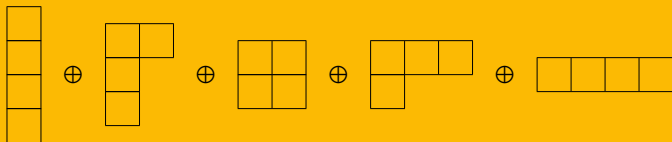


SCHUR FUNCTORS



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I want to explain this sentence:

**The category \mathbf{Schur} is
the free symmetric 2-rig on one generator.**

Joe Moeller, Todd Trimble and I proved this here:

- ▶ [Schur functors and categorified plethysm, arXiv:2106.00190.](#)

Objects of **Schur** are usually called ‘Schur functors’, and we’ll see why. Any object $\rho \in \mathbf{Schur}$ acts functorially on the category of vector spaces... or indeed on any 2-rig.

First try this:

$\mathbb{Z}[x]$ is the free ring on one generator.

This means that given any ring R and element $r \in R$, there exists a unique homomorphism

$$f: \mathbb{Z}[x] \rightarrow R$$

with

$$f(x) = r$$

The reason: for polynomial $P \in \mathbb{Z}[x]$, we must have

$$f(P) = P(r)$$

We will ‘categorify’ this simple result: that is, replace rings by certain *categories* that resemble rings!

For example **Vect**, the category of complex vector spaces and linear maps, resembles a ring.

Given $V, W \in \mathbf{Vect}$ we can ‘add’ them and get $V \oplus W$, and ‘multiply’ them and get $V \otimes W$.

Most of the usual ring axioms hold *up to isomorphism*:

$$U \oplus V \cong V \oplus U \quad (U \oplus V) \oplus W \cong U \oplus (V \oplus W)$$

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

$$U \otimes (V \oplus W) \cong U \otimes V \oplus U \otimes W$$

$$0 \oplus V \cong V \quad \mathbb{C} \otimes V \cong V$$

We could try to define a ‘2-ring’ to be a category that resembles a ring. But there’s no such thing as the *negative* of vector space!

$$V \oplus W \cong \{0\} \Rightarrow V, W \cong \{0\}$$

So, **Vect** is more like a ‘rig’ than a ring.

A **rig** is a ‘ring without negatives’: $(R, +, 0, \cdot, 1)$ such that

- ▶ $+$ is commutative and associative with unit 0 .
- ▶ \cdot is associative with unit 1 .
- ▶ $r \cdot (s + t) = r \cdot s + r \cdot t$, $(s + t) \cdot r = s \cdot r + t \cdot r$.
- ▶ $r \cdot 0 = 0 = 0 \cdot r$.

The rig of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ plays a role in rig theory that \mathbb{Z} plays in ring theory.

- ▶ The initial ring is \mathbb{Z} : given any ring there exists a unique ring homomorphism $f: \mathbb{Z} \rightarrow R$.
- ▶ The initial rig is \mathbb{N} : given any rig there exists a unique rig homomorphism $f: \mathbb{N} \rightarrow R$.

Indeed:

$\mathbb{N}[x]$ is the free rig on one generator.

Given any rig R and element $r \in R$, there exists a unique homomorphism

$$f: \mathbb{N}[x] \rightarrow R$$

with

$$f(x) = r$$

The reason: for any polynomial $P \in \mathbb{N}[x]$, we have

$$f(P) = P(r)$$

There are different concepts of '2-rig', or categorified rig. Today we want one that handles examples like these:

- ▶ **Vect** with the usual \oplus and \otimes .
- ▶ The category of vector bundles on a manifold with the usual \oplus and \otimes .
- ▶ The category of chain complexes of vector spaces with the usual \oplus and \otimes .
- ▶ The category **Rep**(G) of representations of a group G with the usual \oplus and \otimes .
- ▶ The category **Schur** of Schur functors.

All these are 'symmetric' 2-rigs: we have isomorphisms $V \otimes W \cong W \otimes V$ obeying some obvious rules.

$\mathbf{Rep}(G)$

A **representation** ρ of a group G is a (finite-dimensional) vector space V with a linear map $\rho(g): V \rightarrow V$ for each $g \in G$, obeying

$$\rho(gh) = \rho(g) \circ \rho(h) \quad \rho(1) = 1_V$$

A **morphism** of representations from (V, ρ) to (W, ψ) is a linear map $f: V \rightarrow W$ with

$$f \circ \rho(g) = \psi(g) \circ f \quad \forall g \in G$$

These are the objects and morphisms of the category $\mathbf{Rep}(G)$.

$\mathbf{Rep}(G)$ is a 2-rig with

$$(\rho \oplus \psi)(g) = \rho(g) \oplus \psi(g)$$

$$(\rho \otimes \psi)(g) = \rho(g) \otimes \psi(g)$$

The category **Schur** combines the categories **Rep**(S_n) for all $n \in \mathbb{N}$ in one big category!

In the category **Schur**:

- ▶ an object ρ is a list of representations ρ_n of all the groups S_n , where all but finitely many are zero-dimensional;
- ▶ a morphism is a list of morphisms of representations.

In **Schur**, \oplus is defined componentwise:

$$(\rho \oplus \psi)_n = \rho_n \oplus \psi_n$$

A representation of S_k can be thought of as an object ρ of **Schur** that's zero-dimensional except at $k = n$. We get

$$\mathbf{Rep}(S_k) \hookrightarrow \mathbf{Schur}$$

Every object of **Schur** is a finite direct sum of representations of S_k 's.

In **Schur**, the tensor product \otimes is *not* defined componentwise!

It's enough to define it on representations of symmetric groups.

$\rho \in \mathbf{Rep}(S_k)$ and $\psi \in \mathbf{Rep}(S_\ell)$ give $\rho \boxtimes \psi \in \mathbf{Rep}(S_k \times S_\ell)$ by

$$(\rho \boxtimes \psi)(g, h) = \rho(g) \otimes \psi(h) \quad g \in S_k, h \in S_\ell$$

From this we then **induce** a representation $\rho_k \otimes \rho_\ell$ of $S_{k+\ell}$, using the inclusion

$$S_k \times S_\ell \hookrightarrow S_{k+\ell}$$

which gives adjoint functors

$$\mathbf{Rep}(S_k \times S_\ell) \begin{array}{c} \xrightarrow{\text{induction}} \\ \xleftarrow{\text{restriction}} \end{array} \mathbf{Rep}(S_{k+\ell})$$

What is a symmetric 2-rig, *exactly*, and in what sense is **Schur** the free 2-rig on one generator?

A **symmetric monoidal category** \mathbf{C} is a category with a functor $\otimes: \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, an object $I \in \mathbf{C}$, and natural isomorphisms

$$\alpha: (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$

$$\lambda: I \otimes V \xrightarrow{\sim} V \qquad \rho: V \otimes I \xrightarrow{\sim} V$$

$$\sigma: V \otimes W \xrightarrow{\sim} W \otimes V$$

obeying some **well-known equations**.

A **linear category** is a category where the hom-sets are vector spaces and composition $(f, g) \mapsto f \circ g$ is linear in each argument.

A linear category \mathbf{C} is **Cauchy complete** if

- ▶ \mathbf{C} has biproducts: \oplus
- ▶ \mathbf{C} has a zero object: 0
- ▶ every idempotent morphism has a kernel and cokernel.

A **symmetric monoidal linear category** is a linear category that is symmetric monoidal such that the tensor product of morphisms $(f, g) \mapsto f \otimes g$ is linear in each argument.

A **symmetric 2-rig** is a symmetric monoidal linear category that is Cauchy complete.

To say that **Schur** is the free symmetric 2-rig on one object we also need *maps* between symmetric 2-rigs.

A **symmetric monoidal functor** $F: \mathbf{C} \rightarrow \mathbf{D}$ is a functor between symmetric monoidal categories that preserves tensor products and the unit object up to natural isomorphisms

$$\varphi: F(c) \otimes F(c') \xrightarrow{\sim} F(c \otimes c')$$

$$\varphi_0: F(I) \xrightarrow{\sim} I$$

obeying some **well-known equations**.

A **linear functor** is a functor between linear categories that is linear on hom-sets: $F(cf + c'f') = cF(f) + c'F(f')$.

Given symmetric 2-rigs \mathbf{C} and \mathbf{D} , a **map** $F: \mathbf{C} \rightarrow \mathbf{D}$ is a symmetric monoidal functor that is also linear.

Theorem (Baez–Moeller–Trimble)

Suppose \mathbf{R} is a symmetric 2-rig and $r \in \mathbf{R}$. Then there is a map of 2-rigs

$$F: \mathbf{Schur} \rightarrow \mathbf{R}$$

with

$$F(x) = r$$

where $x \in \mathbf{Schur}$ is the 1-dimensional representation of S_1 . Moreover F is unique up to a *symmetric monoidal natural isomorphism*.

We know that $F(x) = r$ for our chosen object $r \in \mathbf{R}$, but what is $F(\rho)$ for *any* object $\rho \in \mathbf{Schur}$?

Suppose $\rho \in \mathbf{Schur}$ corresponds to the list ρ_n of representations of the groups S_n on vector spaces V_n . Then

$$F(\rho) = \bigoplus_{n=0}^{\infty} V_n \otimes_{\mathbb{C}[S_n]} r^{\otimes n}$$

This is just like: if $F: \mathbb{Z}[x] \rightarrow R$ is a homomorphism with $F(x) = r$, then for any $P(x) = \sum_n a_n x^n \in \mathbb{Z}[x]$ we have

$$F(P) = \sum_{n=0}^{\infty} a_n r^n$$

So, we see that given $\rho \in \mathbf{Schur}$ and any object in any 2-rig — for example a vector bundle, or a group representation, etc. — we get a *new* object in that 2-rig. Thus objects of \mathbf{Schur} act on 2-rigs... and they do so *functorially*, so people call them **Schur functors**.

This is just the beginning of a wonderful analogy between rings and 2-rigs, polynomials and Schur functors, etc.

It's part of “categorified commutative algebra”.