SCATTERING AND COMPLETE INTEGRABILITY IN CONFORMALLY INVARIANT NONLINEAR THEORIES

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published in *Jour. Math. Phys.* **31** (1990), 757-762.

ABSTRACT

We study conformally invariant nonlinear wave equations in four dimensions corresponding to multicomponent massless scalar fields with a quartic interaction. We prove that the scattering operator S on the space $\mathbf{H}$ of finite-Einstein-energy Cauchy data has infinitely many fixed points, as well as periodic points of all orders. There are also $\Phi \in \mathbf{H}$ such that $S^n\Phi$ is almost periodic but not periodic, and $\Phi \in \mathbf{H}$ such that $S^n\Phi$ is not almost periodic. We also prove that $\mathbf{H}$ admits no conformally invariant Kähler metrics but infinitely many distinct Kähler metrics invariant under the Poincaré group and scale transformations. Moreover, we prove that time evolution for these nonlinear wave equations is completely integrable on the space $\mathbf{H}$.

I. INTRODUCTION

For most relativistic nonlinear wave equations the construction of a scattering operator defined on a physically natural space of solutions presently requires quite careful arguments. For conformally invariant nonlinear wave equations, however, the scattering operator arises rather simply in terms of the conformal embedding of Minkowski space in the Einstein universe. This facilitates the study of certain questions which are still intractable for nonlinear wave equations that are Poincaré- but not conformally invariant. In this paper we consider massless scalar fields with a quartic interaction, i.e., equations of the form

$$\Box f + q'(f) = 0\,, \tag{1}$$

where q is a non-negative homogeneous polynomial of degree four in the multicomponent scalar field f. We work with the space $\mathbf{H}$ of 'finite-Einstein-energy solutions'. A solution f has finite Einstein energy if the sum of its energy and that of its transform under conformal inversion is finite, or equivalently, if f extends to a finite-energy solution of the corresponding equation on the Einstein universe, as described below. More concretely, a solution f has finite Einstein energy if

$$\int_{x_0=0} \left\{ (1+\frac{r^2}{4})((\nabla f)^2 + \dot{f}^2) - \frac{1}{2}f^2 \right\} d^3x$$

is finite. The advantage of the space $\mathbf{H}$ is that it is preserved by the action of the conformal group, which is not the case for the traditional space of finite-energy solutions.

Some time ago I. Segal pointed out the significance of the physical vacuum in classical field theories as an elliptic fixed point for Minkowski time evolution and hence for the scattering operator[1]. For $\Box f + q'(f) = 0$ with $q \neq 0$ we show that there are infinitely many other fixed points for the scattering operator, as well as periodic points of all orders. The question of the stability of these fixed points and periodic points is closely related to Floquet theory, and using classical results on intervals of instability the presence of 'pseudo-hyperbolic' fixed points can be established, which in turn implies the existence of solutions Φ such that $S^n\Phi$ fails to be almost periodic. This contrasts interestingly with investigations of conformally invariant quantum field theories[2], which suggest (in part on a nonrigorous basis) that the generator of Einstein time evolution should have pure point spectrum, implying almost periodicity of $S^n\Phi$ for all states Φ of the quantized system. The situation is reminiscent of the prevalence of bounded but not almost periodic trajectories in finite-dimensional classical systems whose corresponding quantized Hamiltonians have pure point spectrum (the 'absence of quantum chaos').

A related question concerns the existence of invariant Kähler metrics on the solution manifold of a classical field theory. The solution manifold typically has a natural symplectic structure, and a choice of 'Kählerization' of the solution manifold is closely related to a choice of vacuum state for the quantized theory[3]. The solution manifold of the linear equation $\Box f = 0$, for example, has a flat Kähler metric that is invariant under the conformal group, and states of the corresponding quantized theory may be represented as antiholomorphic functions on the solution manifold via the 'complex wave' or 'Fock-Bargmann-Segal' representation[4]. We show that for $q \neq 0$ the symplectic structure of the solution manifold $\mathbf{H}$ of $\Box f + q'(f) = 0$ has no conformally invariant Kählerizations, but has infinitely many that are Poincaré-invariant and flat. The proof does not rule out the existence of a conformally invariant Kähler metric on a dense open subset of $\mathbf{H}$, nor does it rule out the existence of conformally invariant polarizations that are not Kähler. Thus a manifestly conformally invariant 'geometric quantization' may still be possible[5].

We also show that Minkowski time evolution for $\Box f + q'(f) = 0$ is completely integrable on the space $\mathbf{H}$. This contrasts with the results of Nikolaevskii and Shchur[6] (see also Savvidy[7] and Steeb *et al.*[8]). They embed the manifold of space-independent solutions of $\Box f + q'(f) = 0$, where $f = (f_1, f_2)$ and $q(f) = f_1^2 f_2^2$, in the space of smooth solutions of the Yang-Mills equations. By proving that time evolution on this manifold is nonintegrable, they conclude that the Yang-Mills equations are not integrable. However, these space-independent solutions are not of finite Einstein energy, nor of finite energy in the usual sense, and their relevance to the physics of localized systems is presumably only indirect. There is thus no contradiction: while the complete integrability of some well-known equations in two-dimensional spacetime is rather robust with respect to the space of solutions considered[9], there is no *a priori* reason to expect this in general. Indeed, it is possible that the Yang-Mills equations are completely integrable on a suitable space of solutions modulo gauge transformations. In a separate paper[10] we show using scattering theory that there are infinitely many gauge-invariant conserved quantities for sufficiently regular solutions of these equations.

II. REVIEW

We begin with a review of the Goursat problem approach to scattering theory for conformally invariant nonlinear wave equations[11–15]. Let $\mathbf{M}_0$ denote Minkowski space and let $\widetilde{\mathbf{M}}$ denote the universal cover of the conformal compactification of $\mathbf{M}_0$, which we identify with the 'Einstein universe' $I\!R \times S^3$ and give the coordinates (τ, u). Let V denote a finite-dimensional real Hilbert space and let q be a homogeneous polynomial of degree four on V, bounded below. Let $\mathbf{H} = L^{2,1}(S^3, V) \oplus L^2(S^3, V)$ as a real Hilbert space with inner product given as in Ref. 15, where $L^{2,1}(S^3, V)$ consists of functions which together with their first derivatives lie in $L^2(S^3, V)$. Elements of $\mathbf{H}$, the space of *finite-Einstein-energy Cauchy data*, will be denoted by capital Greek letters, e.g. $\Phi = (\Phi_1, \Phi_2)$. Given $\Phi \in \mathbf{H}$ there is a function $\varphi \colon \widetilde{\mathbf{M}} \to V$, unique modulo functions vanishing a.e., such that

$$(\Box + 1)\varphi + q'(\varphi) = 0 \tag{2}$$

and $(\varphi, \dot{\varphi})|_{\tau=0} = \Phi$, these equations being taken in the sense of distributions. We call such φ a *finite-Einstein-energy solution* of (2) and denote the space of such by $\mathbf{E}_q$, which may be given a Hilbert manifold structure such that the map $\Phi \mapsto \varphi$ is a diffeomorphism.

The group $\widetilde{\mathbf{G}} = \widetilde{SO}(2,4)$ acts as conformal diffeomorphisms of $\widetilde{\mathbf{M}}$, and the subgroup preserving $\mathbf{M}_0$ is the universal cover of $\mathbf{P}$, the extension of the Poincaré group $\mathbf{P}_0$ by scale transformations. The group $\widetilde{\mathbf{G}}$ acts on $\mathbf{E}_q$ via

$$(g\varphi)(x) = \|dg_x^{-1}\| \varphi(g^{-1}(x)) \,, \qquad g \in \widetilde{\mathbf{G}}, \quad x \in \widetilde{\mathbf{M}}, \tag{3}$$

and this action gives rise to an action U_q of $\widetilde{\mathbf{G}}$ as diffeomorphisms of $\mathbf{H}$.

Let $\iota \colon \mathbf{M}_0 \to \widetilde{\mathbf{M}}$ denote the conformal embedding. Given a finite-Einstein-energy solution φ of (2), and defining $f \colon \mathbf{M}_0 \to V$ by $f(x) = p(x)\varphi(\iota(x))$, where p is the conformal factor associated with ι, then (1) holds in the distributional sense, and f is said to be a *finite-Einstein-energy solution* of (1). This correspondence allows one to study scattering of finite-Einstein-energy solutions of (1) in terms of equation (2). The *Minkowski time* evolution subgroup of $\widetilde{\mathbf{G}}$, with generator $\mathbf{T}_0$, acts on $\widetilde{\mathbf{M}}$ in such a way that

$$\exp(t\mathbf{T}_0)\, \iota(x_0, \vec{x}) = \iota(x_0 + t, \vec{x})$$

for all $(x_0, \vec{x}) \in \mathbf{M}_0$. Given $\Phi \in \mathbf{H}$ there exist 'in and out fields' $\Phi_\pm \in \mathbf{H}$ such that

$$\lim_{t \to \pm\infty} \|U_q(\exp(-t\mathbf{T}_0))\Phi - U(\exp(-t\mathbf{T}_0))\Phi_\pm\| = 0 \,,$$

where $U = U_0$. (Note here that $U_q(\exp(-t\mathbf{T}_0))$ evolves a Cauchy datum *forward* by time t due to the presence of g^{-1} in equation (3). Also, while the convergence above was shown in Ref. 14 only for a weaker topology, a similar argument proves convergence in the norm topology of $\mathbf{H}$.) The maps $\Phi \mapsto \Phi_\pm$ are diffeomorphisms. Thus for any given q there is a unique diffeomorphism $S\colon \mathbf{H} \to \mathbf{H}$, the *scattering operator*, such that $S\Phi_- = \Phi_+$.

The boundary of $\mathbf{M}_0$ as embedded in $\widetilde{\mathbf{M}}$ is the union of the 'cones at future and past infinity'

$$C_\pm = \{(\tau, u) \in I\!R \times S^3 : \ \rho = \pm(\pi - \tau)\} ,$$

where ρ is the angle from u to the north pole of S^3. We identify C_- with S^3 by means of the map $(\tau, u) \mapsto u$, but identify C_+ with S^3 by $(\tau, u) \mapsto -u$. Let $\mathbf{H}(C)$, the space of *finite-Einstein-energy Goursat data,* denote $L^{2,1}(S^3, V)$ as a real Hilbert space with inner product given as in Ref. 15. There are diffeomorphisms $W_{q,\pm}\colon \mathbf{H} \to \mathbf{H}(C)$, the *wave transforms,* determined by the property that

$$W_{\pm,q}\Phi = \varphi|_{C_\pm}$$

for all $\Phi \in \mathbf{H}$ such that the corresponding $\varphi \in \mathbf{E}_q$ is continuous. For all $\Phi \in \mathbf{H}$,

$$\Phi_\pm = W_\pm^{-1} W_{\pm,q}\Phi , \tag{4}$$

where $W_\pm = W_{\pm,0}$.

The center of $\widetilde{\mathbf{G}}$ is isomorphic to $Z\!\!Z \times Z\!\!Z_2$, and the generator ζ of the $Z\!\!Z$ factor acts on $\widetilde{\mathbf{M}}$ by

$$\zeta(\tau, u) = (\tau + \pi, -u) .$$

In particular, ζ maps C_- onto C_+, and

$$U_q(\zeta) = W_{+,q}^{-1} W_{-,q} . \tag{5}$$

This gives rise to a close relationship between ζ and scattering. Equation (4) implies that $S = W_+^{-1}\, W_{+,q}\, W_{-,q}^{-1}\, W_-$. Using (5) and the fact that $U(\zeta) = -I$, it follows that

$$S = (W_-^{-1}\, W_{-,q})\; (-U_q(\zeta^{-1}))\; (W_-^{-1}\, W_{-,q})^{-1} . \tag{6}$$

Thus S is conjugate by a diffeomorphism to $-U_q(\zeta^{-1})$.

For $g \in \widetilde{\mathbf{P}}$,

$$(U_q(g)\Phi)_\pm = U(g)\Phi_\pm . \tag{7}$$

It follows that $U(g)\, S = S\, U(g)$ for all $g \in \widetilde{\mathbf{P}}$, which expresses the $\widetilde{\mathbf{P}}$-invariance of the scattering operator.

III. STATEMENT OF RESULTS

Recall that given a Banach space $\mathbf{X}$, a sequence $\{x_n\}_{n\in\mathbb{Z}}$ in $\mathbf{X}$ is said to be *almost periodic* if the closure in $L^\infty(\mathbb{Z},\mathbf{X})$ of the set of translates of the sequence $\{x_n\}$ is compact. (For equivalent definitions see Ref. 16.) Given a homeomorphism $F\colon\mathbf{X}\to\mathbf{X}$, the point $x\in\mathbf{X}$ is said to be *almost periodic for* F if the sequence $\{F^n x\}$ is almost periodic, and *weakly almost periodic for* F if $\{\lambda(F^n x)\}$ is almost periodic for every $\lambda\in\mathbf{X}^*$. A point $x\in\mathbf{X}$ is said to be *periodic of order* n *for* F if n is the least positive integer such that $F^n(x)=x$. Also, we say that two elements of $\mathbf{H}$ are *equivalent* if for some $g\in\widetilde{\mathbf{P}}$, $U(g)$ maps one into the other. We shall prove:

Theorem 1. *Assume $q\neq 0$. Then there exists a family $\Psi_a\in\mathbf{H}$, $a\geq 0$, such that: 1. If $a\neq b$ then Ψ_a and Ψ_b are inequivalent. 2. For each $n\geq 1$ there are infinitely many values of a such that Ψ_a is a periodic point of order n for S. 3. For uncountably many values of a, Ψ_a is almost periodic but not periodic for S. 4. For infinitely many values of a, every neighborhood of Ψ_a in $\mathbf{H}$ contains a point that is not weakly almost periodic for S.*

Next we recall some facts from Ref. 15 about the geometry of $\mathbf{H}$. Given a non-negative integer n, let $\mathbf{H}_n$ denote the space of C^n vectors for the action U of $\widetilde{\mathbf{G}}$ on $\mathbf{H}$. $\mathbf{H}_n$ has a natural Banach space structure and is densely embedded in $\mathbf{H}$. In fact, the elements of $\mathbf{H}_n$ are precisely the C^n vectors for U_q regardless of the value of q, and for any q the action U_q restricts to an action of $\widetilde{\mathbf{G}}$ as diffeomorphisms of $\mathbf{H}_n$. The space $\mathbf{H}_n$ has a symplectic form

$$\omega(\Phi,\Psi)=\int_{S^3}\Phi_1\cdot\Psi_2-\Phi_2\cdot\Psi_1\,,$$

where Φ,Ψ are tangent vectors identified with vectors in $\mathbf{H}_n$. (We shall use the definitions of symplectic forms, Riemann metrics, and Kähler metrics that require only weak nondegeneracy, where a bilinear form A is *weakly nondegenerate* if for every $u\neq 0$ there exists v such that $A(u,v)\neq 0$.) The symplectic form ω is invariant under the action U_q of $\widetilde{\mathbf{G}}$ for any q. By the following theorem ω has infinitely many distinct extensions to a $\widetilde{\mathbf{P}}$-invariant Kähler metric but no extensions to a $\widetilde{\mathbf{G}}$-invariant Kähler metric:

Theorem 2. *Assume $q\neq 0$ and let n be a non-negative integer. Let N be any $U_q(\zeta)$-invariant open neighborhood of $0\in\mathbf{H}_n$. Then there does not exist a Riemann metric on N invariant under the action of $U_q(\zeta)$. There exist infinitely many distinct Kähler metrics on $\mathbf{H}_n$ that are flat, have imaginary part equal to ω, and are invariant under the action*

U_q *of* $\widetilde{\mathbf{P}}$.

Lastly, the Minkowski time evolution for (1) is completely integrable in the following sense. Recall that if $(\mathbf{X},\omega)$ is a symplectic manifold, the subspace $L \subset T_x\mathbf{X}$ is *isotropic* if the restriction of ω to L vanishes.

Theorem 3. *There exists a sequence of real-analytic functions* $F_i\colon \mathbf{H} \to I\!R$ *such that for all* i *and all* t, $U_q(\exp(t\mathbf{T}_0))^* F_i = F_i$. *There exist real-analytic vector fields* v_i *on* $\mathbf{H}$ *such that for all* i,

$$dF_i = \omega(v_i, \cdot)\,.$$

Moreover: 1. For all i *and* j, $[v_i, v_j] = 0$. *2. Generically, i.e., for all* Φ *in a set of second category in* $\mathbf{H}$, *the subspace* $L = \{\Psi \in T_\Phi\mathbf{H}\colon \forall i\; dF_i(\Psi) = 0\}$ *is isotropic in* $T_\Phi\mathbf{H}$.

In short, the functions F_i are a complete set of conserved quantities for equation (1), with pairwise vanishing Poisson brackets.

IV. SPACE-INDEPENDENT SOLUTIONS ON $\widetilde{\mathbf{M}}$

Topological arguments have been used to prove the existence of periodic solutions of a general class of wave equations of the form $(\Box + 1)\varphi + f'(\varphi) = 0$ on $\widetilde{\mathbf{M}}$; unfortunately, the case of f quartic represents a 'critical exponent' to which these methods do not so far apply[17]. Instead of taking an abstract approach, we will exploit the properties of explicit 'space-independent' periodic solutions of $(\Box + 1)\varphi + q'(\varphi) = 0$ on $\widetilde{\mathbf{M}}$ to prove theorems 1 and 2.

We recall that the one-parameter *Einstein time* evolution subgroup of $\widetilde{\mathbf{G}}$, with generator $\mathbf{X}_0$, acts on $\widetilde{\mathbf{M}}$ by

$$\exp(t\mathbf{X}_0)(\tau, u) = (\tau + t, u)\,,$$

where $(\tau, u) \in I\!R \times S^3$. Thus ζ equals $\xi \exp(\pi \mathbf{X}_0)$, where $\xi \in \widetilde{\mathbf{G}}$ acts on $\widetilde{\mathbf{M}}$ by $\xi(\tau, u) = (\tau, -u)$.

Proof of Theorem 1 - We first prove the theorem in the special case $V = I\!R$, $q(\varphi) = \frac{1}{4}\lambda\varphi^4$, where $\lambda > 0$. If the function g_a is defined as the solution of $\ddot{g} + g + \lambda g^3 = 0$ with $g_a(0) = a > 0$ and $\dot{g}_a(0) = 0$, then the function $\varphi_a\colon \widetilde{\mathbf{M}} \to I\!R$ given by $\varphi_a(\tau, u) = g_a(\tau)$ satisfies $(\Box + 1)\varphi_a + \lambda\varphi_a^3 = 0$. The function g_a is easily seen to have period

$$P(a) = 4\int_0^1 \frac{dx}{\sqrt{(1 - x^2) + \lambda a^2(1 - x^4)/2}}\,.$$

The period $P(a)$ is continuous and strictly decreasing in a, with $P(0) = 2\pi$ and $P(a) \to 0$ as $a \to +\infty$. Let $\Phi_a = (a, 0) \in \mathbf{H}$ and let

$$\Psi_a = (\Phi_a)_- \,.$$

To prove statement 1, note that if $U(g)\Psi_a = \Psi_b$ for $g \in \widetilde{\mathbf{P}}$ then $U_q(g)\Phi_a = \Phi_b$ by equation (7). This is impossible if $a \neq b$, as can be seen by inspection of the action of the group $\widetilde{\mathbf{G}}$ on functions of the form φ_a as given in equation (3).

From (6) and the fact that $U_q(\xi)\Phi_a = \Phi_a$ it follows that

$$S^n \Psi_a = \big((-U_q(\zeta^{-1}))^n \, \Phi_a\big)_- = ((-U_q(\exp(-\pi \mathbf{X}_0))^n \, \Phi_a)_- \;. \tag{8}$$

Thus $S^n\Psi_a = \Psi_a$ if and only if $(-U_q(\exp(-\pi\mathbf{X}_0))^n \, \Phi_a = \Phi_a$, and Ψ_a is almost periodic for S if and only if Φ_a is almost periodic for $-U_q(\exp(-\pi\mathbf{X}_0))$. Since for all $a > 0$ the function $U_q(\exp(-t\mathbf{X}_0))\Phi_a$ is periodic in t, it follows that Φ_a is almost periodic for $-U_q(\exp(-\pi\mathbf{X}_0))$. For n even, $(-U_q(\exp(-\pi\mathbf{X}_0))^n\Phi_a = \Phi_a$ if and only if $kP(a) = n\pi$ for some integer k. For n odd, $(-U_q(\exp(-\pi\mathbf{X}_0))^n\Phi_a = \Phi_a$ if and only if $(k+\frac{1}{2})P(a) = n\pi$ for some integer k. Using the fact that P(a) is continuous and strictly decreasing, it follows that for any n there are infinitely many values of a for which Ψ_a is a periodic point of order n for S, and that there are uncountably many values of a for which Ψ_a is almost periodic but not periodic for S. Thus statements 2 and 3 have been shown.

To prove statement 4, we consider small perturbations of the space-independent solutions:

Lemma 1. *Suppose that $a > 0$ and $P(a)/\pi$ is rational. Then $U_q(\exp(-2n\pi\mathbf{X}_0))\Phi_a = \Phi_a$ for some n, and the differential of $U_q(\exp(-2n\pi\mathbf{X}_0))$ is a bounded operator on the tangent space of $\mathbf{H}$ at Φ_a. This operator is of the form $I + K$ with K compact, and decomposes as the direct sum of linear operators T_i on 2-dimensional subspaces of the tangent space of $\mathbf{H}$ at Φ_a. For infinitely many such a one of the summands T_i has a real eigenvalue of modulus less than one.*

Proof - The differential of the Einstein time evolution operator $U_q(\exp(-t\mathbf{X}_0))$ can be computed by solving the variational differential equation as in Ref. 15. Thus if Ψ is a tangent vector at the point $\Phi_a \in \mathbf{H}$ and $\Psi(t) = dU_q(\exp(-t\mathbf{X}_0))\Psi$, then $\Psi(t)$, under its canonical identification with a vector in $\mathbf{H}$, satisfies the integral equation

$$\Psi(t) = e^{tA}\,\Psi + \int_0^t e^{(t-s)A} B(s)\,\Psi(s)\,ds\,, \tag{9}$$

where the skew-adjoint operator A on $\mathbf{H}$ is given by $A(f_1, f_2) = (f_2, -(\Delta+1)f_1)$, and the smooth maps $B(t)\colon \mathbf{H} \to \mathbf{H}$ are given by $B(t)(f_1, f_2) = (0, -3\lambda g_a(t)^2 f_1)$.

One can solve (9) using a time-ordered exponential. Letting $\Psi_{int}(t) = e^{-tA}\Psi(t)$, it follows that

$$\Psi_{int}(t) = \left(I + \sum_{n\geq 1} \int_{0\leq t_1 \leq \cdots \leq t_n \leq t} C(t_n)\cdots C(t_1)\, dt_1 \cdots dt_n \right) \Psi \quad ,$$

where $C(t) = e^{-tA}B(t)e^{tA}$ and the sum is norm-convergent. In fact, since $C(t)$ is compact for all t and norm-continuous as a function of t, the operator $\Psi \mapsto \Psi_{int}(t)$ is of the form $I+K$ with K compact. The spectrum of $(\Delta+1)$ on $L_2(S^3)$ consists of eigenvalues N^2, $N \geq 1$, with finite multiplicity. Thus $e^{2\pi n A} = I$, so $\Psi(2\pi n) = \Psi_{int}(2\pi n)$. It follows that the operator $\Psi \mapsto \Psi(2\pi n)$ is of the form $I+K$ with K compact.

Let f_i be an orthonormal basis of eigenfunctions of $(\Delta+1)$ on $L_2(S^3)$. If $\Psi = (\alpha_1 f_i, \alpha_2 f_i)$ where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $(\Delta+1)f_i = N^2 f_i$, then (9) implies that $\Psi(t) = (y(t)f_i, \dot{y}(t)f_i)$, where $y\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\partial_t^2 y(t) + (N^2 + 3\lambda\, g_a(t)^2)y(t) = 0 \tag{10}$$

with $y(0) = \alpha_1$ and $\dot{y}(0) = \alpha_2$. Let $T_i\colon (\alpha_1, \alpha_2) \mapsto (y(0), \dot{y}(0))$. Then the operator $\Psi \mapsto \Psi(2\pi n)$ is a direct sum of the 2×2 matrices T_i.

Next we use some results from Floquet theory[18]. First, to show that one of the matrices T_i has a real eigenvalue of modulus less than one, it suffices to show that for some integer $N > 1$ equation (10) has unbounded solutions, or that N^2 lies in one of the 'intervals of instability' $(\lambda_{2j-1}, \lambda_{2j})$. We use an explicit formula for the function g_a above to show that these intervals of instability are nonempty. By the definition of the Jacobi elliptic function $\mathrm{cn}(t|m)$ as the solution of the differential equation

$$\partial_t^2\, \mathrm{cn}(t|m) + (1-2m)\, \mathrm{cn}(t|m) + 2m\, \mathrm{cn}^3(t|m) = 0$$

with initial conditions $\mathrm{cn}(t|m) = 1$ and $\partial_t\, \mathrm{cn}(t|m) = 0$, it follows that $g_a(t) = a\, \mathrm{cn}(bt|m)$, where

$$b = \sqrt{c+1}\,,\ m = \frac{c}{2(c+1)}\,,\ c = \lambda a^2\,.$$

By the identity $\mathrm{sn}^2(t|m) + \mathrm{cn}^2(t|m) = 1$, equation (10) can be seen to be equivalent to Lamé's equation:

$$\partial_s^2 y(s) + \big(k^2 - p(p+1)m^2\, \mathrm{sn}^2(s|m)\big)y(s) = 0\,,$$

where $k^2 = (N^2 + 3c)/(c+1)$, $p(p+1) = 12(1 + c^{-1})$, and $s = bt$. This equation is of intrinsic interest in Floquet theory, and appears in the inverse spectral approach to space-periodic solutions of the Kortweg-deVries equation[9]. It is known that none of the intervals of instability of Lamé's equation is empty unless p is an integer. By the definition of p above, this does not occur if $c \geq 1$.

We will treat the endpoints of the intervals of instability for (10) as functions of a. The above remarks imply that the endpoints of the $2j$th interval of instability of (10), $(\lambda_{2j-1}(a), \lambda_{2j}(a))$, satisfy $\lambda_{2j-1}(a) < \lambda_{2j}(a)$ if $c \geq 1$. It follows from Floquet theory that the $\lambda_j(a)$ are continuous functions of a. Moreover, the following bounds hold:

$$|\sqrt{\lambda_{2j-1}(a)} - \frac{4\pi j}{P(a)}| \, , \, |\sqrt{\lambda_{2j}(a)} - \frac{4\pi j}{P(a)}| \leq \epsilon_j(a) \, ,$$

where $\epsilon_j(a) \to 0$ as $j \to \infty$, uniformly in a for $a > 0$ in any compact interval.

For any sufficiently small $P > 0$, there exist a_1, a_2 with $2P(a_1) = P(a_2) = P$, $a_2 > a_1$, and $\lambda a_1^2, \lambda a_2^2 \geq 1$. Choose j large enough that $\epsilon_j(a) \leq 1$ for all $a \in [a_1, a_2]$. Thus we have

$$\sqrt{\lambda_{2j}(a_1)} \leq \frac{2\pi j}{P} + 1 \, ; \quad \sqrt{\lambda_{2j-1}(a_2)} \geq \frac{4\pi j}{P} - 1 \, .$$

By the intermediate value theorem this implies that for P sufficiently small, some integer N lies in $(\sqrt{\lambda_{2j-1}(a)}, \sqrt{\lambda_{2j}(a)})$ for all a in some nonempty open subinterval of $[a_1, a_2]$. Thus infinitely many $a \in [a_1, a_2]$ have $P(a)/\pi$ rational and $N^2 \in (\lambda_{2j-1}(a), \lambda_{2j}(a))$. $\square$

Lemma 2. *For infinitely many values of a, $S^{2n}\Psi_a = \Psi_a$ for some n, and the differential of S^{2n} at Ψ_a is of the form $I + K$ with K compact, and has a real eigenvalue of modulus less than one.*

Proof - If $P(a)/\pi$ is rational, then $S^{2n}\Psi_a = \Psi_a$ for some n. By (8) the differential of S^{2n} at Ψ_a is conjugate to the differential of $U_q(\exp(-2n\pi \mathbf{X}_0))$ at Φ_a. By Lemma 1 this implies that the differential of S^{2n} at Ψ_a is of the form $I + K$ with K compact, and is conjugate to the direct sum of the matrices T_i. Choosing a as in Lemma 1, it follows that the differential of S^{2n} at Φ_a has a real eigenvalue of modulus less than one. $\square$

To prove statement 4 of the theorem in the special case we are considering, choose a as in Lemma 2. By the lemma, for some constant $0 < r < 1$ the spectrum of $(dS^{2n})_{\Psi_a}$ lies off the circle $|z| = r$ and has non-empty intersection with $\{|z| < r\}$, i.e., Ψ_a is an r-pseudo-hyperbolic fixed point for S^{2n}. By the invariant manifold theorem for pseudo-hyperbolic fixed points[19], in any neighborhood of Ψ_a there exists $\Psi' \neq \Psi_a$ such that

$$\lim_{n\to+\infty} \|S^{2n}\Psi' - \Psi_a\| = 0\,.$$

This implies that $\{S^n\Psi'\}$ cannot be weakly almost periodic, as the only convergent weakly almost periodic sequences are constant[16].

Lastly, we make the following observation. Given q on V, let V' be a one-dimensional subspace of V such that the maximum of q on the unit ball of V occurs at the intersection of V' with the unit sphere. Let e be a unit vector in V', and let $\lambda = 4q(e)$. If $\varphi\colon \widetilde{\mathbf{M}} \to I\!R$ is a finite-Einstein-energy solution of $(\Box + 1)\varphi + \lambda\varphi^3 = 0$, then $e\varphi\colon \widetilde{\mathbf{M}} \to V$ is a finite-Einstein-energy solution of (2). This embedding is easily seen to reduce the proof of the theorem to the special case we have considered. $\Box$

Corollary. *Assume that $q \neq 0$ and n is a non-negative integer. Then there exists a family $\Phi_a \in \mathbf{H}_n$, $a \geq 0$, such that: 1. For each $m \geq 1$ there are infinitely many values of a such that Φ_a is a periodic point of order m for $U_q(\zeta)$. 2. For uncountably many values of a, Φ_a is almost periodic but not periodic for $U_q(\zeta)$. 3. For infinitely many values of a, every neighborhood of Φ_a in $\mathbf{H}_n$ contains a point that is not weakly almost periodic for $U_q(\zeta)$.*

Proof - This follows from the proof of Theorem 1. $\Box$

Proof of Theorem 2 - For the first statement it suffices to show that there is no Riemann metric g on N invariant under $U_q(\zeta^{2n}) = U_q(\exp(2n\pi\mathbf{X}_0))$. If there were, the metric on N given by

$$d(x,y) = \inf\{\int_0^1 g(f'(t), f'(t))^{1/2}dt\colon\ f\colon[0,1] \to N \text{ is smooth},\ f(0) = x,\ f(1) = y\}$$

would be continuous in the $\mathbf{H}_n$ topology and invariant under $U_q(\exp(2n\pi\mathbf{X}_0))$. However by the following lemma this would contradict the properties of the family $\{\Phi_a\}$ established in the proof of Theorem 1:

Lemma 3. *Let $\mathbf{X}$ be a topological space with a continuous metric d (not necessarily inducing the topology of $\mathbf{X}$) and let U_t be a continuous action of $I\!R$ on $\mathbf{X}$. Suppose that for integer values of t the map U_t preserves d. Then there cannot exist a family $\Phi_a \in \mathbf{X}$, $a \geq 0$, such that: 1) Φ_a depends continuously on a, 2) Φ_a is periodic for U_t with period $P(a)$, and 3) $P(a)$ is continuous and strictly monotone.*

Proof - Choose $a \geq 0$ such that $P(a)$ is rational and nonzero. For some n, $U_n(\Phi_a) = \Phi_a$. Let $D = \sup_{t\in I\!R} d(\Phi_a, U_t\Phi_a)$; clearly $0 < D < \infty$. Choose b such that $P(b)$ is irrational, and close enough to a that $d(\Phi_a, \Phi_b) \leq D/4$ and $d(\Phi_b, U_t\Phi_b) > 3D/4$ for some $t \in [0, P(b)]$. By Kronecker's theorem, for any $\delta > 0$ there exists an integer m that $|nm \bmod P(b) - t| \leq \delta$. Choosing δ sufficiently small, one obtains m such that $d(\Phi_b, U_{nm}\Phi_b) \geq 3D/4$. Thus we have

$$\begin{aligned} d(U_{nm}\Phi_a, U_{nm}\Phi_b) &= d(\Phi_a, U_{nm}\Phi_b) \\ &\geq d(\Phi_b, U_{nm}\Phi_b) - d(\Phi_a, \Phi_b) \\ &\geq \frac{D}{2}\,, \end{aligned}$$

contradicting the assumption that U_{nm} preserves d, which implies

$$d(U_{nm}\Phi_a, U_{nm}\Phi_b) = d(\Phi_a, \Phi_b) \leq \frac{D}{4}\,. \qquad \square$$

The existence of a flat Kähler structure g^- on $\mathbf{H}_n$ with imaginary part equal to ω and invariant under the action U_q of $\widetilde{\mathbf{P}}$ is shown in Ref. 15 in the special case of a one-component field φ, and the proof generalizes straightforwardly. Since $\zeta \in \widetilde{\mathbf{G}}$ is central it follows that all the flat Kähler structures $U_q(\zeta^k)^* g^-$ are similarly $\widetilde{\mathbf{P}}$-invariant, and since $U_q(\zeta)$ is symplectic they all have imaginary part equal to ω. Since $U_q(\zeta^{2n})$ preserves no Riemann metric they are all distinct. $\square$

V. COMPLETE INTEGRABILITY

We begin by introducing a representation of the space $\mathbf{H}$ in terms of Fourier transforms of finite-Einstein-energy solutions of $\Box f = 0$. Let V^c denote the complexification of V, and let $\mathcal{S}(I\!R^3, V)$ denote the Schwartz space of V-valued functions on $I\!R^3$.

Lemma 4. *Given $\Phi \in \mathbf{H}$, let φ be the solution of $(\Box + 1)\varphi = 0$ on $\widetilde{\mathbf{M}}$ with $(\varphi, \dot\varphi)|_{\tau=0} = \Phi$, and let f be the solution of $\Box f = 0$ on $\mathbf{M}_0$ given by $f(x) = p(x)\varphi(\iota(x))$. Given $\vec{k} \in I\!R^3$ let $k_0 = \|\vec{k}\|$. Then there is a unique function $\widehat{\Phi} \in L^2(I\!R^3, k_0^{-1} d^3k, V^c)$ such that*

$$f(x) = (2\pi)^{-\frac{3}{2}}\, \mathrm{Re} \int \widehat{\Phi}(\vec{k})\, e^{i(k_0 x_0 - \vec{k}\cdot\vec{x})}\ \ k_0^{-1} d^3k\,, \tag{11}$$

The map $\Phi \mapsto \widehat{\Phi}$ is one-to-one and continuous from $\mathbf{H}$ to $L^2(I\!R^3, k_0^{-1} d^3k, V^c)$; denote its range as $\widehat{\mathbf{H}}$, and give $\widehat{\mathbf{H}}$ the norm such that $\|\widehat{\Phi}\|_{\widehat{\mathbf{H}}} = \|\Phi\|_{\mathbf{H}}$. There is a continuous embedding of $C_0^\infty(I\!R^3 - 0, V^c)$ in $\widehat{\mathbf{H}}$.

Proof - If $\Phi \in \mathbf{H}$, f is defined as above, and $\widehat{\Phi}$ is defined by

$$\widehat{\Phi}(\vec{k}) = \int (k_0 f(0,\vec{x}) - i\dot{f}(0,\vec{x}))\, e^{i\vec{k}\cdot\vec{x}}\, d^3x \tag{12}$$

(in the distributional sense), then by results of Ref. 20,

$$\begin{aligned}
\int |\widehat{\Phi}|^2 k_0^{-1} d^3k &= \int_{x_0=0} ((\Delta^{1/4} f)^2 + (\Delta^{-1/4}\dot{f})^2) d^3x \\
&\leq \int \left\{ (1+\frac{r^2}{4})((\nabla f)^2 + \dot{f}^2) - \frac{1}{2} f^2 \right\} d^3x \\
&= \|\Phi\|^2 ,
\end{aligned}$$

so $\Phi \mapsto \widehat{\Phi}$ is continuous from $\mathbf{H}$ to $L^2(I\!\!R^3, k_0^{-1} d^3k, V^c)$. Taking Fourier transforms, (11) follows, which implies that the map $\Phi \mapsto \widehat{\Phi}$ is one-to-one.

If f is a solution of $\Box f = 0$ with $(f, \dot{f})|_{x_0=0} \in \mathcal{S}(I\!\!R^3, V) \oplus \mathcal{S}(I\!\!R^3, V)$, then f is a finite-Einstein-energy solution, and the map $\mathcal{S}(I\!\!R^3, V) \oplus \mathcal{S}(I\!\!R^3, V) \to \mathbf{H}$ is continuous. By (11) this implies that there is a continuous embedding of $C_0^\infty(I\!\!R^3 - 0, V^c)$ in $\widehat{\mathbf{H}}$. $\Box$

Lemma 5. *If Ψ_1, Ψ_2 are tangent vectors at a point of* $\mathbf{H}$, *identified with vectors in* $\mathbf{H}$, *we have*

$$\omega(\Psi_1, \Psi_2) = \mathrm{Im} \int \widehat{\Psi}_1 \cdot \overline{\widehat{\Psi}}_2 \;\; k_0^{-1} d^3k \; .$$

Proof - This is well-known; see for example Ref. 20. $\Box$

Proof of Theorem 3 - By Lemma 4, for any $h \in C_0^\infty(I\!\!R^3, V^c)$ the function $F_h: \mathbf{H} \to I\!\!R$ given by

$$F_h(\Phi) = \int |h \cdot \widehat{\Phi}|^2 \, k_0^{-1} d^3k$$

is real-analytic, and for all t

$$U(\exp(t\mathbf{T}_0))^* F_h = F_h \; ,$$

i.e., F_h is a conserved quantity for the free Minkowski time evolution on $\mathbf{H}$. We obtain analogous conserved quantities for the interacting Minkowski time evolution as follows. Define $F_{h,q}: \mathbf{H} \to I\!\!R$ by

$$F_{h,q}(\Phi) = F_h(\Phi_-) \; ,$$

where $\Phi_- = W_-^{-1} W_{-,q} \Phi$ is the 'in field'. Since F_h is real-analytic and the map $\Phi \mapsto \Phi_-$ is real-analytic[21], it follows that $F_{h,q}$ is real-analytic. Recalling that $U(g)\Phi_- = (U_q(g)\Phi)_-$

for all $\Phi \in \mathbf{H}$ and $g \in \widetilde{\mathbf{P}}$, it follows that for all t,

$$F_{h,q}(U_q(\exp(t\mathbf{T}_0))\Phi) = F_h((U_q(\exp(t\mathbf{T}_0))\Phi)_-) = F_h(U(\exp(t\mathbf{T}_0))\Phi_-) = F_h(\Phi_-)$$
$$= F_{h,q}(\Phi)\ ,$$

so $U_q(\exp(t\mathbf{T}_0))^* F_{h,q} = F_{h,q}$.

Let $\{e_\lambda\}$ be an orthonormal basis of V, and let h_i be a sequence in $C_0^\infty(I\!R^3 - 0, V)$ forming an orthonormal basis of $L^2(I\!R^3, V^c)$, such that each h_i is of the form ke_λ for some λ and some real-valued $k \in C_0^\infty(I\!R^3 - 0)$. Defining $F_i = F_{h_i,q}$, it follows from the above remarks that the F_i are real-analytic and invariant under $U_q(\exp(t\mathbf{T}_0))$, as desired.

Next we construct the corresponding vector fields v_i. Let $R\Phi = \widehat{\Phi_-}$. Since the map $\Phi \mapsto \Phi_-$ is a real-analytic diffeomorphism, the same is true of $R\colon \mathbf{H} \to \widehat{\mathbf{H}}$. A computation using the Fourier transform shows that the map $\widehat{\Phi} \mapsto 2i(h_i \cdot \widehat{\Phi})\overline{h}_i$ is a bounded linear transformation of $\widehat{\mathbf{H}}$. Let $v_i \in T_\Phi\mathbf{H}$ be the tangent vector such that $dR(v_i) = 2i(h_i \cdot R\Phi)\overline{h}_i$. As a function of Φ, v_i defines a real-analytic vector field on $\mathbf{H}$.

Let $\Psi \in T_\Phi\mathbf{H}$. It follows from Lemma 5 and the fact that $\Phi \mapsto \Phi_-$ is symplectic[15] that

$$\omega(v_i, \Psi) = \mathrm{Im} \int dR(v_i) \cdot \overline{dR(\Psi)}\, k_0^{-1} d^3k$$
$$= \mathrm{Im} \int 2i(h_i \cdot R\Phi)\, (\overline{h_i \cdot dR(\Psi)})\, k_0^{-1} d^3k\ .$$

By definition,

$$F_i(\Phi) = \int |h_i \cdot R\Phi|^2\, k_0^{-1} d^3k\ ,$$

so

$$dF_i(\Psi) = 2\,\mathrm{Re} \int (h_i \cdot R\Phi)\, \overline{(h_i \cdot dR(\Psi))}\, k_0^{-1} d^3k = \omega(v_i, \Psi)$$

as claimed.

To prove statement 1 it suffices to show that $\omega(v_i, v_j) = 0$, as follows:

$$\omega(v_i, v_j) = \mathrm{Im} \int dR(v_i) \cdot \overline{dR(v_j)}\, k_0^{-1} d^3k$$
$$= 4\,\mathrm{Im} \int (h_i \cdot R\Phi)\, (\overline{h}_i \cdot h_j)\, (\overline{h_j \cdot R\Phi})\, k_0^{-1} d^3k$$
$$= 0\ ,$$

using the fact that h_i and h_j are both of the form ke_λ with k real-valued.

To prove statement 2 we show that the space $L \subset T_\Phi\mathbf{H}$ is isotropic for $\Phi \in D$, where $D \in \mathbf{H}$ is the set of Φ such that $e_\lambda \cdot R\Phi$ is a.e. nonzero for all λ. (Here and below 'almost everywhere' and all integrals are relative to the measure $k_0^{-1} d^3k$.) Suppose that $\Phi \in D$ and

$\Psi \in T_\Phi \mathbf{H}$. Then $\Psi \in L$, that is, $dF_i(\Psi) = 0$ for all i, if and only if $(e_\lambda \cdot R\Phi)\overline{(e_\lambda \cdot dR(\Psi))}$ is a.e. imaginary for all λ. By the definition of D, given $\Psi_1, \Psi_2 \in L$ we may conclude that $(e_\lambda \cdot dR(\Psi_1))\overline{(e_\lambda \cdot dR(\Psi_2))}$ is a.e. real, hence

$$\omega(\Psi_1, \Psi_2) = \mathrm{Im} \int dR(\Psi_1) \cdot \overline{dR(\Psi_2)}\, k_0^{-1} d^3k = 0\,.$$

Thus L is isotropic.

It remains to show that D is of second category. Since $\Phi \mapsto \Phi_-$ is a diffeomorphism, it suffices to show that for each λ, the set of $\Phi \in \mathbf{H}$ such that $e_\lambda \cdot \widehat{\Phi}$ is a.e. nonzero is of second category. Thus it is enough to show that in the case $V = I\!R$, the set D' of $\Phi \in \mathbf{H}$ such that $\widehat{\Phi}$ is a.e. nonzero is of second category. Note that $D' = \bigcap_{m,n} D_{m,n}$, where $D_{n,m}$ is the the set of $\Phi \in \mathbf{H}$ such that $\widehat{\Phi}$ vanishes on a set of measure less than $1/n$ in the ball of radius m. Thus it suffices to show that each $D_{n,m}$ is open and dense in $\mathbf{H}$. We will show that for all n, m the complement $D^c_{n,m}$ is nowhere dense in $\mathbf{H}$.

First we show that the interior of $D^c_{n,m}$ is empty. Given any $\Phi \in H$, by Lemma 4 there exists $\Psi \in \mathbf{H}$ of arbitrarily small norm such that $\widehat{\Phi} + \widehat{\Psi}$ vanishes on a set of measure less than $1/n$ in the ball of radius m, so that $\Phi + \Psi \in D_{n,m}$.

Next we show that $D^c_{n,m}$ is closed. Suppose that $\Phi_i \in D^c_{n,m}$ and $\Phi_i \to \Phi$ but $\Phi \in D_{n,m}$. Then for each i the set

$$A_i = \{\vec{k} \in I\!R^3 \colon\ \|\vec{k}\| \leq m,\ |\widehat{\Phi}_i| = 0\}$$

has measure greater than or equal to $1/n$, but for some constant $c > 0$ the set

$$A = \{\vec{k} \in I\!R^3 \colon\ \|\vec{k}\| \leq m,\ |\widehat{\Phi}| \leq c\}$$

has measure less than $1/n$, since the ball of radius m has finite measure. We obtain a contradiction as follows. On the one hand $\widehat{\Phi}_i \to \widehat{\Phi}$ in L^2 by Lemma 5, so

$$\int_{A_i} |\widehat{\Phi}|^2 = \int_{A_i} |\widehat{\Phi} - \widehat{\Phi}_i|^2 \to 0\,.$$

On the other hand

$$\int_{A_i} |\widehat{\Phi}|^2 \geq \int_{A_i - A} |\widehat{\Phi}|^2 \geq c^2 \int_{A_i - A} 1\,,$$

and the measure of $A_i - A$ is bounded below by a positive constant. $\Box$

ACKNOWLEDGEMENTS

The author is indebted to I. Segal and Z. Zhou for many discussions, especially concerning invariant Kähler metrics on solution manifolds. He would also like to thank J. Stafney for help with the category argument.

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