# The Beauty of Roots

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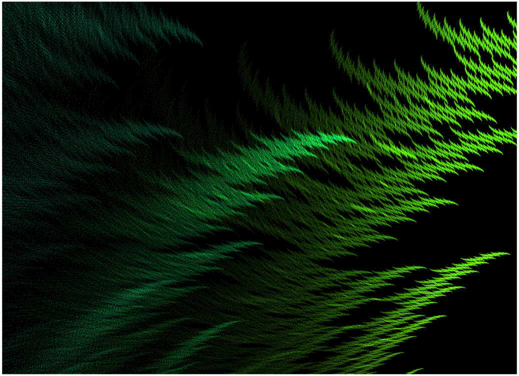
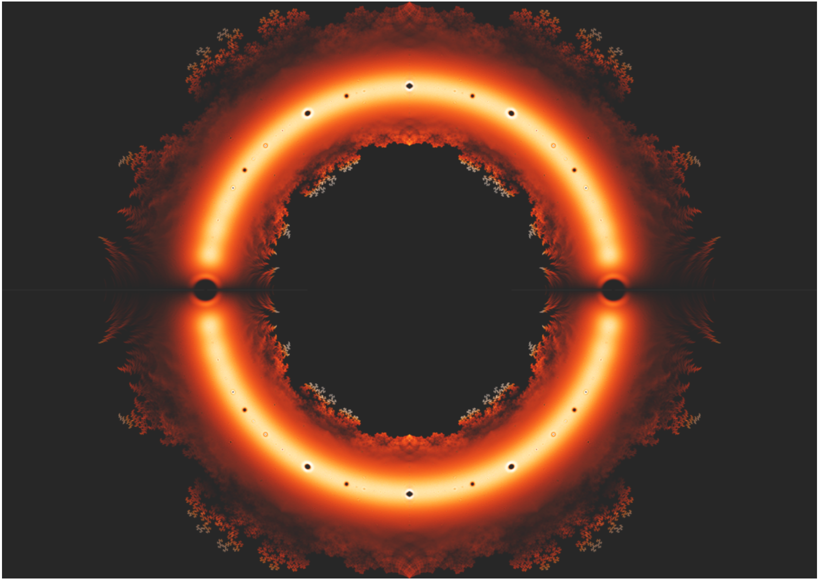
with lots of help from Greg Egan

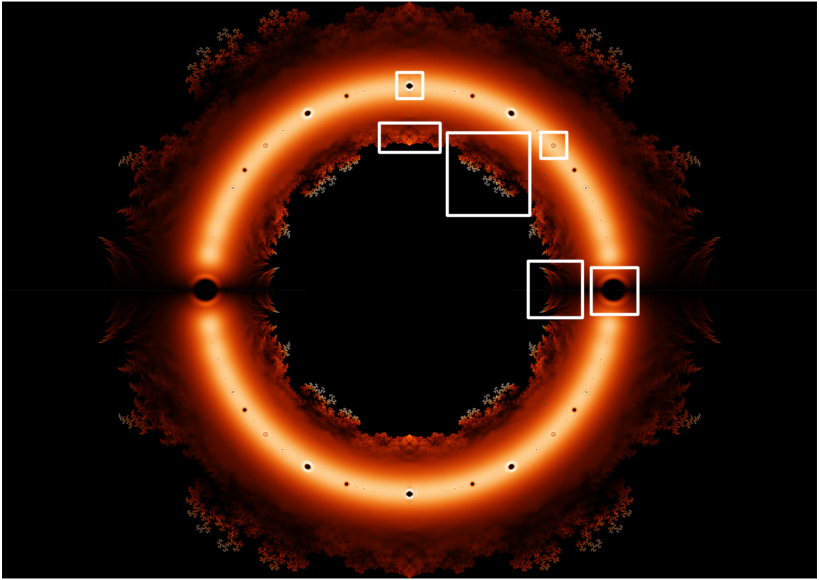


**Definition.** A **Littlewood polynomial** is a polynomial whose coefficients are all 1 and -1.

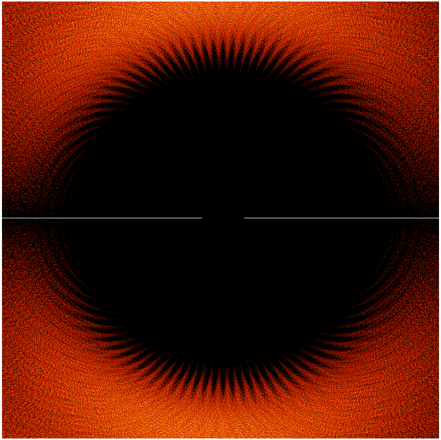
Let's draw all roots of all Littlewood polynomials!



Certain regions seem particularly interesting:

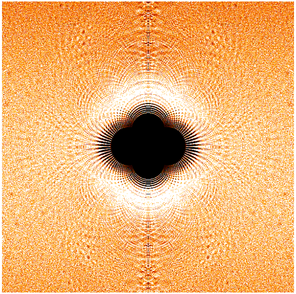


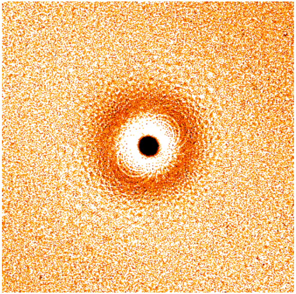
The hole at 1:



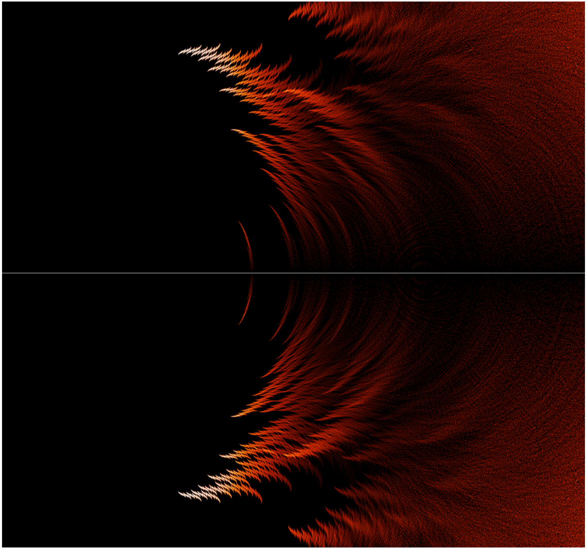
Note the line along the real axis: more Littlewood polynomials have real roots than *nearly* real roots.

The holes at $i$ and $e^{i\pi/4}$:

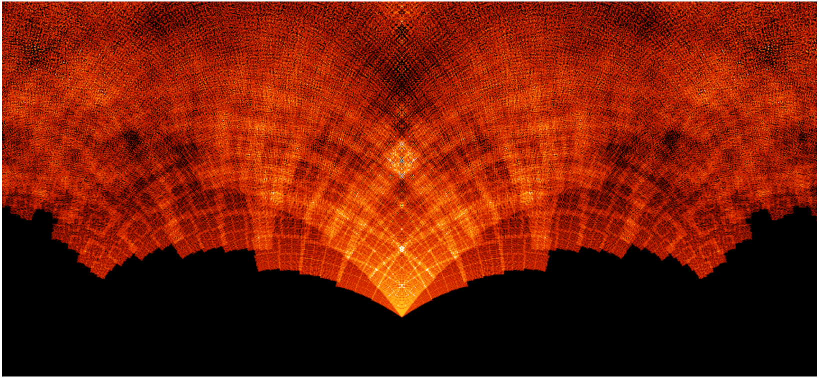




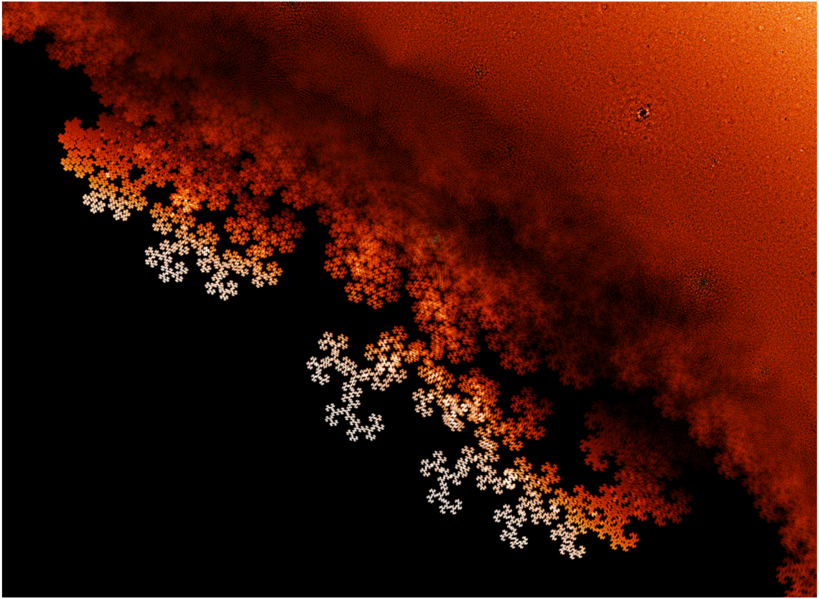
This plot is centered at the point $\frac{4}{5}$:



This is centered at the point $\frac{4}{5}i$:



This is centered at $\frac{1}{2}e^{i/5}$:



Can we understand these pictures? Let

$$\mathbf{D} = \{z \in \mathbb{C} \ : \ z \text{ is the root of some Littlewood polynomial}\}$$

Here are some theorems, which I won't prove in this 'easy' version of my talk slides.

**Theorem 1.** $\mathbf{D}$ is contained in the annulus $\{1/2 < |z| < 2\}$

**Proof.** See the detailed version of these slides.

**Theorem 2.** The closure of $\overline{\mathbf{D}}$ contains the annulus $\{2^{-1/4} \leq |z| \leq 2^{1/4}\}$

**Proof.** This was proved by Thierry Bousch in 1988.

The closure $\overline{\mathbf{D}}$ is easier to study than $\mathbf{D}$. For example:

**Theorem 3.** $\overline{\mathbf{D}}$ is connected.

**Proof.** This was proved by Bousch in 1993 — see the detailed version of these slides for more.

Here is the key to understanding the beautiful patterns in the set $\overline{\mathbf{D}}$:

**Definition.** A **Littlewood series** is a power series all of whose coefficients are 1 or $-1$:

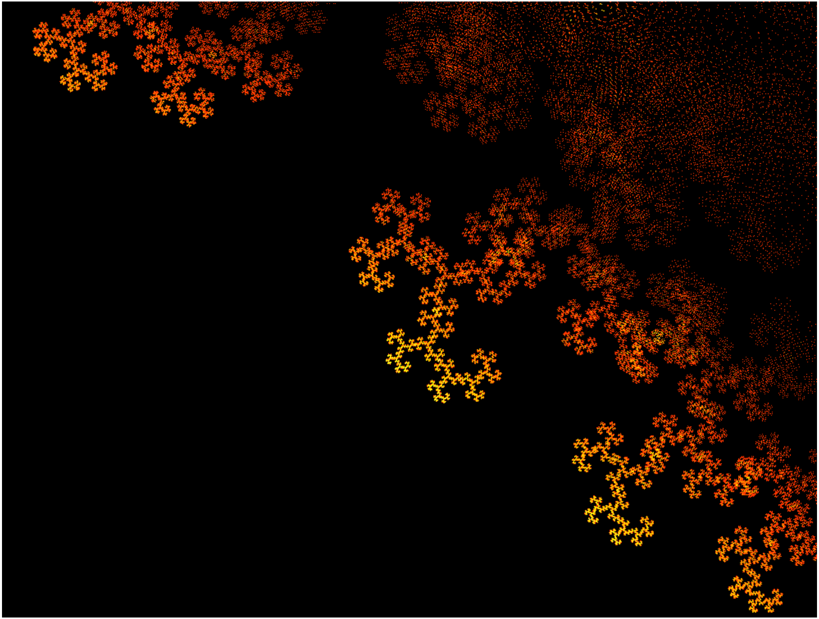
$$f(z) = \pm 1 \pm z \pm z^2 \pm z^3 \cdots$$

**Theorem 4.** The set $\overline{\mathbf{D}}$ is the set of *roots* of all Littlewood series.

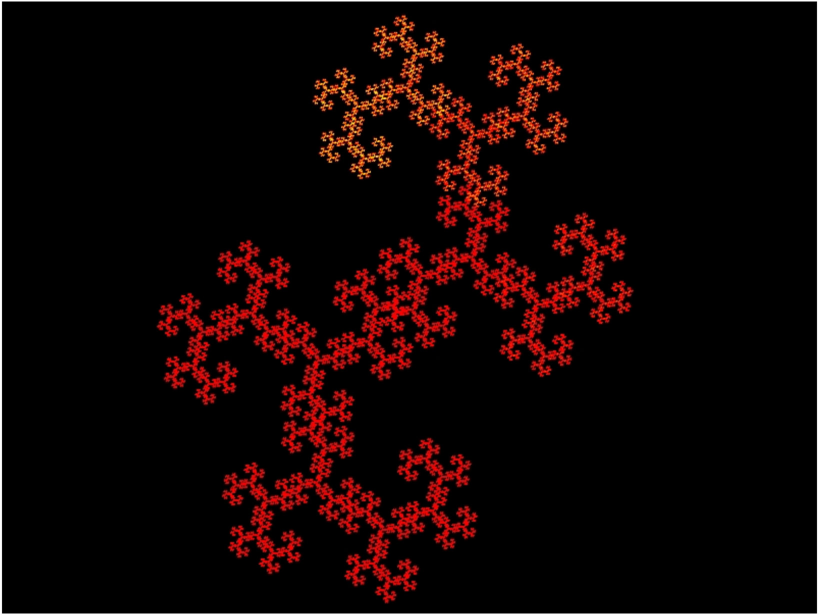
**Definition.** The **dragon** $D_q$ is the set of *values* of all Littlewood series at the point $q$.

And here's the marvelous fact: *the portion of* $\overline{\mathbf{D}}$ *in a small neighborhood of* $q \in \mathbb{C}$ *tends to look like* $D_q$. Let's see an example...

Here's the set $\overline{\mathbf{D}}$ near $q = 0.375453 + 0.544825i$:



And here's the dragon $D_q$ for $q = 0.375453 + 0.544825i$:

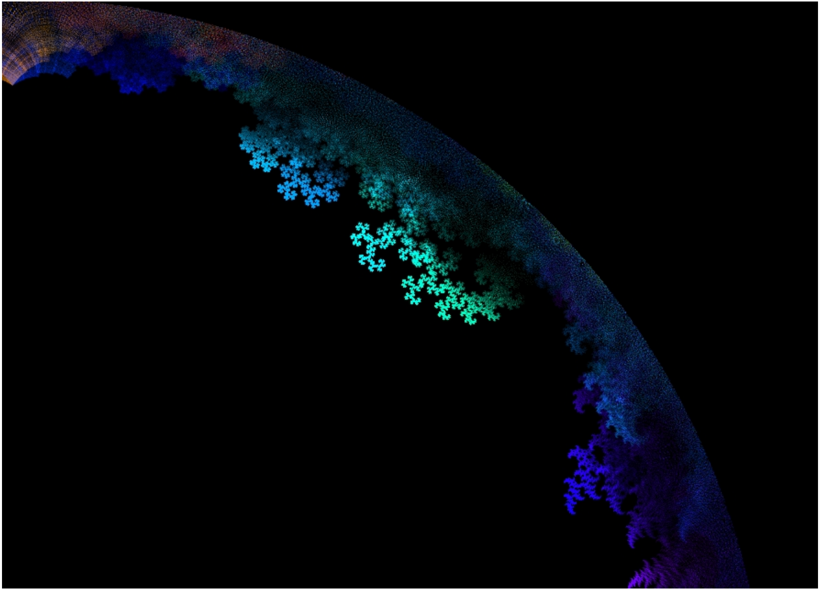


Let’s zoom in on the set of roots of Littlewood polynomials of degree 20. When we zoom in enough, we’ll see it’s a discrete set!

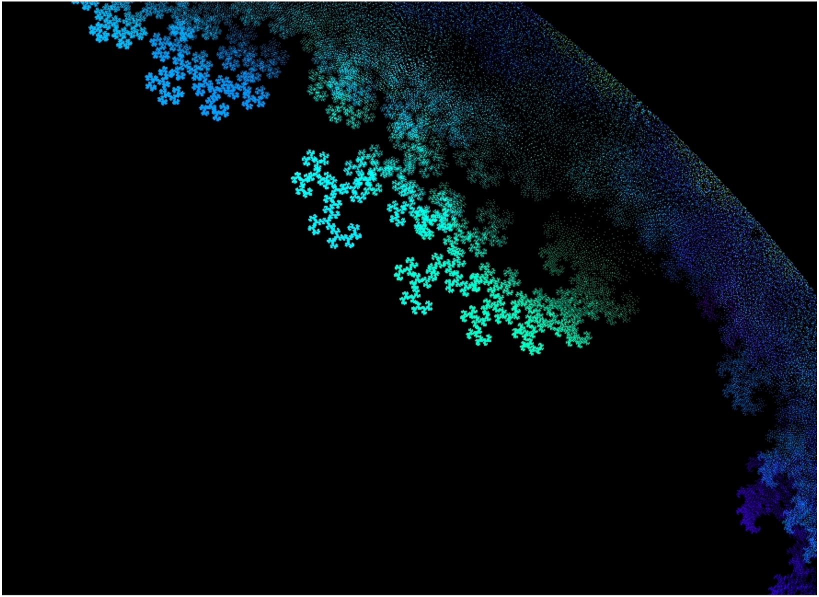
Then we’ll increase the degree and see how the set ‘fills in’.

Then we’ll switch to a zoomed-in view of the corresponding dragon, and then zoom out.

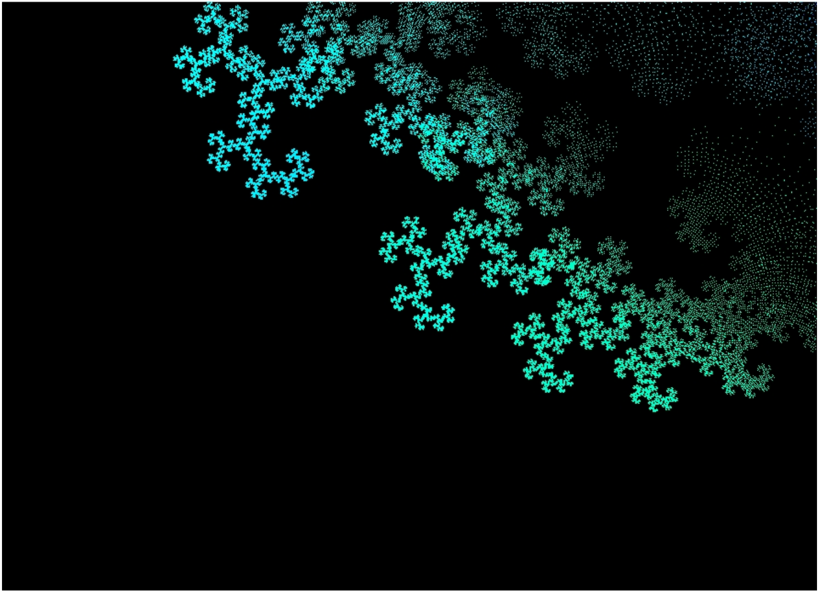
center 0.42065 + 0.48354$i$, height .62508, degree 20



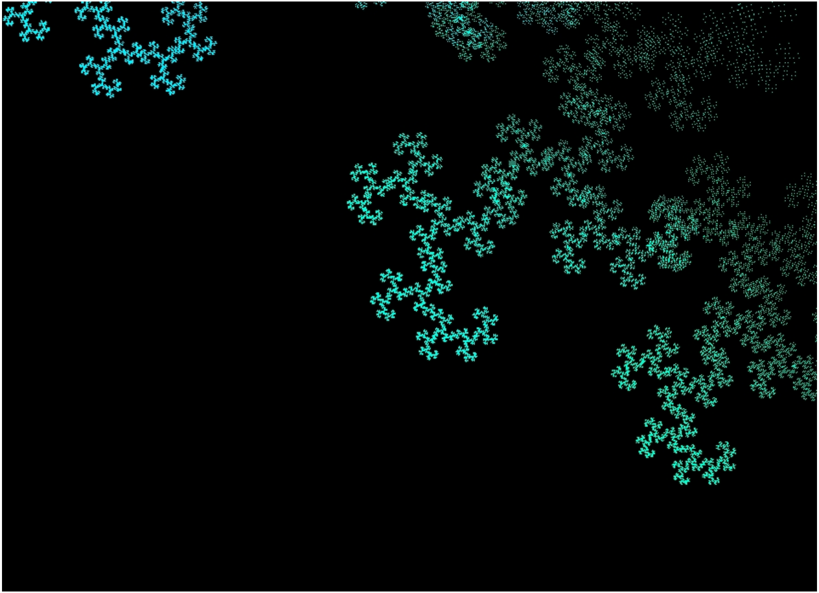
center $0.42065 + 0.48354i$, height .31304, degree 20



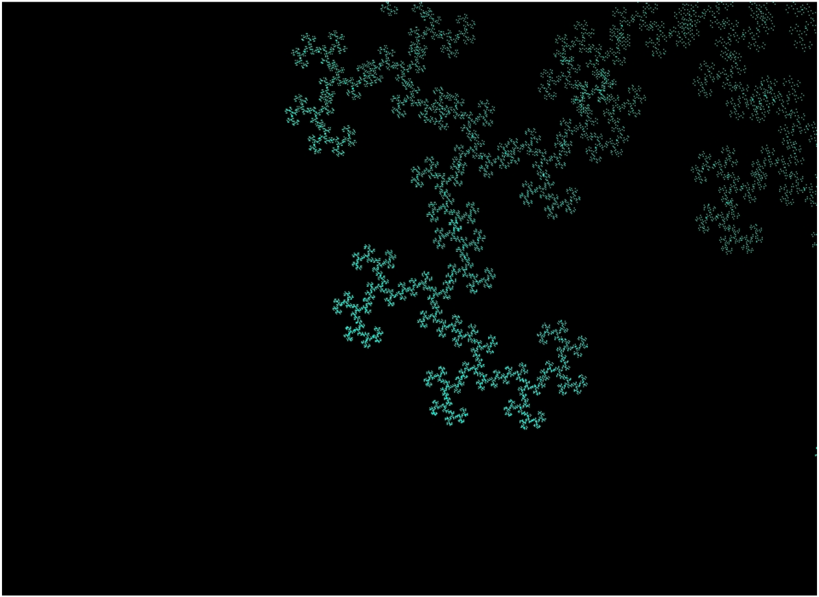
center $0.42065 + 0.48354i$, height .15652, degree 20



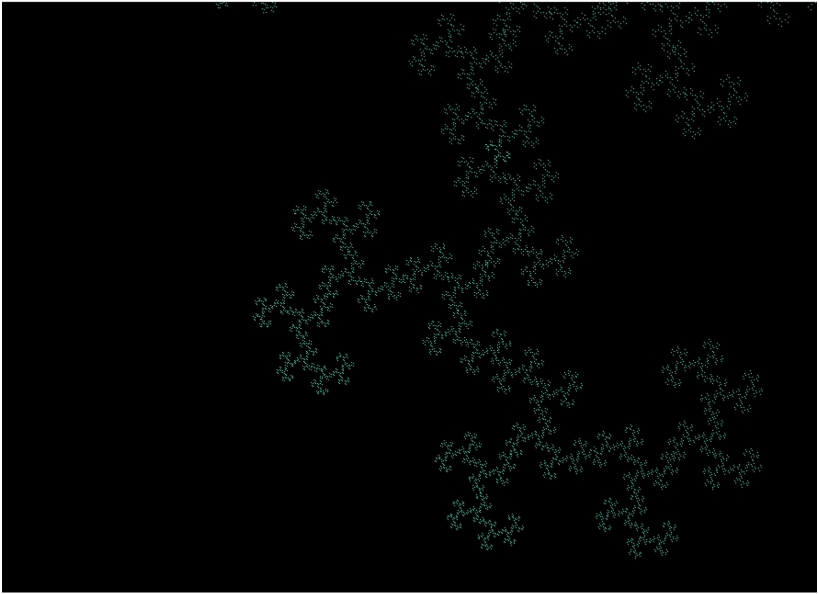
center 0.42065 + 0.48354$i$, height .07826, degree 20



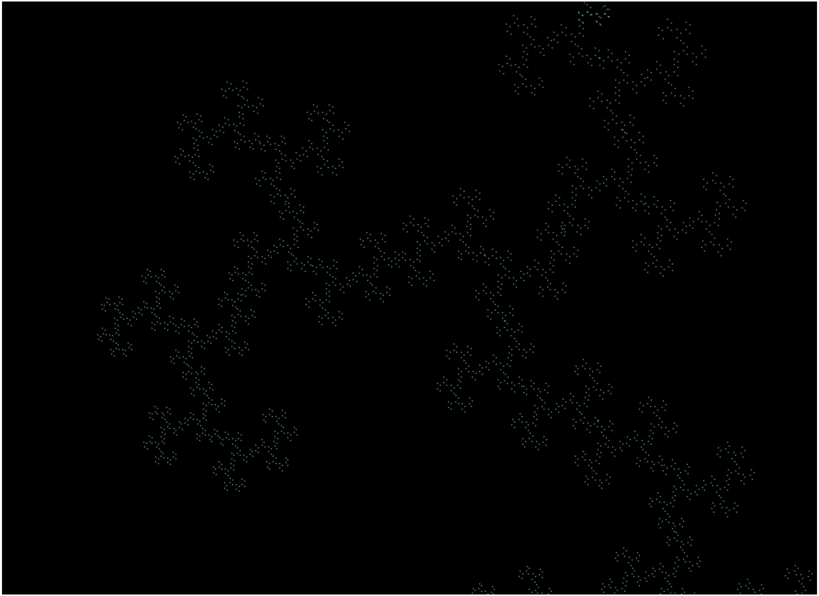
center $0.42065 + 0.48354i$, height .03913, degree 20



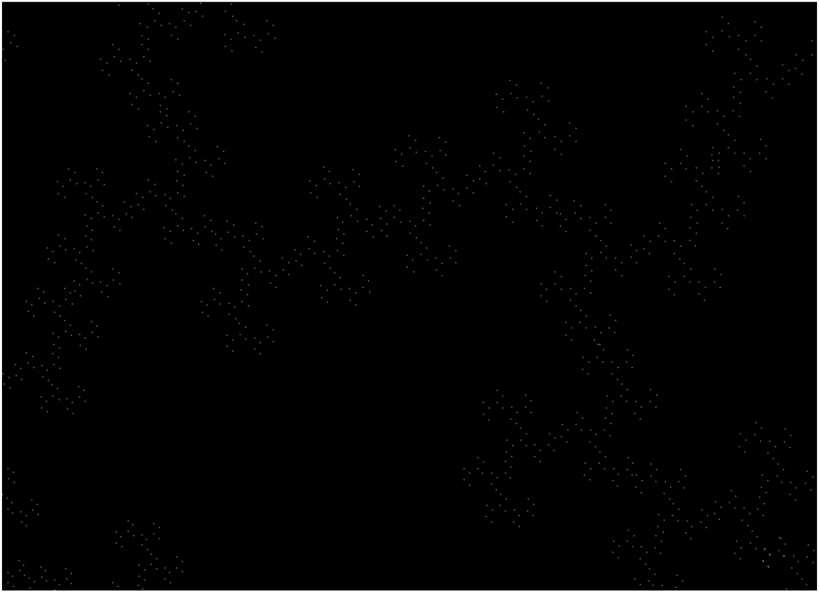
center $0.42065 + 0.48354i$, height .019565, degree 20



center $0.42065 + 0.48354i$, height .0097825, degree 20



center $0.42065 + 0.48354i$, height .0048912, degree 20

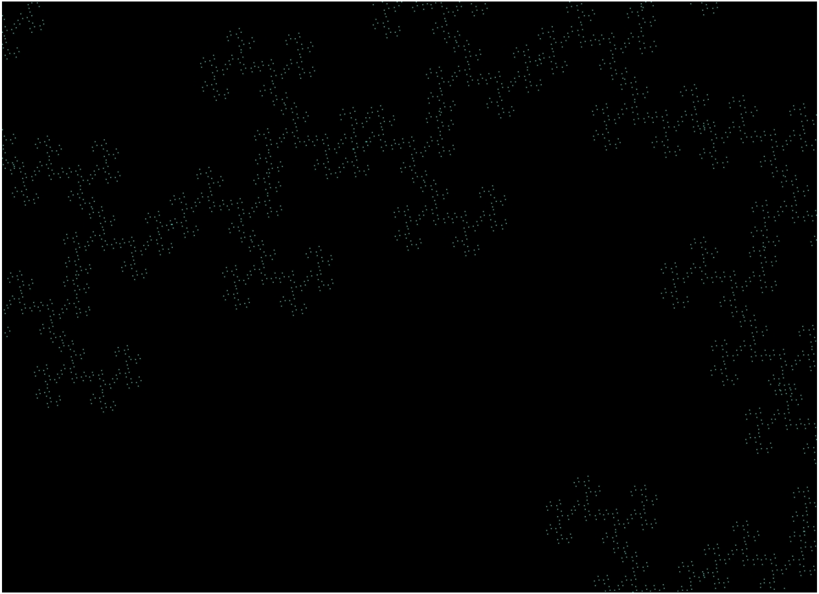


center $0.42065 + 0.48354i$, height $.0024456$, degree 20

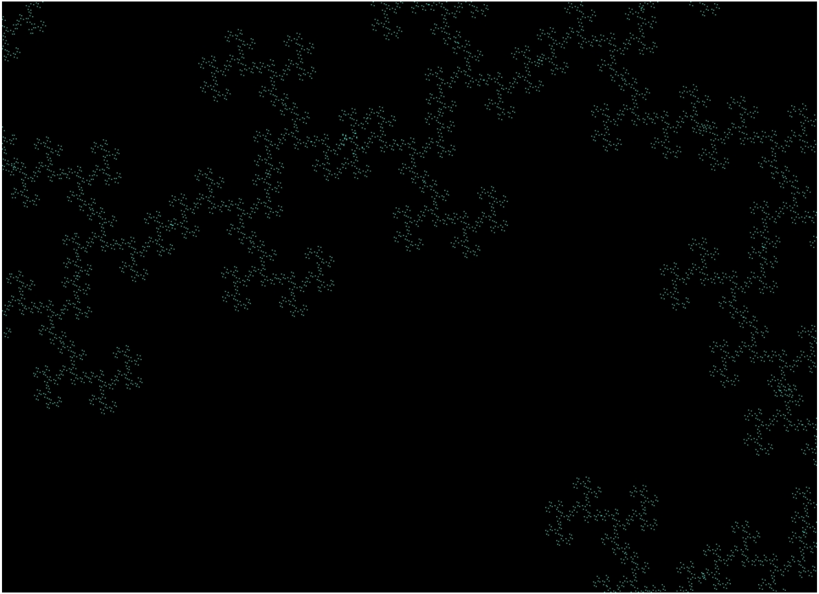
center $0.42065 + 0.48354i$, height .0024456, degree 21

center $0.42065 + 0.48354i$, height .0024456, degree 22

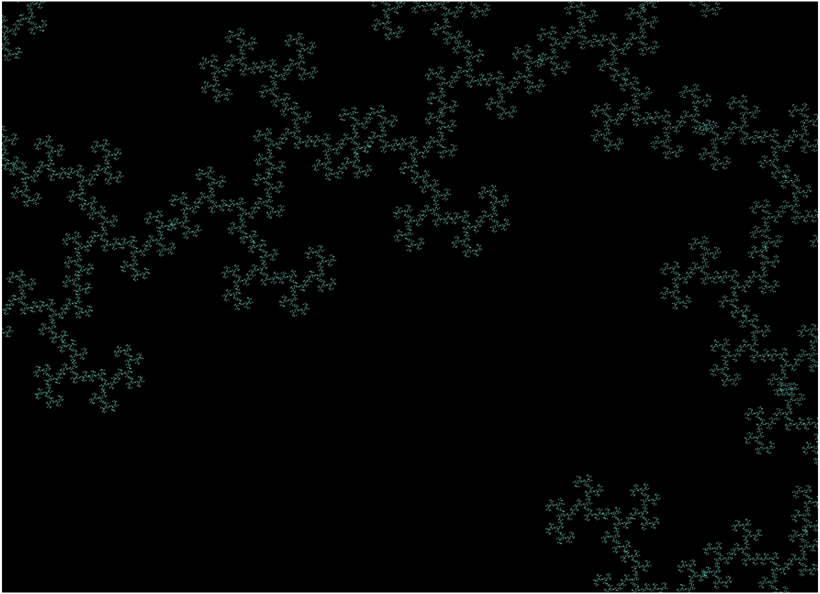
center $0.42065 + 0.48354i$, height .0024456, degree 23



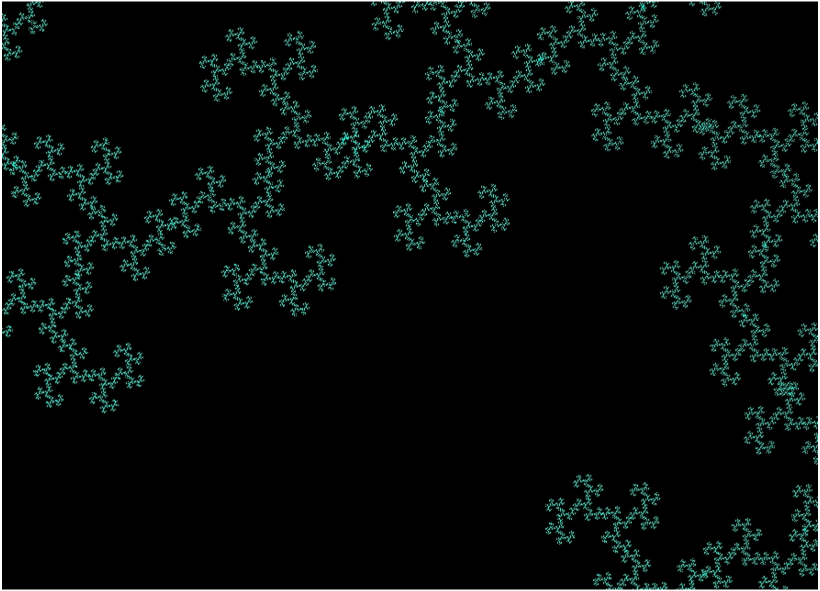
center $0.42065 + 0.48354i$, height .0024456, degree 24



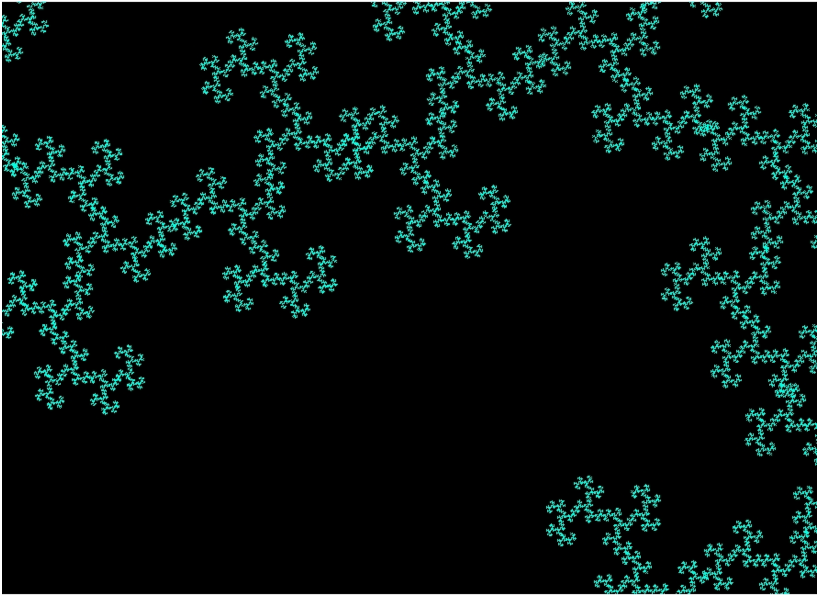
center $0.42065 + 0.48354i$, height .0024456, degree 25



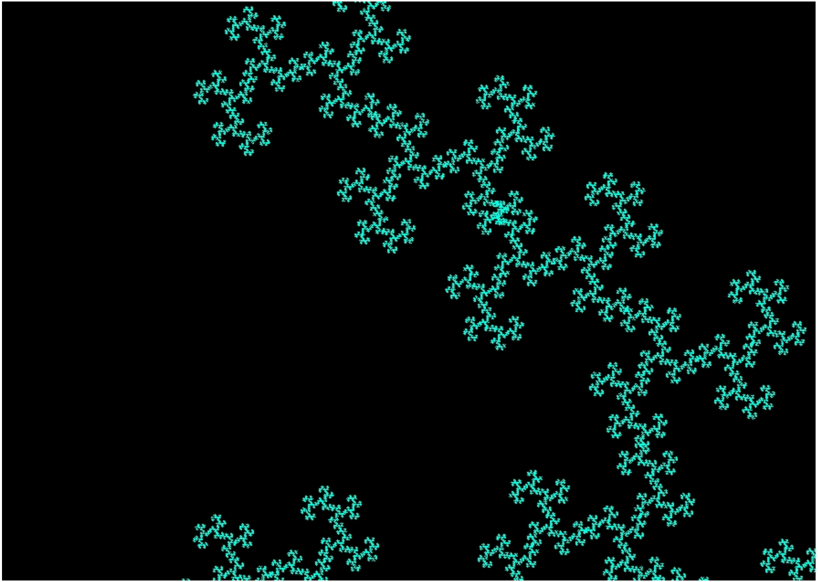
center $0.42065 + 0.48354i$, height .0024456, degree 26



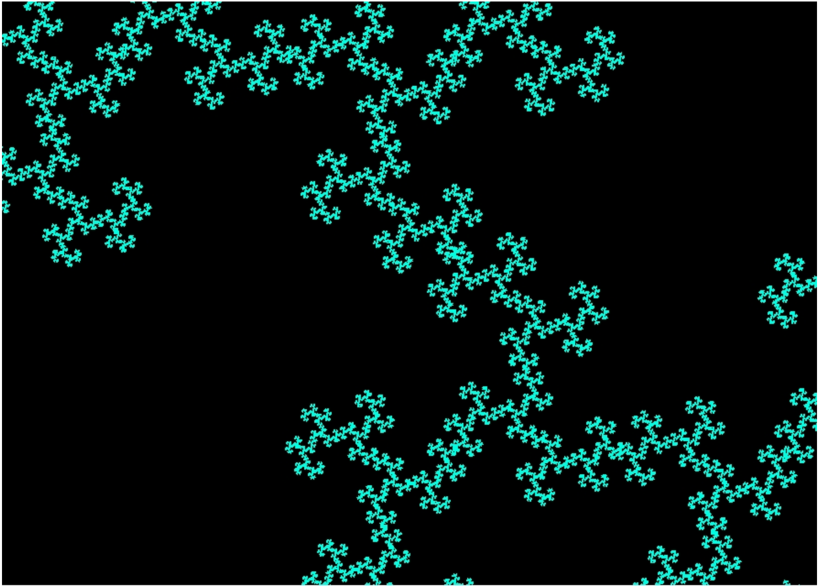
center $0.42065 + 0.48354i$, height .0024456, degree 27



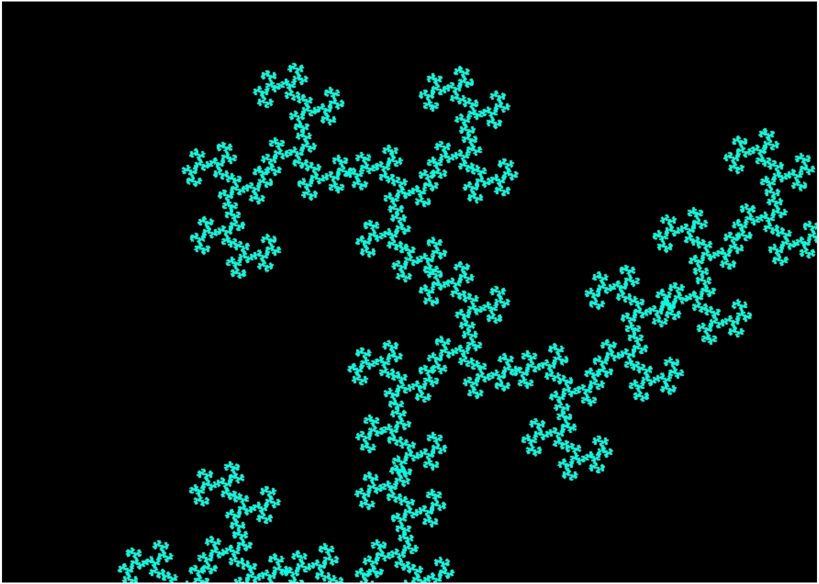
## dragon for $0.42065 + 0.48354i$, height .0024456



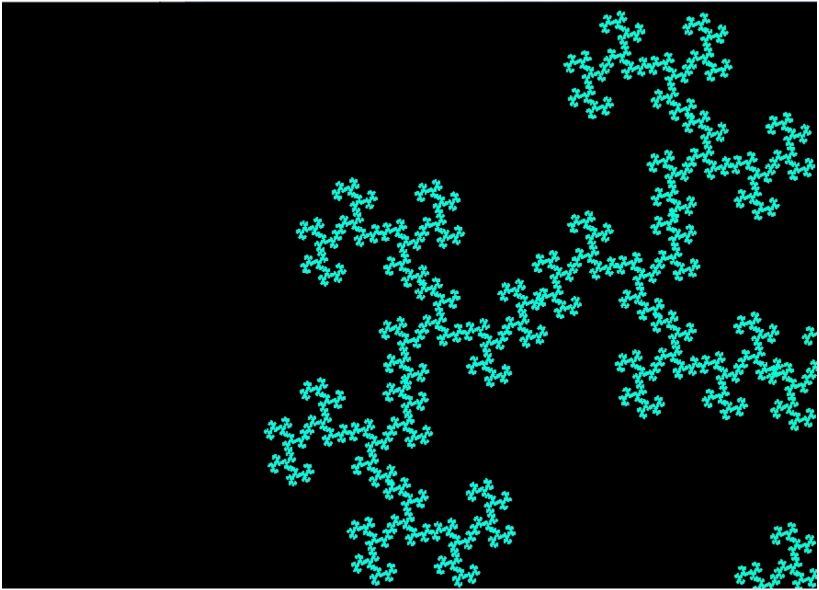
dragon for $0.42065 + 0.48354i$, height .0048912



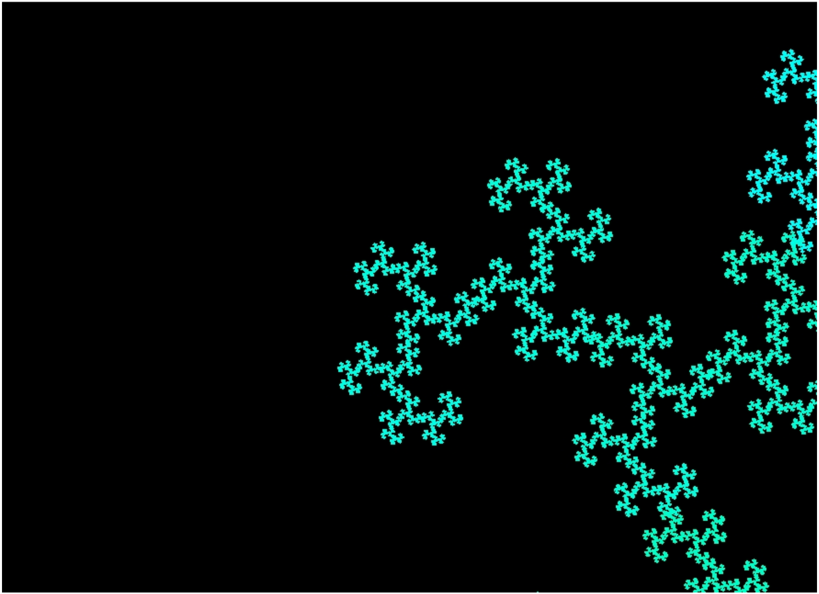
dragon for $0.42065 + 0.48354i$, height .0097825



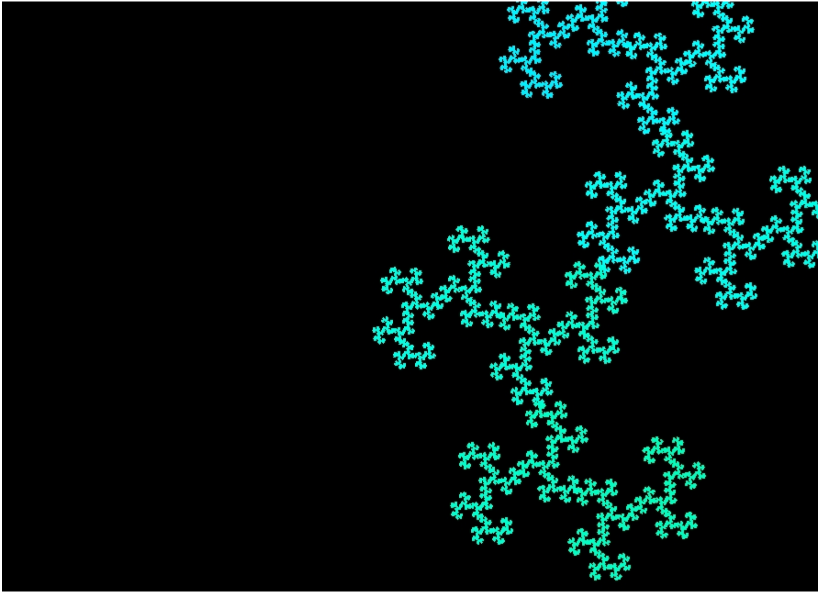
dragon for $0.42065 + 0.48354i$, height .019565



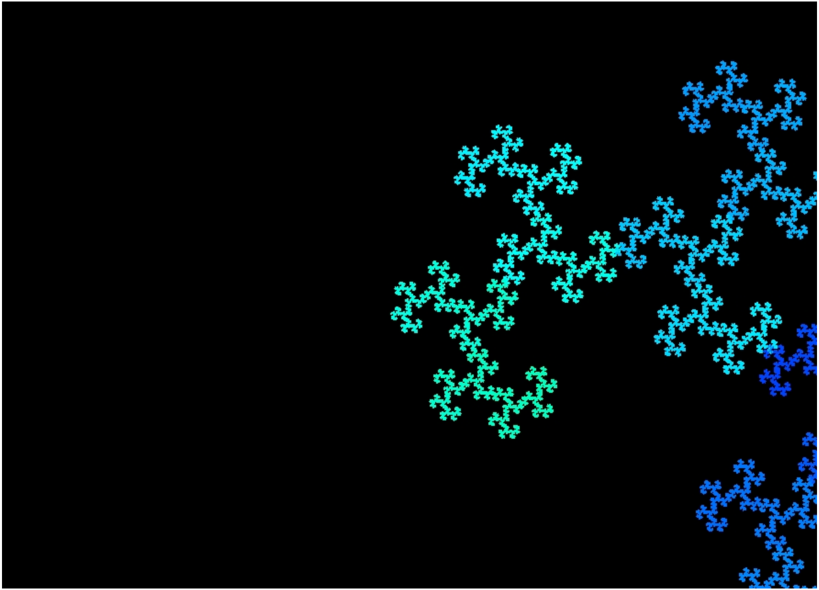
dragon for $0.42065 + 0.48354i$, height .03913



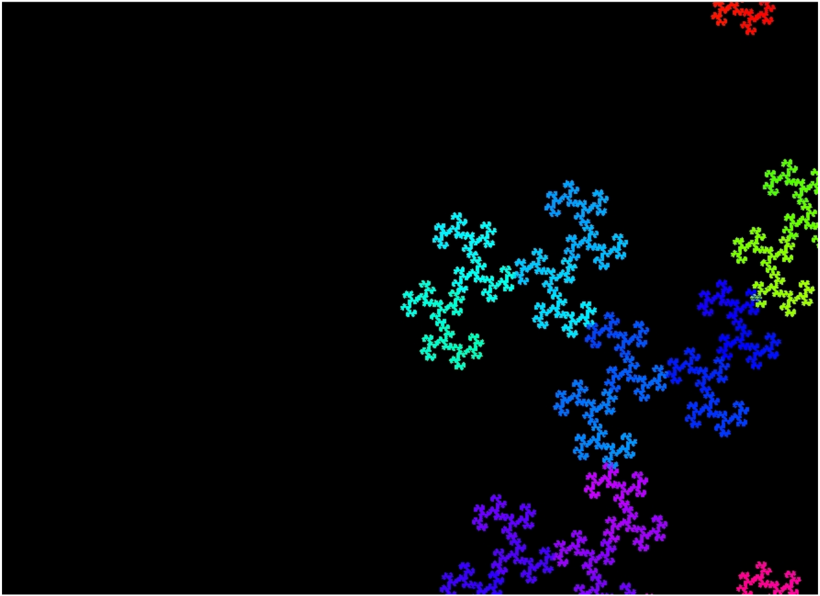
dragon for $0.42065 + 0.48354i$, height .07826



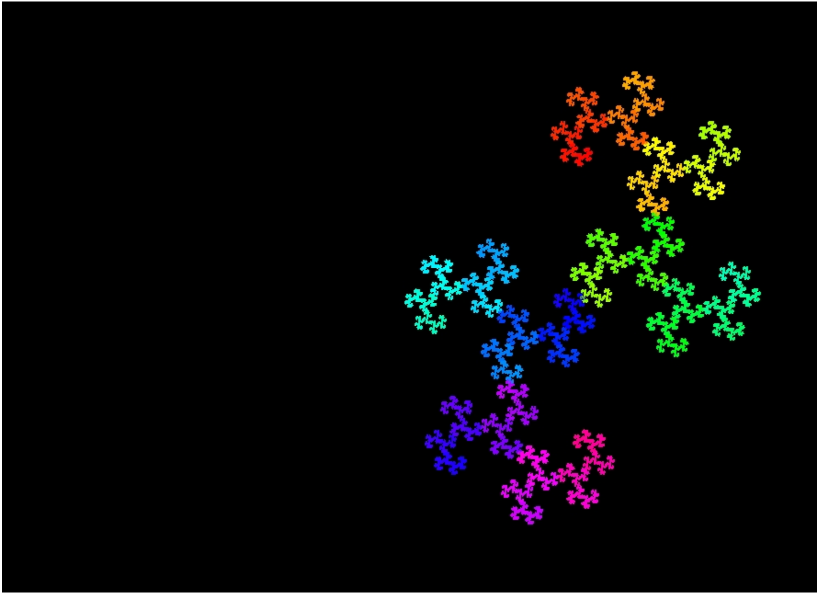
dragon $0.42065 + 0.48354i$, height .15652



dragon for $0.42065 + 0.48354i$, height .31304



dragon for $0.42065 + 0.48354i$, height .62508



But why does $\overline{\mathbf{D}}$ near $q$ tend to resemble the dragon $D_q$?

If $f$ is a Littlewood series, $f(q)$ is a point in the dragon $D_q$. For $p$ near $q$,

$$f(p) \approx f(q) \; + \; f'(q)\,(p-q)$$

So we expect $f(p) = 0$ when

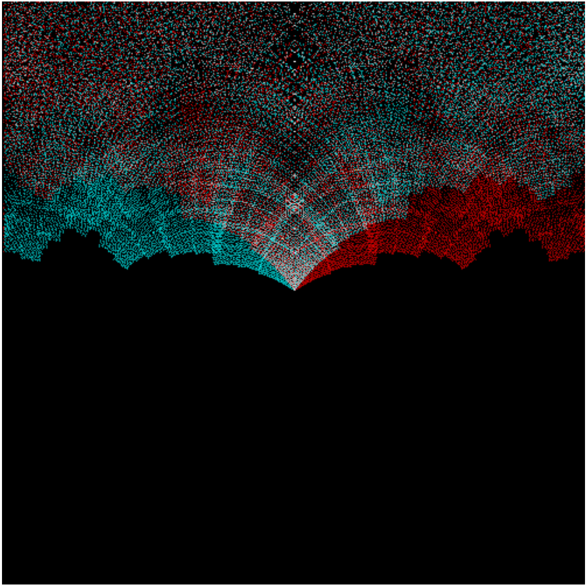
$$p - q \approx -\frac{f(q)}{f'(q)}$$

If this reasoning is good, this formula approximately gives points $p$ in $\overline{\mathbf{D}}$ near $q$ from points $f(q) \in D_q$.

So, we expect that near $q$, the set $\overline{\mathbf{D}}$ will *approximately* look like a somewhat distorted copy of the dragon $D_q$, or sometimes a union of such copies.

We're working on stating this precisely and proving it.

$\overline{D}$ near $q = \frac{4}{5}i$:



union of distorted dragons for $q = \frac{4}{5}i$: