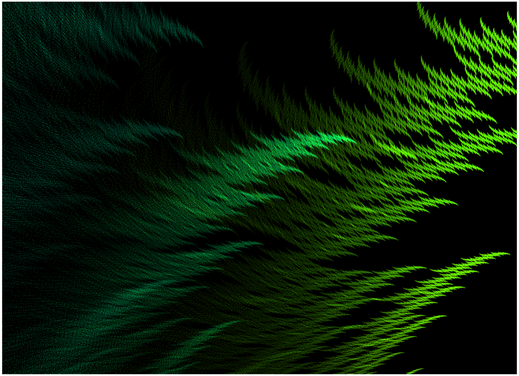
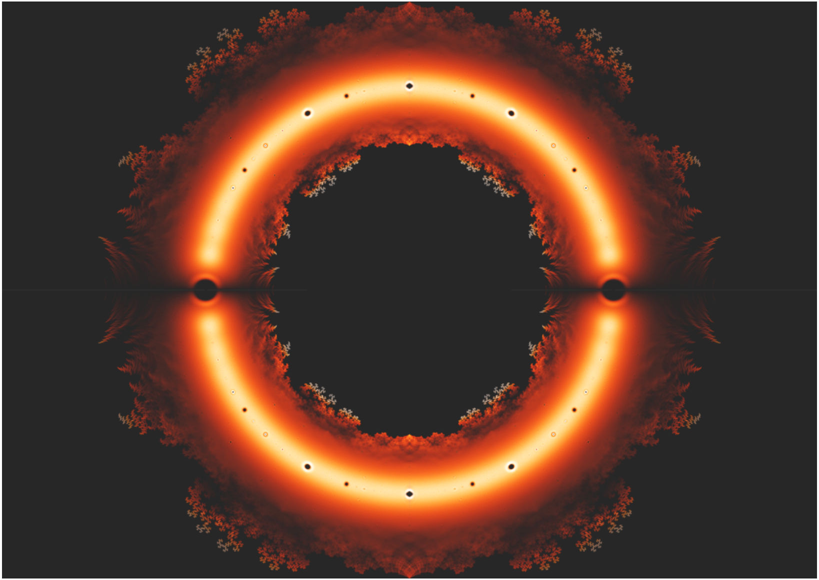
The Beauty of Roots

John Baez, Dan Christensen and Sam Derbyshire
with lots of help from Greg Egan

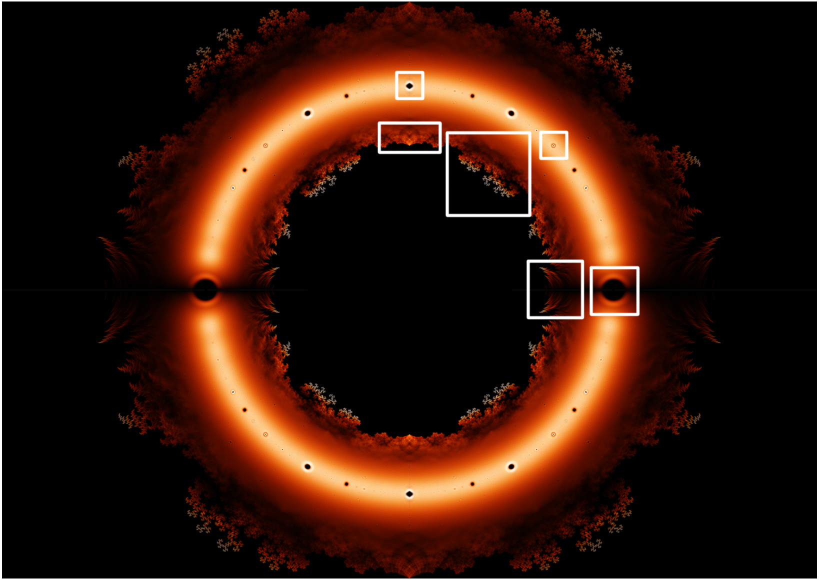


Definition. A **Littlewood polynomial** is a polynomial whose coefficients are all 1 and -1.

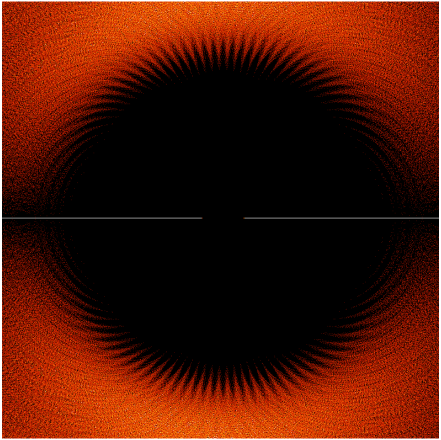
Let's draw all roots of all Littlewood polynomials!



Certain regions seem particularly interesting:

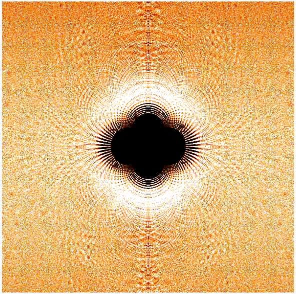


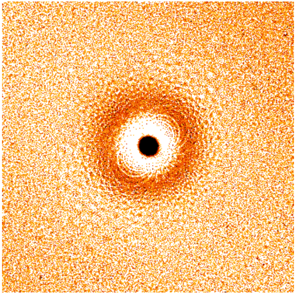
The hole at 1:



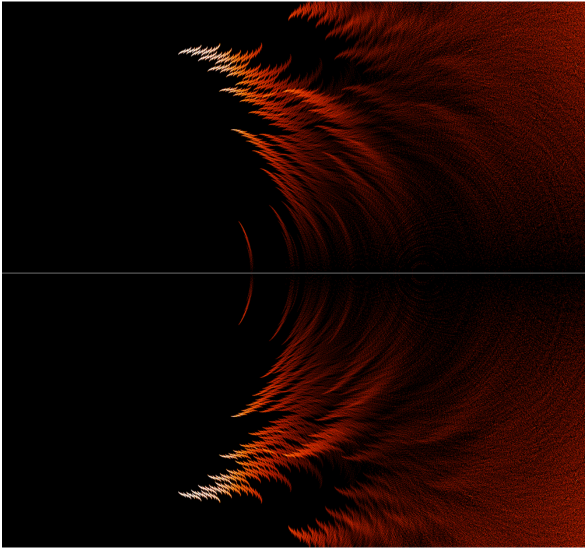
Note the line along the real axis: more Littlewood polynomials have real roots than *nearly* real roots.

The holes at i and $e^{i\pi/4}$:

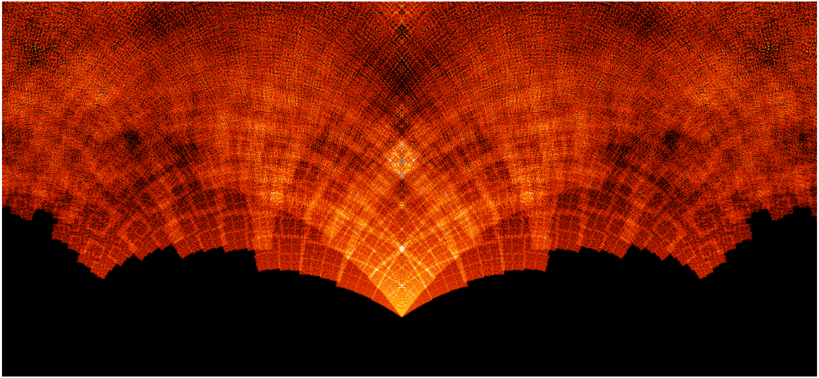




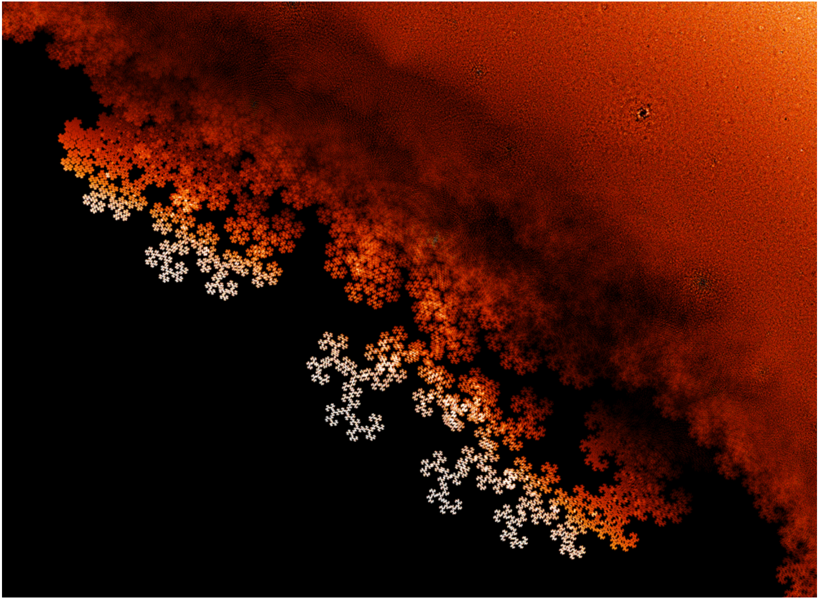
This plot is centered at the point $\frac{4}{5}$:



This is centered at the point $\frac{4}{5}i$:



This is centered at $\frac{1}{2}e^{i/5}$:



Can we understand these pictures? Let

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$$\mathbf{D} = \{z \in \mathbb{C} \ : \ z \text{ is the root of some Littlewood polynomial}\}$$

Theorem 1. $\mathbf{D} \subseteq \{1/2 < |z| < 2\}$

Proof. Suppose z is a root of a Littlewood polynomial. Then

$$1 = \pm z \pm z^2 \pm \cdots \pm z^n$$

If $|z| < 1$ then

$$1 \leq |z| + |z|^2 + \cdots + |z|^n < \frac{|z|}{1 - |z|}$$

so $|z| > 1/2$. Since z is the root of a Littlewood polynomial if and only if z^{-1} is, **D** is contained in the annulus $\frac{1}{2} < |z| < 2$.

Theorem 2. $\{2^{-1/4} \leq |z| \leq 2^{1/4}\} \subseteq \overline{\mathbf{D}}$.

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Proof. This was proved by Thierry Bousch in 1988. We won't prove it here.

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Proof. This was proved by Bousch in 1993. Let's sketch how the proof works. It's enough to show $\overline{\mathbf{D}} \cap \{|z| < r\}$ is connected where r is slightly less than 1.

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Lemma 1. A point $z \in \mathbb{C}$ with $|z| < 1$ lies in $\overline{\mathbf{D}}$ if and only if some Littlewood series vanishes at this point.

A Littlewood polynomial is not a Littlewood series! But any Littlewood polynomial, say

$$P(z) = a_0 + \cdots + a_d z^d$$

gives a Littlewood series having the same roots with $|z| < 1$:

$$\frac{P(z)}{1 - z^{d+1}} = a_0 + \cdots + a_d z^d + a_0 z^{d+1} + \cdots + a_d z^{2d+1} + a_0 z^{2d+2} + \cdots$$

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To do this, let's show that $\mathbf{R}$ is closed and $\mathbf{D}$ is dense in $\mathbf{R}$.

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Lemma 2. Any Littlewood series has finitely many roots in the disc $\{|z| \leq r\}$. The map $\rho : \mathbf{L} \to \mathbf{M}$ sending a Littlewood series to its multiset of roots in this disc is continuous.

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It follows that $\overline{\mathbf{D}} = \mathbf{R}$.

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Suppose $U \subseteq \mathbf{R} \cap \{|z| < r\}$ is closed and open in the relative topology. We want to show U is empty.

Let $\mathbf{L}_U$ be the set of Littlewood series with a root in U. $\mathbf{L}_U$ is a closed and open subset of $\mathbf{L}$. Thus, we can determine whether $f \in \mathbf{L}$ lies in $\mathbf{L}_U$ by looking at its first d terms:

$$f(z) = a_0 + a_1 z + \cdots + a_{d-1} z^{d-1} + a_d z^d + \cdots$$

for some d. Choose the *smallest* d with this property.

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We will get a contradiction if U, and thus $\mathbf{L}_U$, is nonempty! We'll show $d - 1$ has the same property.

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There are two cases, $a_d = 1$ and $a_d = -1$. We'll just do the first, since the second is similar. So:

$$f(z) = 1 + a_1 z + \cdots + a_{d-1} z^{d-1} + z^d + \cdots$$

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has a root in U, and we want to show the same for any $g \in \mathbf{L}$ with the same first $d - 1$ terms, say

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We may assume g differs from f in its dth term:

$$g(z) = 1 + a_1 z + \cdots + a_{d-1} z^{d-1} - z^d + b_{d+1} z^d + \cdots$$

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It suffices to show that $\tilde{g}$ has a root in U:

$$\tilde{g}(z) = \left(1 + a_1 z + \cdots + a_{d-1} z^{d-1}\right) / \left(1 + z^d\right)$$

since this has the same first d terms as g.

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since this has the same first d terms as g. Since f has a root in U, so does

$$\tilde{f}(z) = \left(1 + a_1 z + \cdots + a_{d-1} z^{d-1}\right) / \left(1 - z^d\right)$$

since this has the same first d terms as f.

$$\tilde{g}(z) = \left(1 + a_1 z + \cdots + a_{d-1} z^{d-1}\right) / \left(1 + z^d\right)$$

and

$$\tilde{f}(z) = \left(1 + a_1 z + \cdots + a_{d-1} z^{d-1}\right) / \left(1 - z^d\right)$$

so

$$\tilde{g}(z) = \left(\frac{1 - z^d}{1 + z^d}\right) \tilde{f}(z)$$

Since $\tilde{f}$ has a root in U, so does $\tilde{g}$. QED!

Here is the key to understanding the beautiful patterns in the set $\overline{\mathbf{D}}$. Define two functions from the complex plane to itself, depending on a complex parameter q:

$$\begin{aligned} f_{+q}(z) &= 1 + qz \\ f_{-q}(z) &= 1 - qz \end{aligned}$$

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When $|q| < 1$ these are both contraction mappings, so by Hutchinson's theorem on iterated function systems there's a unique nonempty compact set $D_q \subseteq \mathbb{C}$ with

$$D_q = f_{+q}(D_q) \cup f_{-q}(D_q)$$

We call this set a **dragon.**

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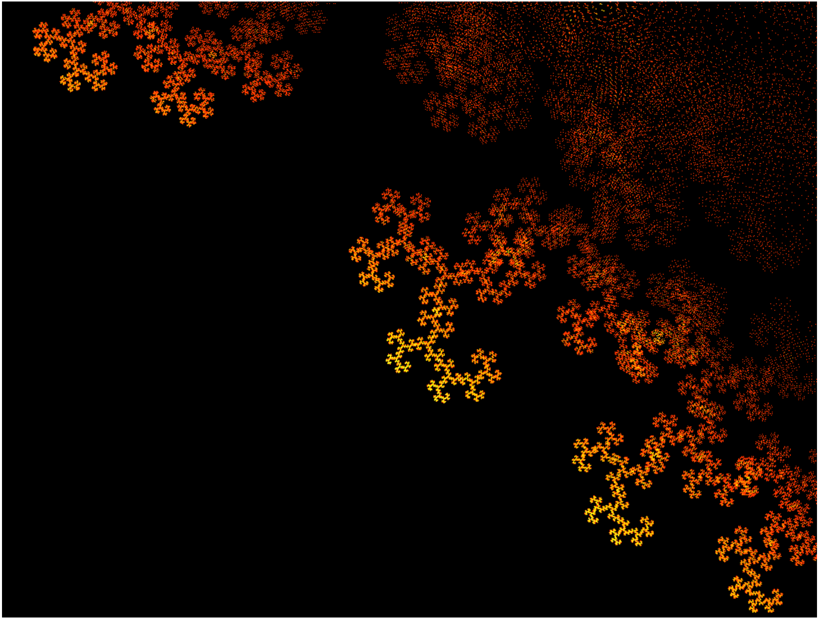
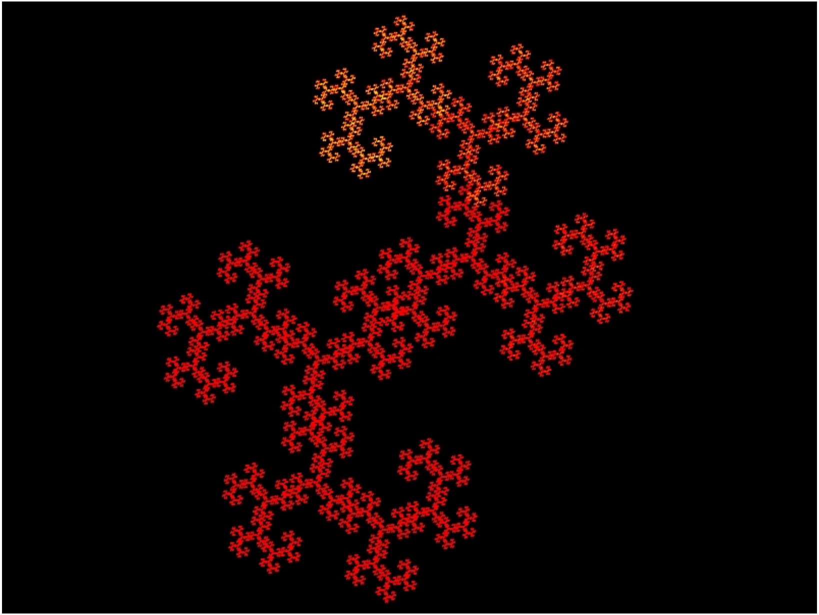
Here's the marvelous fact: *the portion of* $\overline{\mathbf{D}}$ *in a small neighborhood of* $q \in \mathbb{C}$ *tends to look like* D_q.

For example, here's the set $\overline{\mathbf{D}}$ near $q = 0.375453 + 0.544825i$:



And here's the dragon D_q for $q = 0.375453 + 0.544825i$:



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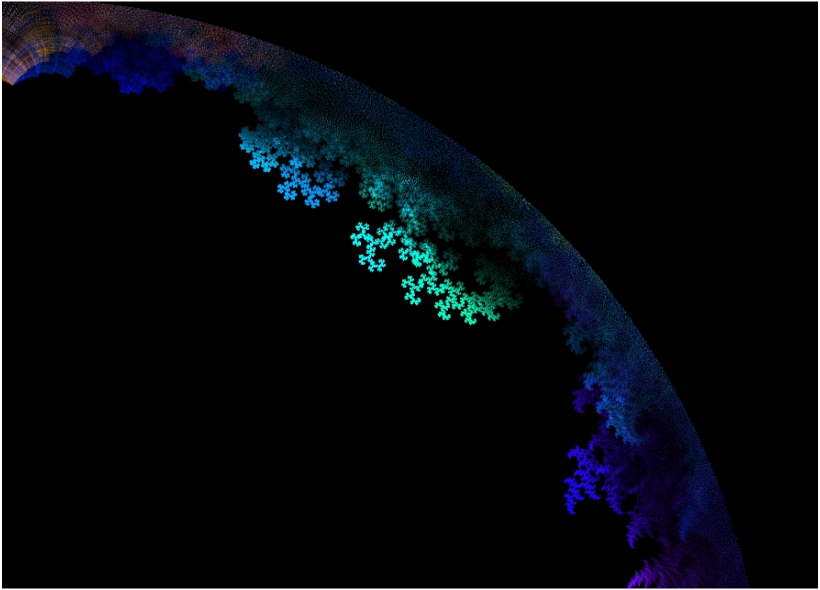
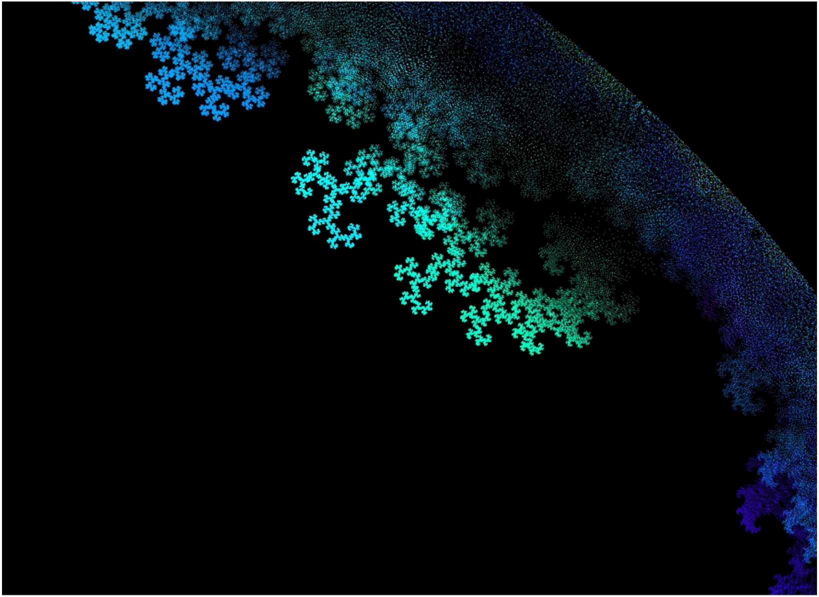
Then we’ll increase the degree and see how the set ‘fills in’.

Then we’ll switch to a zoomed-in view of the corresponding dragon, and then zoom out.

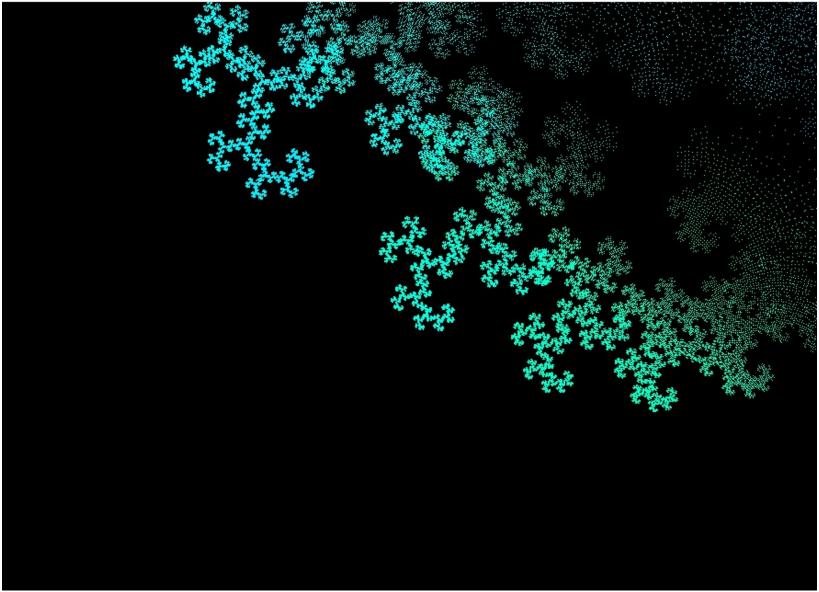
center $0.42065 + 0.48354i$, height .62508, degree 20



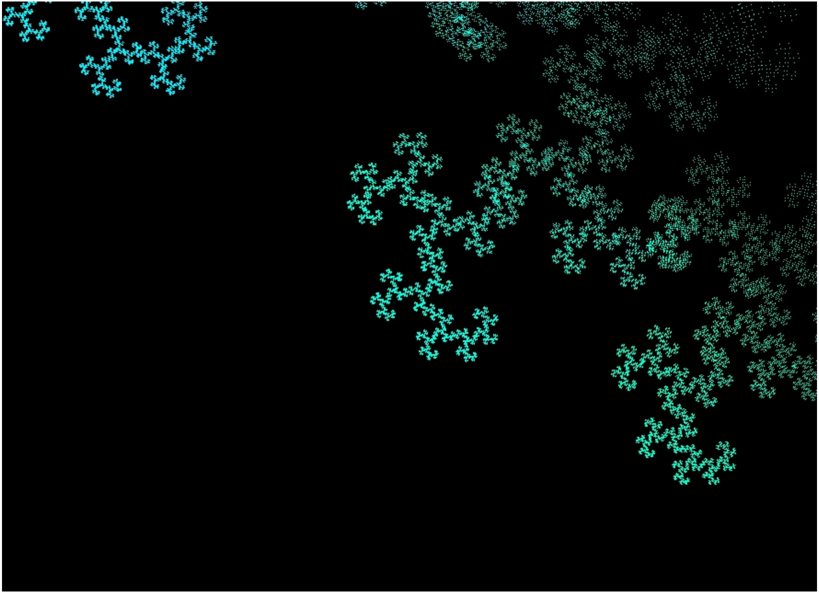
center $0.42065 + 0.48354i$, height .31304, degree 20



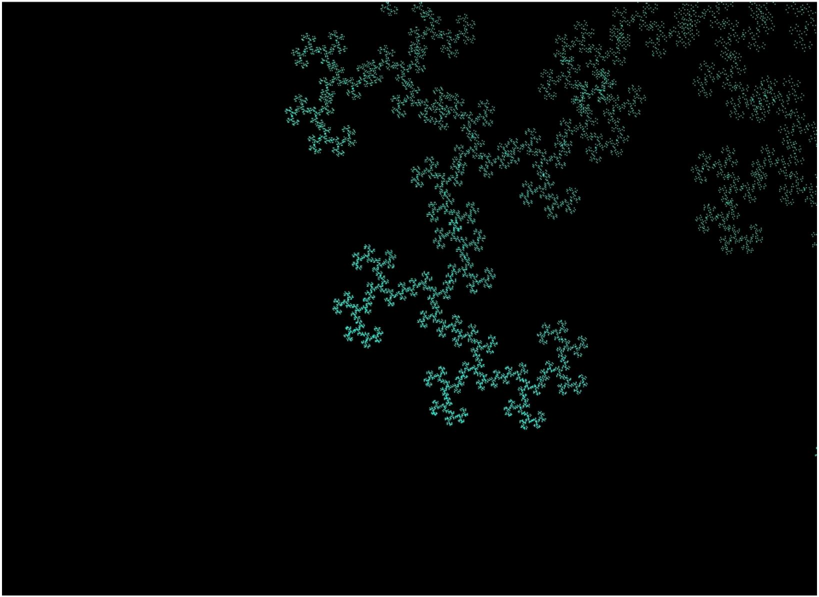
center $0.42065 + 0.48354i$, height .15652, degree 20



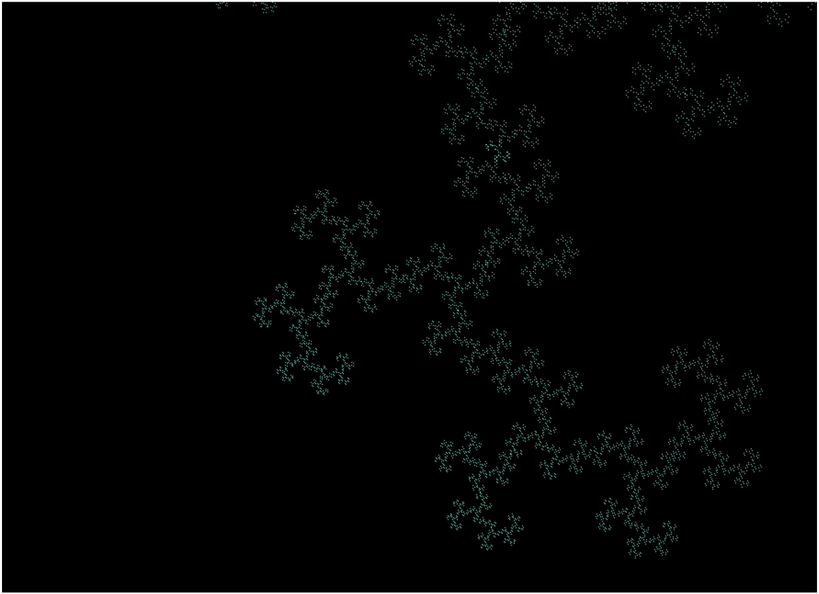
center $0.42065 + 0.48354i$, height .07826, degree 20



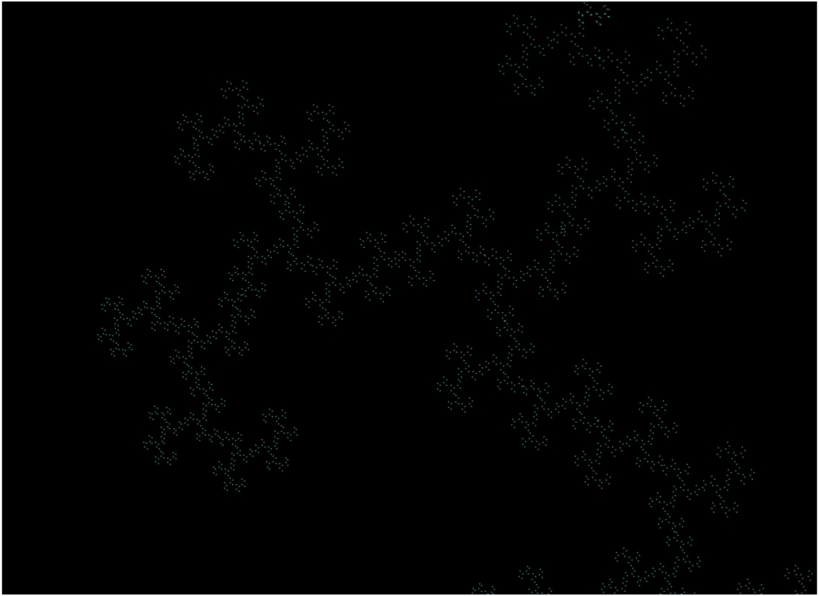
center $0.42065 + 0.48354i$, height .03913, degree 20



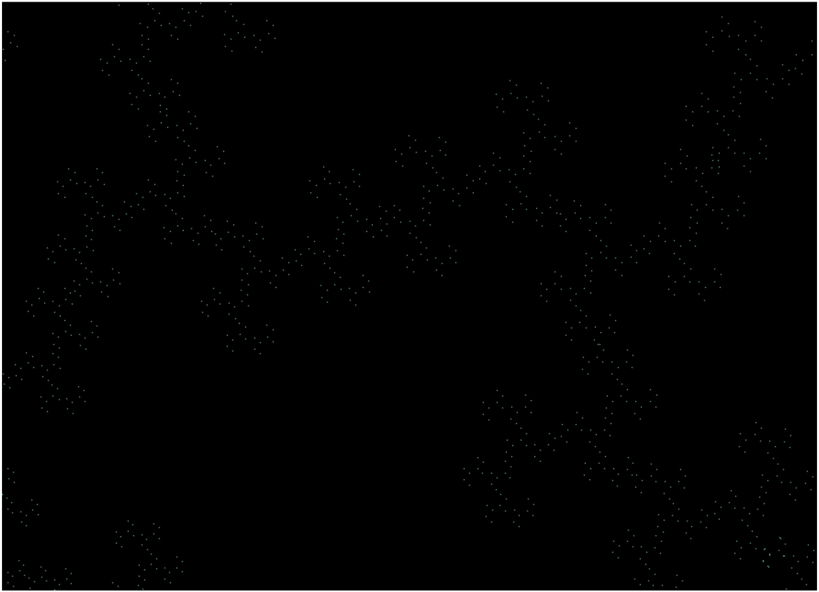
center $0.42065 + 0.48354i$, height .019565, degree 20



center $0.42065 + 0.48354i$, height .0097825, degree 20



center $0.42065 + 0.48354i$, height .0048912, degree 20



center $0.42065 + 0.48354i$, height .0024456, degree 20

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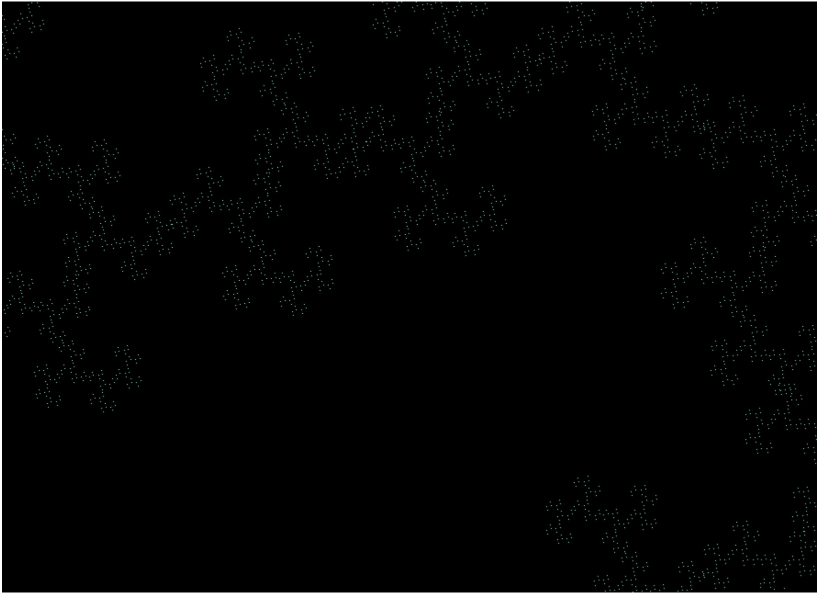
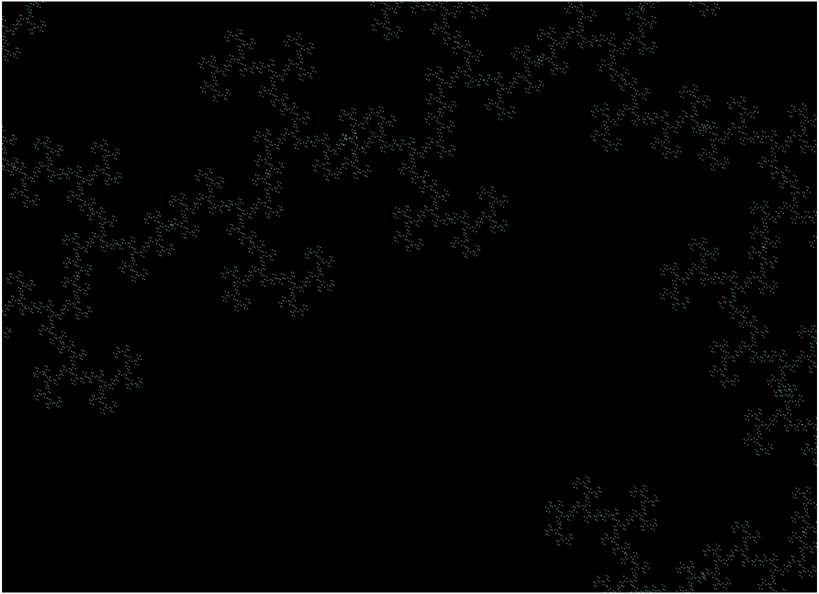
center $0.42065 + 0.48354i$, height .0024456, degree 22

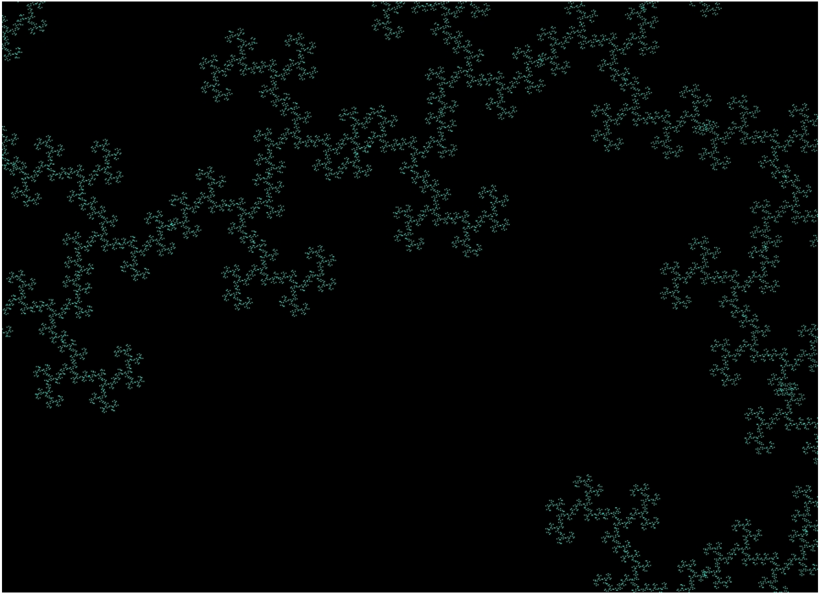
center $0.42065 + 0.48354i$, height .0024456, degree 23



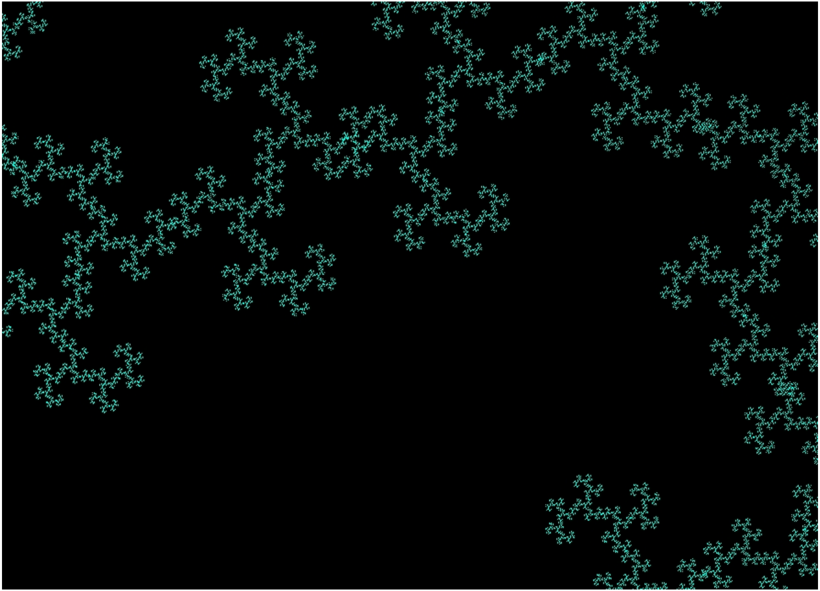
center $0.42065 + 0.48354i$, height .0024456, degree 24



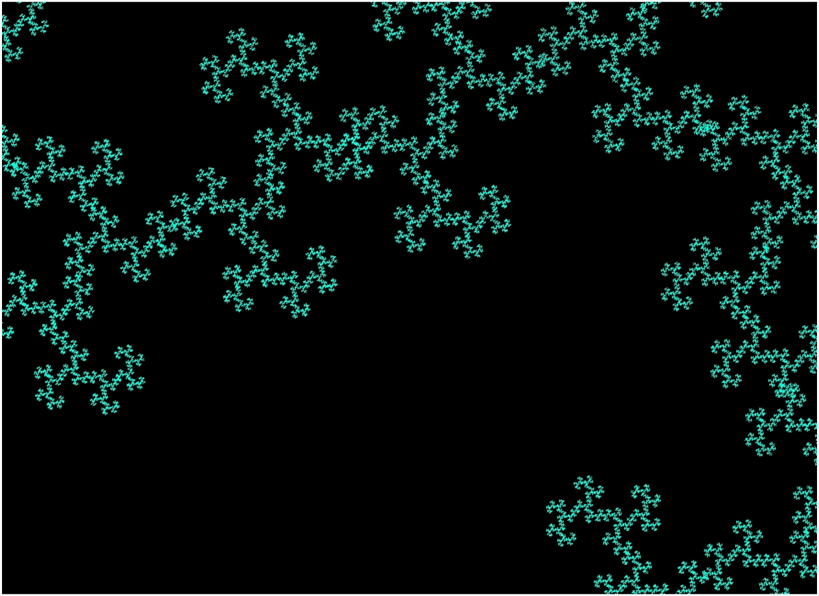
center $0.42065 + 0.48354i$, height .0024456, degree 25



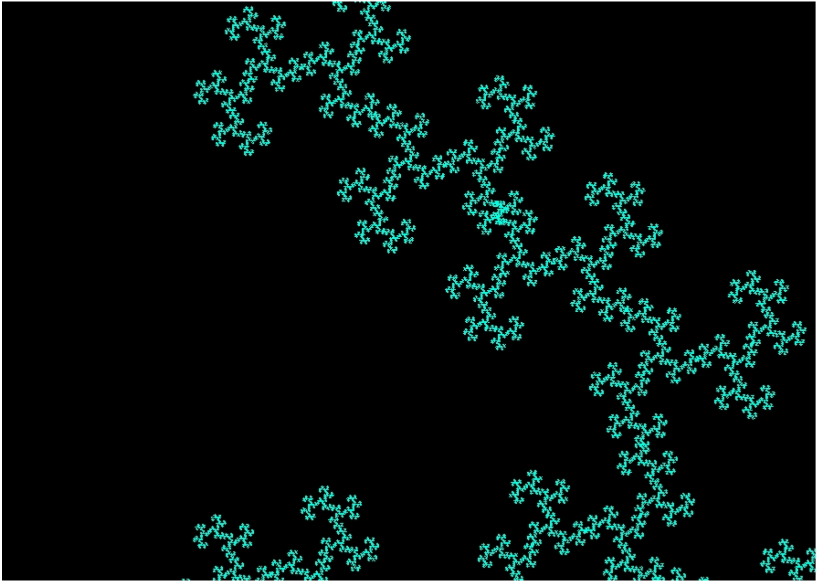
center $0.42065 + 0.48354i$, height .0024456, degree 26



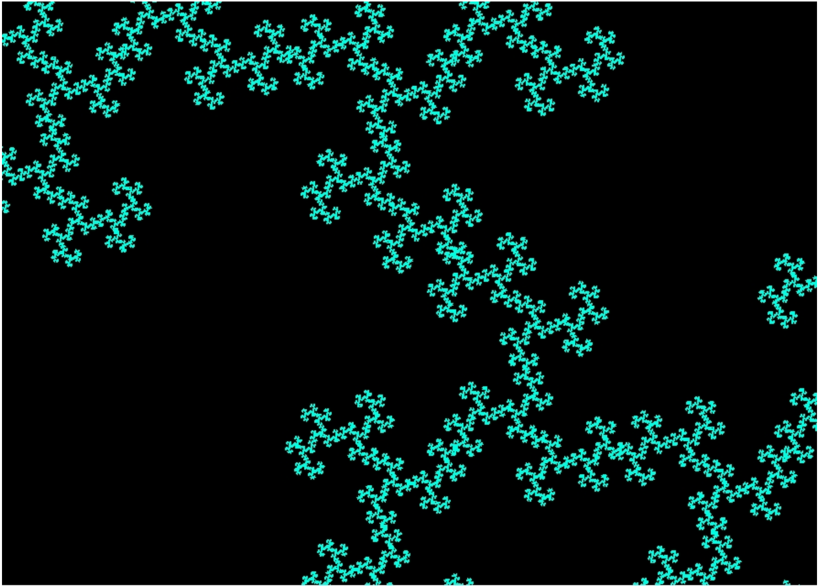
center $0.42065 + 0.48354i$, height .0024456, degree 27



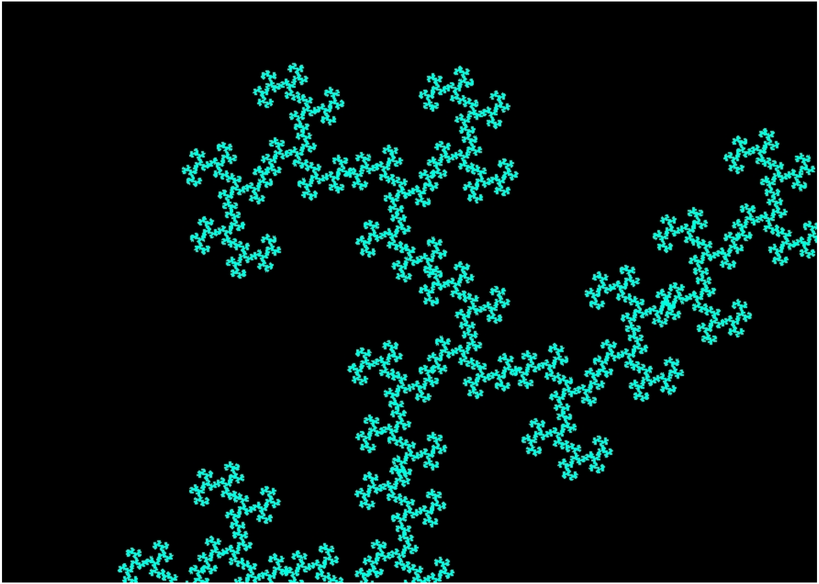
dragon for $0.42065 + 0.48354i$, height .0024456



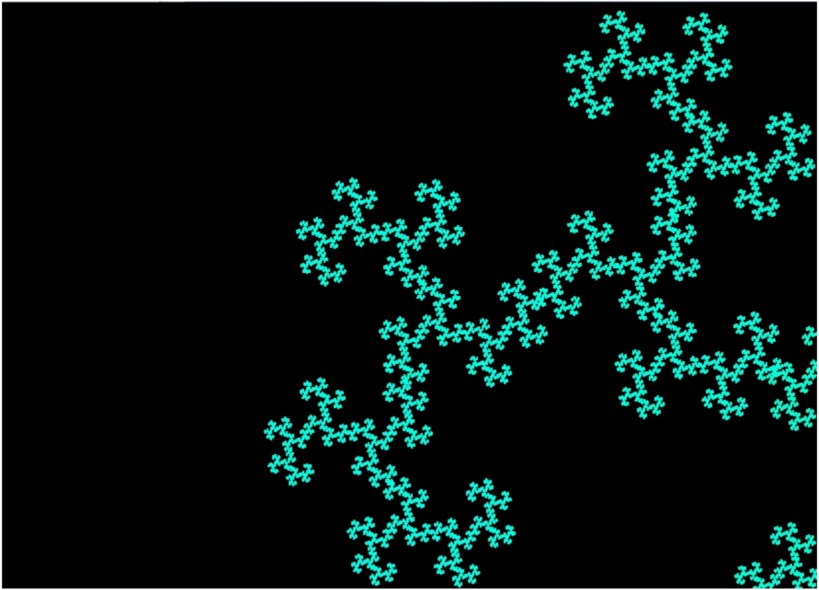
dragon for $0.42065 + 0.48354i$, height .0048912



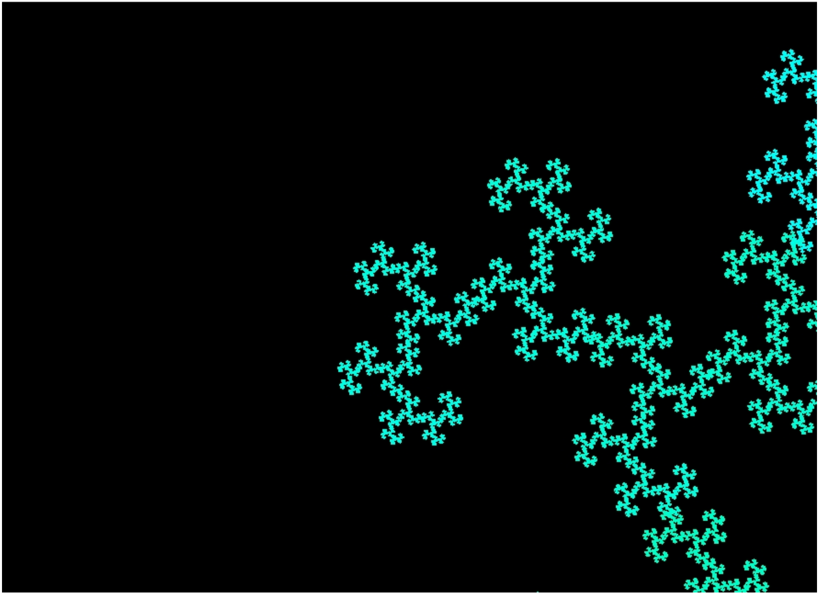
dragon for $0.42065 + 0.48354i$, height .0097825



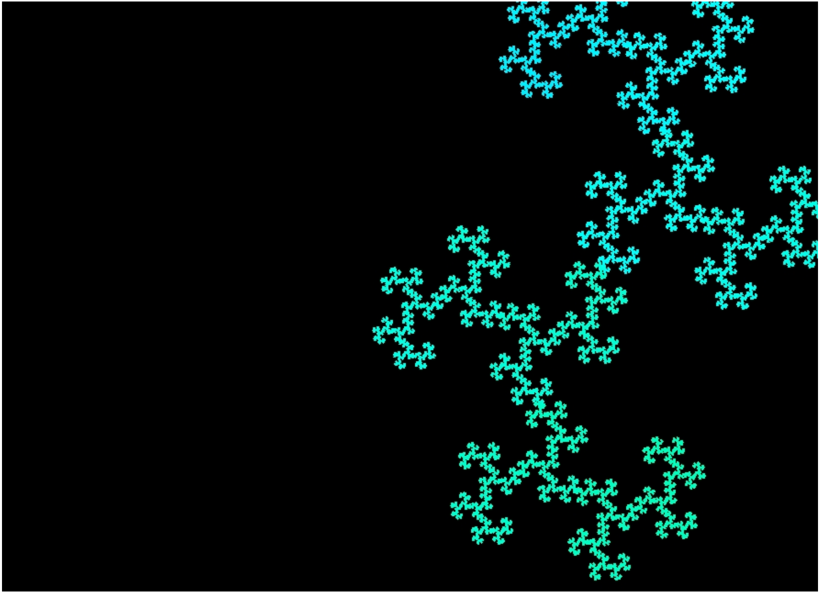
dragon for $0.42065 + 0.48354i$, height .019565



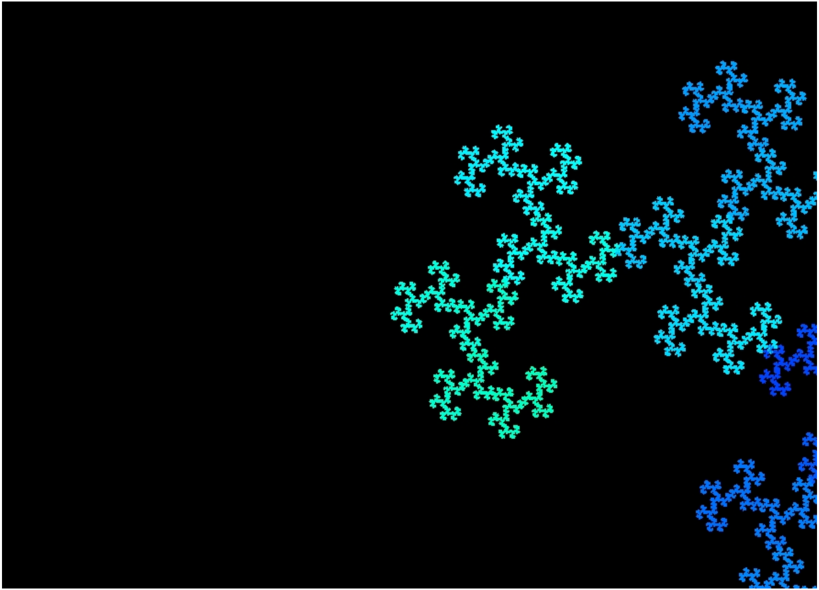
dragon for $0.42065 + 0.48354i$, height .03913



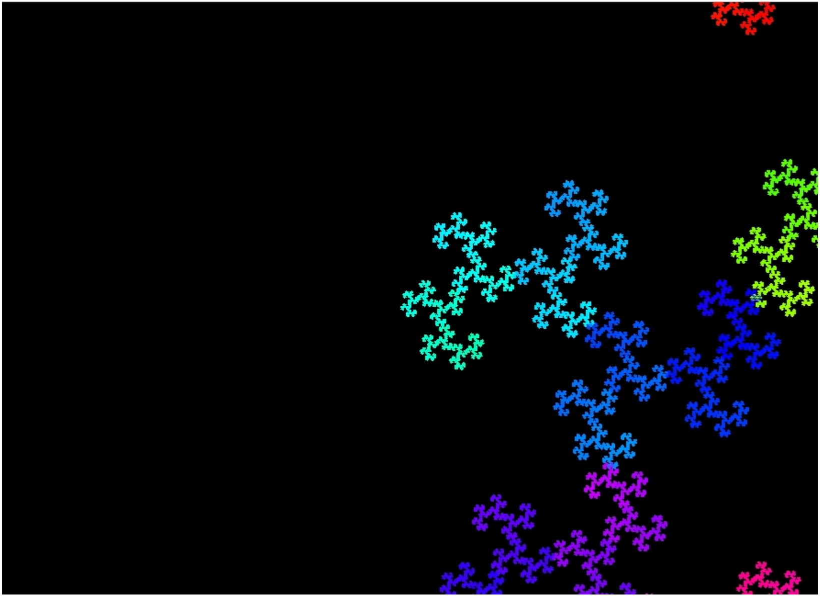
dragon for $0.42065 + 0.48354i$, height .07826



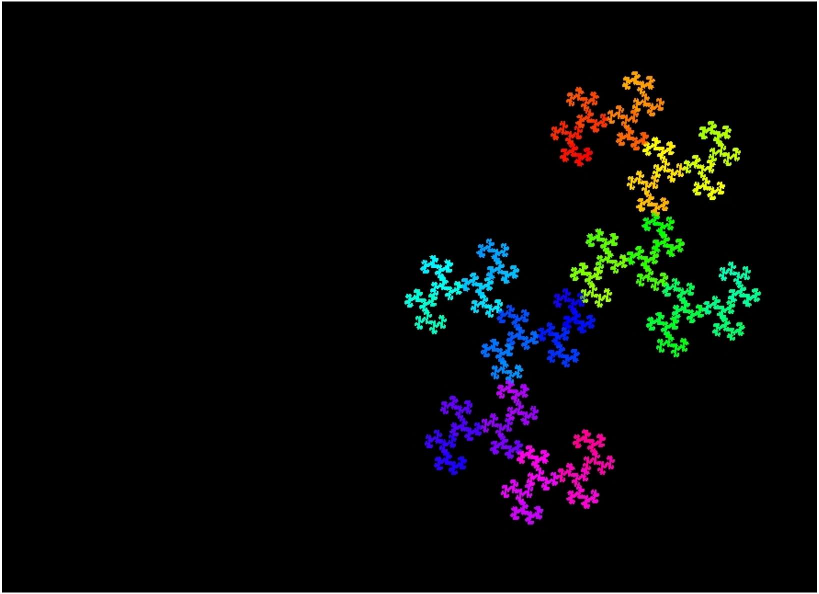
dragon $0.42065 + 0.48354i$, height .15652



dragon for $0.42065 + 0.48354i$, height .31304



dragon for $0.42065 + 0.48354i$, height .62508



The set $\overline{\mathbf{D}}$ is the set of *roots* of all Littlewood series. The set D_q is the set of *values* of all Littlewood series at the point q:

Theorem 4. For $|q| < 1$, $D_q = \{f(q) \colon f \in \mathbf{L}\}$.

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Theorem 4. For $|q| < 1$, $D_q = \{f(q) \colon f \in \mathbf{L}\}$.

This is easy to show.

But why does $\overline{\mathbf{D}}$ near q tend to resemble D_q?

Each Littlewood series f maps q to a point $f(q) \in D_q$. For p near q,

$$f(p) \approx f(q) \; + \; f'(q)\,(p - q)$$

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Thus, we expect $f(p) = 0$ when

$$p - q \approx -\frac{f(q)}{f'(q)}$$

If this reasoning is good, this formula approximately gives points $p \in \overline{\mathbf{D}}$ near q from points $f(q) \in D_q$.

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So, we should expect that near q, the set $\overline{\mathbf{D}}$ will *approximately* look like a somewhat distorted copy of the dragon D_q, or sometimes a union of such copies.

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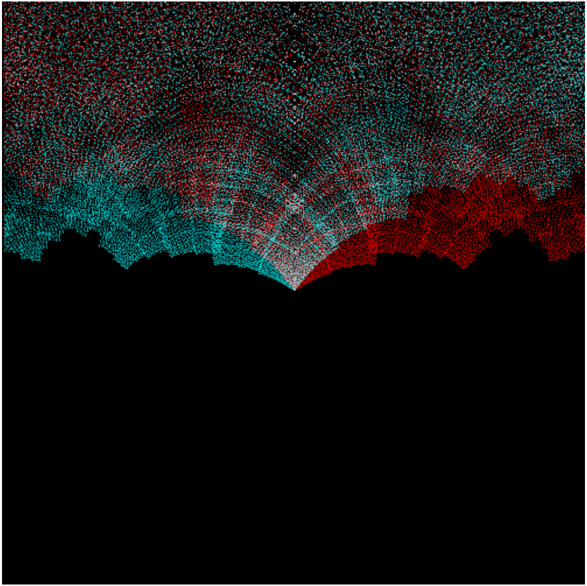
So, we should expect that near q, the set $\overline{\mathbf{D}}$ will *approximately* look like a somewhat distorted copy of the dragon D_q, or sometimes a union of such copies.

We're working on stating this precisely and proving it.

$\overline{D}$ near $q = \frac{4}{5}i$:



union of distorted dragons for $q = \frac{4}{5}i$: