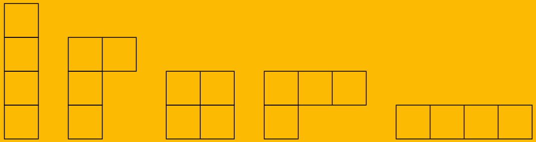
SCHUR FUNCTORS



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Ohio State University
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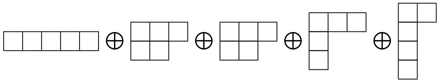

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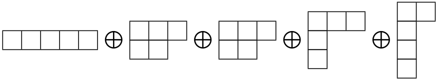
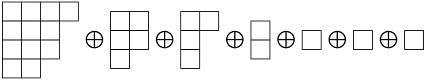


$(n = 5)$

The category **Schur** contains representations of *all* the symmetric groups. A typical object of **Schur**:



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This is a theorem proved by Joe Moeller, Todd Trimble and me.

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The reason: for polynomial $P \in \mathbb{Z}[x]$, we must have

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For example **Vect**, the category of complex vector spaces and linear maps, resembles a ring. Given $V, W \in$ **Vect** we can ‘add’ them and get $V \oplus W$, and ‘multiply’ them and get $V \otimes W$.

Most of the usual ring axioms hold *up to isomorphism*:

$$U \oplus V \cong V \oplus U \qquad (U \oplus V) \oplus W \cong U \oplus (V \oplus W)$$

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

$$U \otimes (V \oplus W) \cong U \otimes V \;\oplus\; U \otimes W$$

$$0 \oplus V \cong V \qquad \mathbb{C} \otimes V \cong V$$

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So, **Vect** is more like a 'rig' than a ring.

A **rig** is a 'ring without negatives': $(R, +, 0, \cdot, 1)$ such that

- $+$ is commutative and associative with unit 0.
- $\cdot$ is associative with unit 1.
- $r \cdot (s + t) = r \cdot s + r \cdot t, \quad (s + t) \cdot r = s \cdot r + t \cdot r$.
- $r \cdot 0 = 0 = 0 \cdot r$.

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- The initial ring is $\mathbb{Z}$: given any ring there exists a unique ring homomorphism $f\colon \mathbb{Z} \to R$.
- The initial rig is $\mathbb{N}$: given any rig there exists a unique rig homomorphism $f\colon \mathbb{N} \to R$.

Indeed:

$\mathbb{N}[x]$ **is the free rig on one generator.**

Given any rig R and element $r \in R$, there exists a unique homomorphism

$$f\colon \mathbb{N}[x] \to R$$

with

$$f(x) = r$$

The reason: for any polynomial $P \in \mathbb{N}[x]$, we have

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- The category **Rep**(G) of representations of a group G with the usual $\oplus$ and $\otimes$.
- The category **Schur** of Schur functors.

All these are 'symmetric' 2-rigs: we have isomorphisms $V \otimes W \cong W \otimes V$ obeying some obvious rules.

Rep(G)

A **representation** ρ of a group G is a (finite-dimensional) vector space V with a linear map $\rho(g) \colon V \to V$ for each $g \in G$, obeying

$$\rho(gh) = \rho(g) \circ \rho(h) \qquad \rho(1) = 1_V$$

A **morphism** of representations from (V, ρ) to (W, ψ) is a linear map $f \colon V \to W$ with

$$f \circ \rho(g) = \psi(g) \circ f \qquad \forall g \in G$$

These are the objects and morphisms of the category **Rep**(G).

Rep(G) is a 2-rig with

$$(\rho \oplus \psi)(g) = \rho(g) \oplus \psi(g)$$

$$(\rho \otimes \psi)(g) = \rho(g) \otimes \psi(g)$$

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In the category **Schur**:

- an object ρ is a list of representations ρ_n of all the groups S_n, where all but finitely many are zero-dimensional;
- a morphism is a list of morphisms of representations.

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A representation of S_k can be thought of as an object ρ of **Schur** that's zero-dimensional except at $k = n$. We get

$$\mathbf{Rep}(S_k) \hookrightarrow \mathbf{Schur}$$

Every object of **Schur** is a finite direct sum of representations of S_k's.

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It's enough to define it on representations of symmetric groups. $\rho \in \mathbf{Rep}(S_k)$ and $\psi \in \mathbf{Rep}(S_\ell)$ give $\rho \boxtimes \psi \in \mathbf{Rep}(S_k \times S_\ell)$ by

$$(\rho \boxtimes \psi)(g,h) = \rho(g) \otimes \psi(h) \qquad g \in S_k, h \in S_\ell$$

From this we then **induce** a representation $\rho_k \otimes \rho_\ell$ of $S_{k+\ell}$, using the inclusion

$$S_k \times S_\ell \hookrightarrow S_{k+\ell}$$

which gives adjoint functors

$$\mathbf{Rep}(S_k \times S_\ell) \underset{\text{restriction}}{\overset{\text{induction}}{\rightleftarrows}} \mathbf{Rep}(S_{k+\ell})$$

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A **symmetric monoidal category** $\mathbf{C}$ is a category with a functor $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, an object $I \in \mathbf{C}$, and natural isomorphisms

$$\alpha \colon (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$

$$\lambda \colon I \otimes V \xrightarrow{\sim} V \qquad\qquad \rho \colon V \otimes I \xrightarrow{\sim} V$$

$$\sigma \colon V \otimes W \xrightarrow{\sim} W \otimes V$$

obeying some well-known equations.

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A **symmetric 2-rig** is a symmetric monoidal linear category that is Cauchy complete.

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$$\varphi\colon\ F(c) \otimes F(c') \xrightarrow{\sim} F(c \otimes c')$$

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Given symmetric 2-rigs $\mathbf{C}$ and $\mathbf{D}$, a **map** $F\colon \mathbf{C} \to \mathbf{D}$ is a symmetric monoidal functor that is also linear.

Theorem (Baez–Moeller–Trimble)

Suppose $\mathbf{R}$ *is a symmetric 2-rig and* $r \in \mathbf{R}$. *Then there is a map of 2-rigs*

$$F\colon \mathbf{Schur} \to \mathbf{R}$$

with

$$F(x) = r$$

where $x \in \mathbf{Schur}$ *is the 1-dimensional representation of* S_1. *Moreover* F *is unique up to a symmetric monoidal natural isomorphism.*

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Suppose $\rho \in \mathbf{Schur}$ corresponds to the list ρ_n of representations of the groups S_n on vector spaces V_n. Then

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This is just like: if $F\colon \mathbb{Z}[x] \to R$ is a homomorphism with $F(x) = r$, then for any $P(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[x]$ we have

$$F(P) = \sum_{n=0}^{\infty} a_n r^n$$

This is just the beginning of a wonderful analogy between rings and 2-rigs, polynomials and Schur functors, etc.

It’s “categorified commutative algebra”.