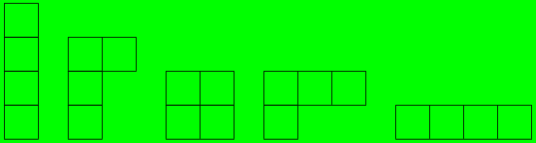
YOUNG DIAGRAMS



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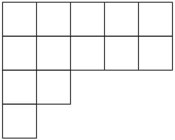
A **Young diagram** is a finite sequence of natural numbers $n_1 \geq n_2 \geq \cdots \geq n_k > 0$.

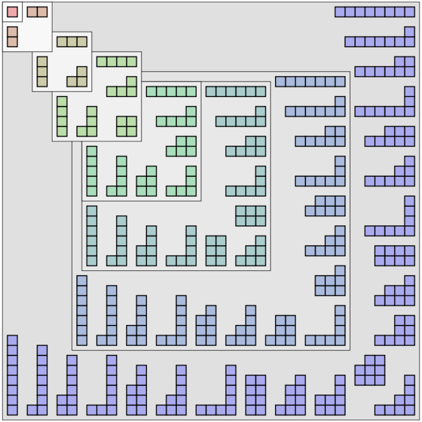
k is the number of **rows** in the Young diagram, n_1 is the number of **columns**, and $n = \sum_i n_i$ is the number of **boxes**.



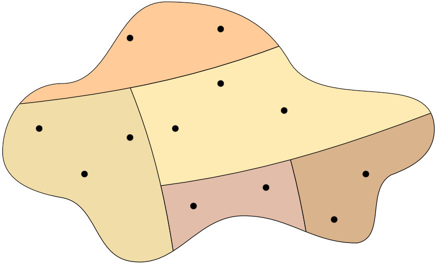
$$4 \geq 3 \geq 2 \geq 2 \geq 2$$

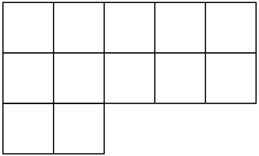
The number of *n*-box Young diagrams is the **partition number *p*(*n*)**:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, \ldots$$

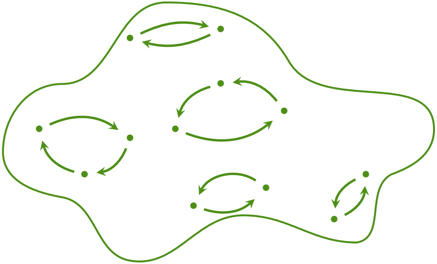


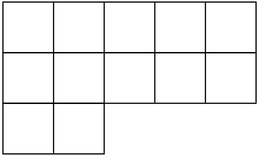
- Young diagrams with n boxes classify partitions of n-element sets up to isomorphism.





- Young diagrams with n boxes classify conjugacy classes in S_n.





A **representation** of a group G on the complex vector space V is a group homomorphism

$$\rho\colon G \to \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ is the group of invertible linear maps from V to itself.

This representation is **irreducible** if the only subspaces of V that are preserved by all the maps $\rho(g)\colon V \to V$ are $\{0\}$ and V. An irreducible representation is called an **irrep**.

Every finite group has as many *conjugacy classes* as it has isomorphism classes of *irreps*. But usually there's no specific best 1-1 correspondence!

But for the symmetric groups S_n, there's a known bijection between conjugacy classes and irreps.

So:

- Young diagrams with n boxes classify irreps of S_n.

Also:

- Young diagrams with $< k$ rows classify finite-dimensional irreps of

$$\mathrm{SL}(k,\mathbb{C}) = \{g \in \mathrm{GL}(\mathbb{C}^k) : \det(g) = 1\}.$$

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And much more! I'll talk about a category **Schur** which consists of *all* finite-dimensional representations of *all* symmetric groups. In this category every object is a direct sum of representations coming from Young diagrams.