Some thoughts on the number 6

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In his book Twelve Geometric Essays, Coxeter has an essay with an ominous title, roughly, "12 points in PG(3, 5) with 95040 self-transformations." In the first two pages of this essay he shows how to see that not all automorphisms of S_6 are inner, by explicitly constructing an outer automorphism. In what follows I recount these first two pages and relate the ideas to the icosahedron. It is very elementary, so finite group experts may be bored to tears, but anyone who has enjoyed the Platonic solids will be able to follow it.

First, buy, borrow or steal, or at least draw, a regular icosahedron! The regular icosahedron is a beautiful thing. It has 20 faces, each an equilateral triangle. Since it is dual to the dodecahedron, it has 12 vertices. (If you don't believe me, count them.) And by Euler's formula,

F - E + V = 2,

it has 30 edges. (Again, I encourage you to count them. In what follows you will need to grok this fact, not just believe it.)

Homework exercise 1: Prove that the regular icosahedron exists.

What? Yes, prove there is a solid with 20 faces all congruent equilateral triangles, five meeting at each vertex! For all you know, it might not exist - the faces might not be able to be precisely equilateral! We wouldn't want to be studying a nonexistent object, after all.

Okay, now think of the regular icosahedron as being centered at the origin. Then each vertex x has an antipodal vertex -x. Draw the line segment from x to -x; let's call it an AXIS. There are 6 axes. If we draw the "top" of the icosahedron, the vertices look like:

and each axis contains one of these 6 vertices as one of its endpoints. We will use this picture to keep track of the axes.

Okay, now let's call a pair of axes a DUAD. Please consult your icosahedron and contemplate a duad. A duad forms the diagonals of a rectangle whose four corners are vertices of the icosahedron.

Exercise 2: Show that this is a golden rectangle, that is, the long side is (1 + sqrt(5))/2 as long as the short side.

Now since a duad is determined by a choice of two axes, there are 6 choose 2 duads, that is, 15 duads. We can easily use duads to count how many symmetries the icosahedron has - rotations, that is, but not reflections. Any symmetry of the icosahedron maps a duad to another duad. We can map a fixed duad to any other duad in four different ways (since a rectangle, hence a duad, has 4-fold symmetry.) If you

think about it, knowing where a fixed duad goes completely determines the rotation of the icosahedron. Thus there are 4 x 15 = 60 symmetries.

Let us call a set of 3 duads, no two having an axis in common, a SYNTHEME. Since there are 6 axes, 2 per duad, a syntheme completely accounts for all the axes. We can call the 3 duads in a syntheme A, B, and C, and draw the syntheme thus:

```
      A

B         B
      A

  C     C
```

(for example). Here we are just showing which of the top 6 vertices lie in which duad. It is infinitely more beautiful, however, to stare at your icosahedron and visualize the 3 interlocking duads in the above syntheme. Each duad determines a golden rectangle, and in the syntheme above the golden rectangles are all perpendicular! Let us call this kind of syntheme a TRUE CROSS. There are 5 true crosses, namely the above one and 4 more obtained by rotating it, e.g.,

```
      B

C         A
      A

  C     B
```

and so on. Note that exercise 2 and the contemplation of the true cross easily allow one to solve:

Exercise 3: Let G = (1 + sqrt(5))/2. Show that the 12 points

```
(+-G, +-1, 0),
(+-1, 0, +-G),
(0, +-G, +-1)
```

are the vertices of a regular icosahedron.

Of course exercise 3 implies exercise 1, but there is an infinitely more simple and charming solution of exercise 1. If you understand how the icosahedron is dual to the dodecahedron, you can easily use exercise 3 to do

Exercise 4: Show that the 20 points

```
(+-1/G, +-G, 0),
(+-G, 0, +-1/G),
(0, +-1/G, +-G)
(+-1, +-1, +-1)
```

are the vertices of a regular dodecahedron.

Now, let's use the fact that there are 5 true crosses to determine the symmetry group of the icosahedron. Any symmetry of the icosahedron (again, not including reflections) determines a permutation of the true crosses, and knowing the permutation allows us to reconstruct the symmetry. It follows that the

symmetry group of the icosahedron is a subgroup of the group of permutations of 5 objects, S_5.

Which permutations can we get? Well, by rotating the pentagon as we were doing above, we clearly get all cyclic permutations. In fact, one can check:

Exercise 5: The symmetries of the icosahedron correspond to the *even* permutations of the 5 true crosses.

Thus the symmetry group of the icosahedron is the group of even permutations of 5 objects, the alternating group A_5. This of course has 5!/2 = 60 elements.

Note that the group A_5 acts as symmetries of the set of 6 axes. This action is transitive, that is, any axis can be mapped to any other. Thus we have an interesting way of thinking of A_5 as a subgroup of S_6.

Exercise 6: Does this inclusion of A_5 in S_6 extend to an inclusion of S_5 in S_6? (See the end of the article for the solution.)

Now, back to synthemes! We have been talking about 5 of them, the true crosses. But there are 5 of a different sort, which look like

```
      A

B         C
      A

  C     B
```

and the rotated versions thereof. By applying other symmetries of the icosahedron to the above one also gets

```
      A

B         C
      A

  B     C
```

and the rotated versions thereof. There are thus 10 synthemes of this sort in all; we'll call such a syntheme a SKEW CROSS. If there is anyone who is taking it easy this summer and wishes to pay me back for all the hard work I put into keeping sci.math and sci.physics interesting (it's a dirty job, but somebody has to do it!), I would be delighted to get some Mathematica code that would print out 2 beautiful pictures of an icosahedron with 1) a true cross, and 2) a skew cross inscribed in it, perhaps with the golden rectangles shaded in gray. I can visualize the true cross quite well, but have some trouble with the skew cross.

Exercise 7: Every syntheme is a true cross or a skew cross.

There are thus 15 synthemes in all. Note that the symmetries of the icosahedron act transitively on the 10 skew crosses, giving us an interesting embedding of A_5 in S_10.

Exercise 8: Every duad lies in 3 synthemes - one true cross and two skew crosses.

Thus we have a nice duality: 15 duads and 15 synthemes, 3 duads in each syntheme, and 3 synthemes containing each duad. In Coxeter's essay he draws a picture of this setup.

There are various ways to pick 5 synthemes such that each duad lies in exactly one of the synthemes. For example, we may choose the 5 true crosses: each duad lies in exactly one of the true crosses. Let us call such a choice of 5 synthemes a PENTAD.

So, one pentad consists of the 5 true crosses:

```
     A               B               C               C               B

B        B      C        A     C        B     B        C      A        C
     A               A               A               A               A

  C    C          C    B         B    A         A    B          B    C
```

Another pentad consists of 5 of the skew crosses, and looks like this:

```
      A               B               C               C               B

 B        B      C        A     B        C     C        B      A        C
      A               A               A               A               A

   C    C          B    C         B    A         A    B          C    B
```

and there are 4 more consisting of rotations of this pentad.

Exercise 9: There are only 6 pentads!

Now any permutation of the 6 axes induces a permutation of the 6 pentads. This defines a homomomorphism from S_6 to itself. As Chris Henrich pointed out, this is the mysterious and sublime OUTER AUTOMORPHISM of S_6. I might as well quote his post, which is infinitely more compact than the above ramblings:

Consider 6 things, e.g. the numerals 123456. There are 15 non-ordered pairs of them. (Call these "duads.") Also there are 15 ways to divide the original set into three duads, e.g. {(12}{34}{56}}. Call these "synthemes." Five synthemes can be chosen so as to contain each duad exactly once; and there turn out to be exactly six ways to make this choice. You can label these six sets ABCDEF. Then a permutation of 123456 induces a permutation of ABCDEF. A two-cycle such as (12) induces a product of three disjoint two-cycles such as (AB)(CD)(EF), so the map from one permutation to the other cannot be an inner automorphism.

Exercise 10: Take the embedding of A_5 in S_6 described above and compose it with the outer automorphism of S_6. What does the image look like as a subgroup of S_6? [I haven't worked it out, but I bet it's the group consisting of even permutations of (123456) which leave the element 1 alone. Here 1,2,3,4,5,6 are the six pentads and 1 is the pentad consisting of the true crosses.]

For more on the icosahedron, and the truncated icosahedron, check out "week79".

Thanks go to Derek Clegg for wondering whether all automorphisms of S_n were inner. Thanks go to

Allen Knutson for saying the answer was yes unless n = 6, and to Dave Sibley for some important clues on the relation to icosahedra. Thanks go to Christopher J. Henrich for summarizing the relevant ideas from Coxeter's essay. Coxeter mentions that he could have titled his essay "Some thoughts on the number 6," so I decided to steal that title for this post. Thanks go to Bradley W. Brock for clarifying the nature of the skew crosses, finding the pentads consisting of skew crosses, and creating exercise 4! Thanks go to Charles Yeomans and Tom R. for some facts that were too sophisticated to find their way into this article. Thanks go to Sylvester for inventing duads and synthemes.

Exercise 6 was solved by Dave Ring:

Does this inclusion of A_5 in S_6 extend to an inclusion of S_5 in S_6?

Yes. Switching the first two true crosses, an odd permutation:

```
        A                B

B          B       C          A
        A                A

    C     C           C     B
```

is generated by switching axes 1 and 5, 2 and 3, 4 and 6, numbering the axes as follows:

```
          1

5           2
          6

    4     3
```

Note that doing this doesn't affect the other three true crosses. The rest of the odd permutations can be reached from this permutation with the known even permutations.