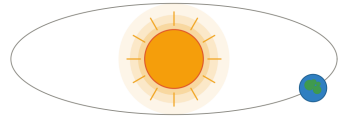
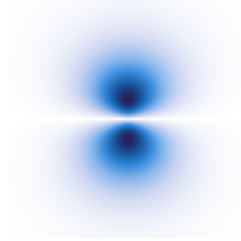
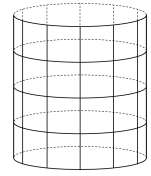
The Solar System,



The Atom,



And The Universe



John Baez

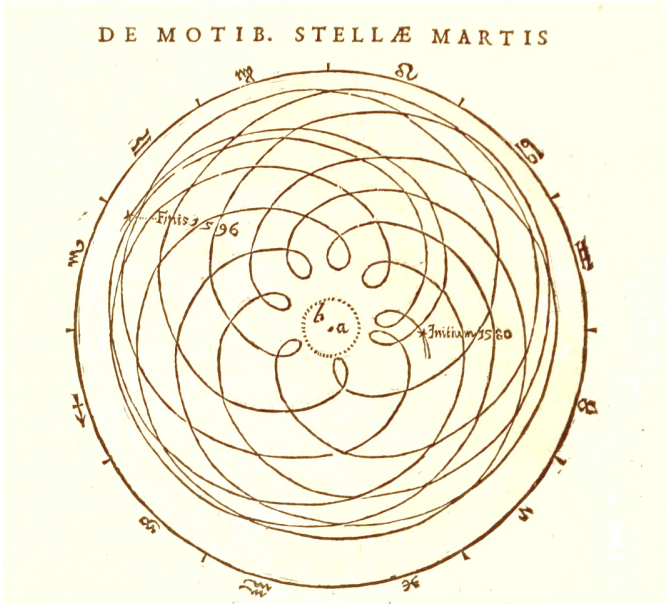
School of Physics and Astronomy, the University of Glasgow

14 October, 2026

The Solar System

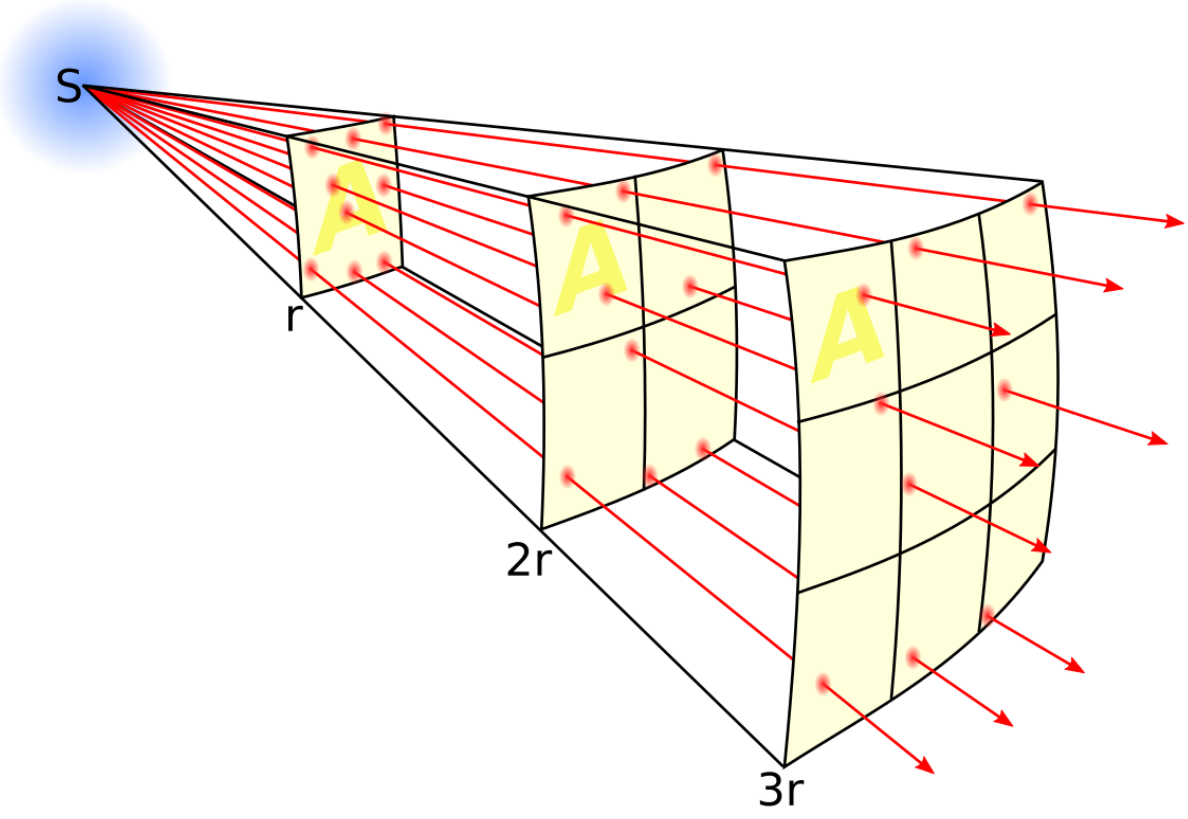
In 1601, Kepler became the imperial mathematician and astrologer in Prague, analyzing data collected by Tycho Brahe.

From Earth the orbits of planets look very complicated:

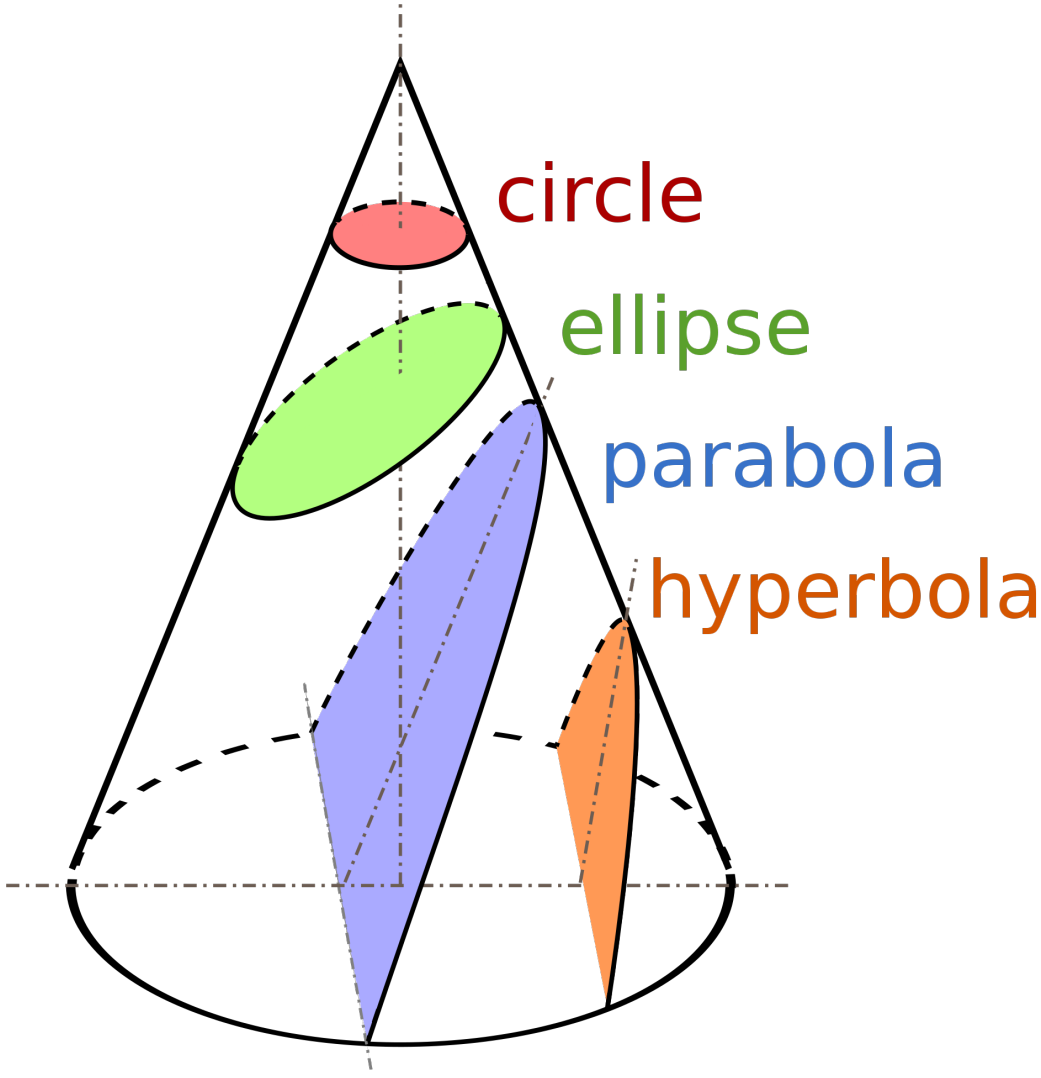


But Kepler believed planets orbit the Sun. And after 4 years of hard calculations, he realized that Mars moves in an elliptical orbit.

In 1680, Robert Hooke suggested to his friend Isaac Newton that gravity obeys an inverse square force law.



By 1686, Newton showed that a object moving in an inverse square force must move in an orbit that's a conic section.



Amazingly, the theory of conic sections had been developed by Apollonius around 200 BC, and developed further by the Arabs around 1000 AD.



This theory was waiting for Kepler and then Newton, who showed planets and comets moving in an inverse square force have orbits that are conic sections!

The **Kepler problem**: how does a particle move in an inverse square force?

The particle has a position $\mathbf{q}(t) \in \mathbb{R}^3$ that obeys Newton's law

$$\mathbf{F} = m\frac{d^2}{dt^2}\mathbf{q}(t)$$

where m is the particle's mass and the force $\mathbf{F}$ is

$$\mathbf{F} = -\frac{k}{\|\mathbf{q}(t)\|^2}$$

for some constant k.

Instead of solving this problem outright, let's look at the conserved quantities!

First, energy is conserved:

$$E = \underbrace{\frac{1}{2} m \|\dot{\mathbf{q}}\|^2}_{\text{kinetic}} - \underbrace{\frac{k}{\|\mathbf{q}(t)\|}}_{\text{potential}}$$

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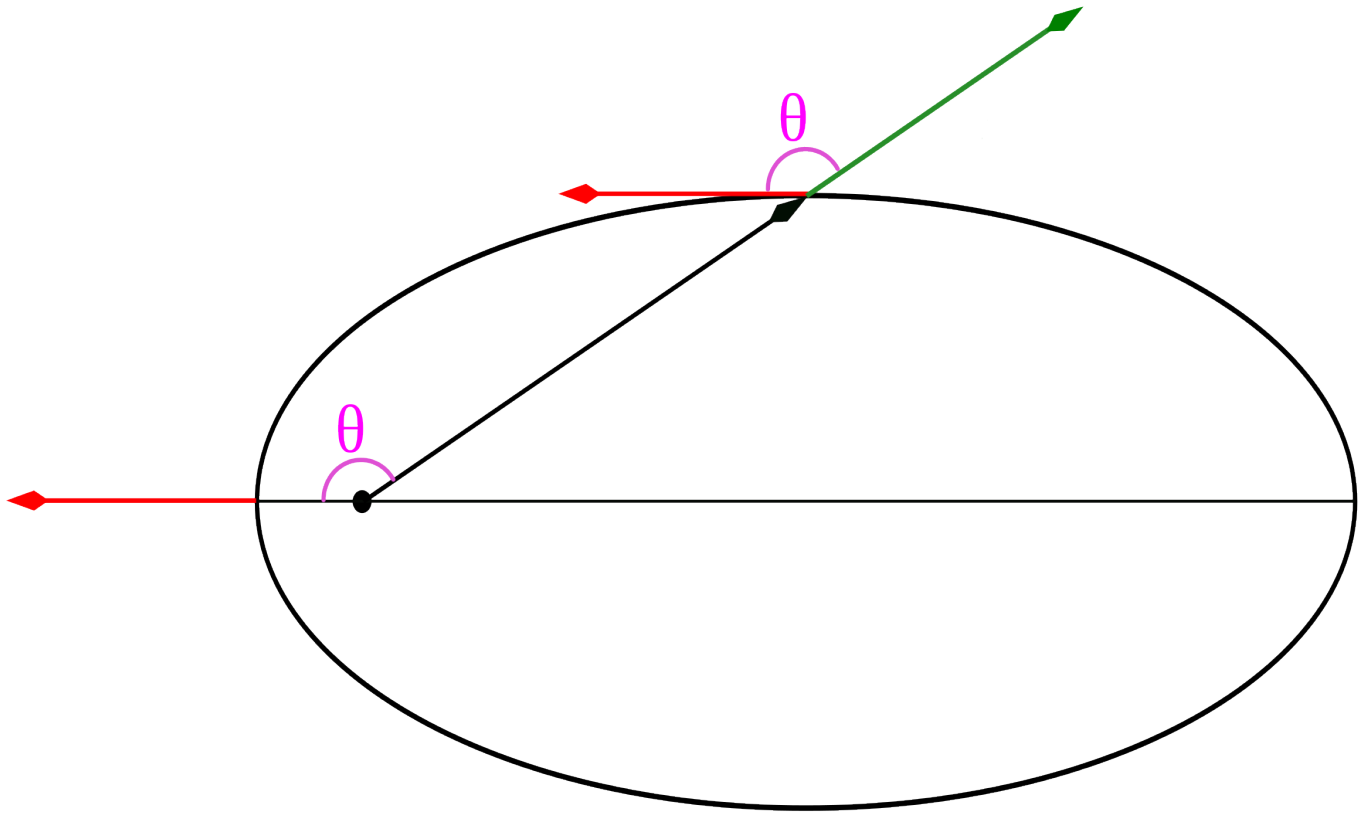
$$E = \underbrace{\frac{1}{2} m \|\dot{\mathbf{q}}\|^2}_{\text{kinetic}} - \underbrace{\frac{k}{\|\mathbf{q}(t)\|}}_{\text{potential}}$$

Second, angular momentum is conserved:

$$\mathbf{L} = \mathbf{q} \times \mathbf{p}$$

where $\mathbf{p} = m\dot{\mathbf{q}}$ is the momentum.

But the Kepler problem also has 3 more conserved quantities, the components of a vector **e**. For a planet in an elliptical orbit, **e** always points toward the perihelion:



and its magnitude is the eccentricity of the ellipse.

This vector **e** was named the ‘Runge–Lenz vector’ after Lenz used it in 1924 to study the hydrogen atom, citing Runge’s popular textbook from five years earlier.

But Runge never claimed any originality: he attributed this vector to Gibbs. Now some call it the ‘Laplace–Runge–Lenz vector’, honoring Laplace’s discussion of it in 1799.

But in fact this vector goes back at least to Jakob Hermann, who wrote about it in 1710, triggering further work by Johann Bernoulli in the same year.

I will call **e** the **eccentricity vector**.

The formula for the eccentricity vector is strange:

$$\mathbf{e} = \mathbf{p} \times \mathbf{L} - mk\frac{\mathbf{q}}{\|\mathbf{q}\|}$$

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You can use the conservation of **L** and **e** to quickly solve the Kepler problem and prove the solutions are conic sections.

But *why* is **e** conserved? Noether's theorem helps us here.



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Time translation symmetry gives conservation of energy, E.

Rotation symmetry gives 3 conserved quantities: the components of angular momentum vector **L**.

But in fact, rotations in *4-dimensional* space act as symmetries on the elliptical orbits of a planet in the inverse square force law!

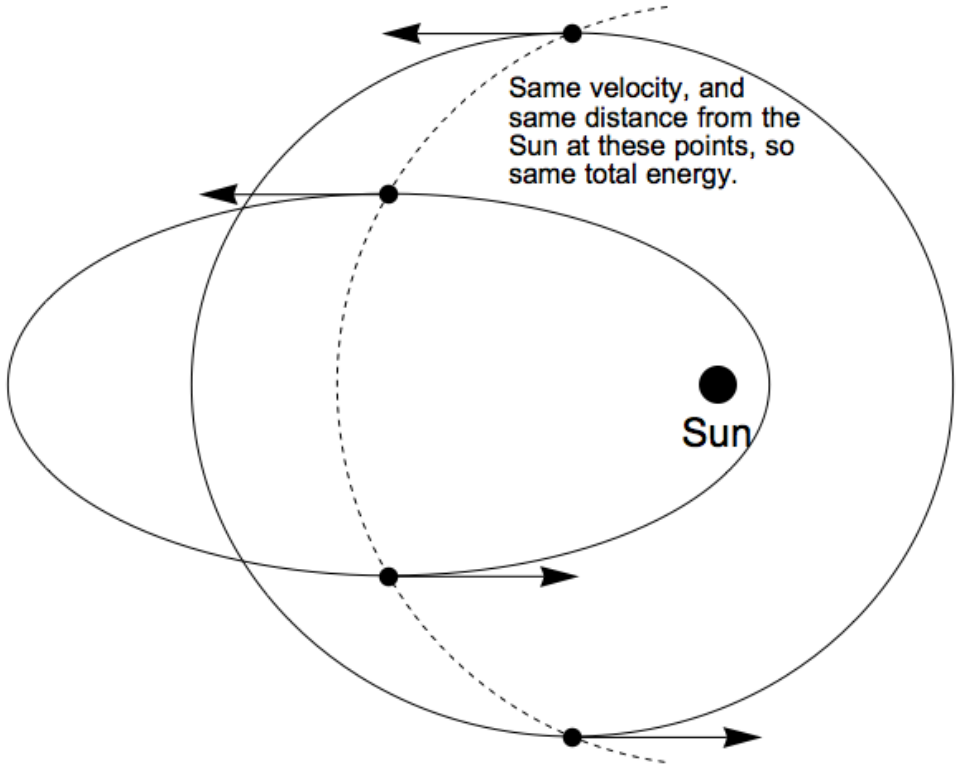
These "hidden symmetries" give 6 conserved quantities: the components of **L** and also the eccentricity vector **e**.

To understand these hidden symmetries, let's go down a dimension and look at the inverse square force law in 2d space — just to make things easier to visualize.

I'll borrow some pictures and ideas from here:

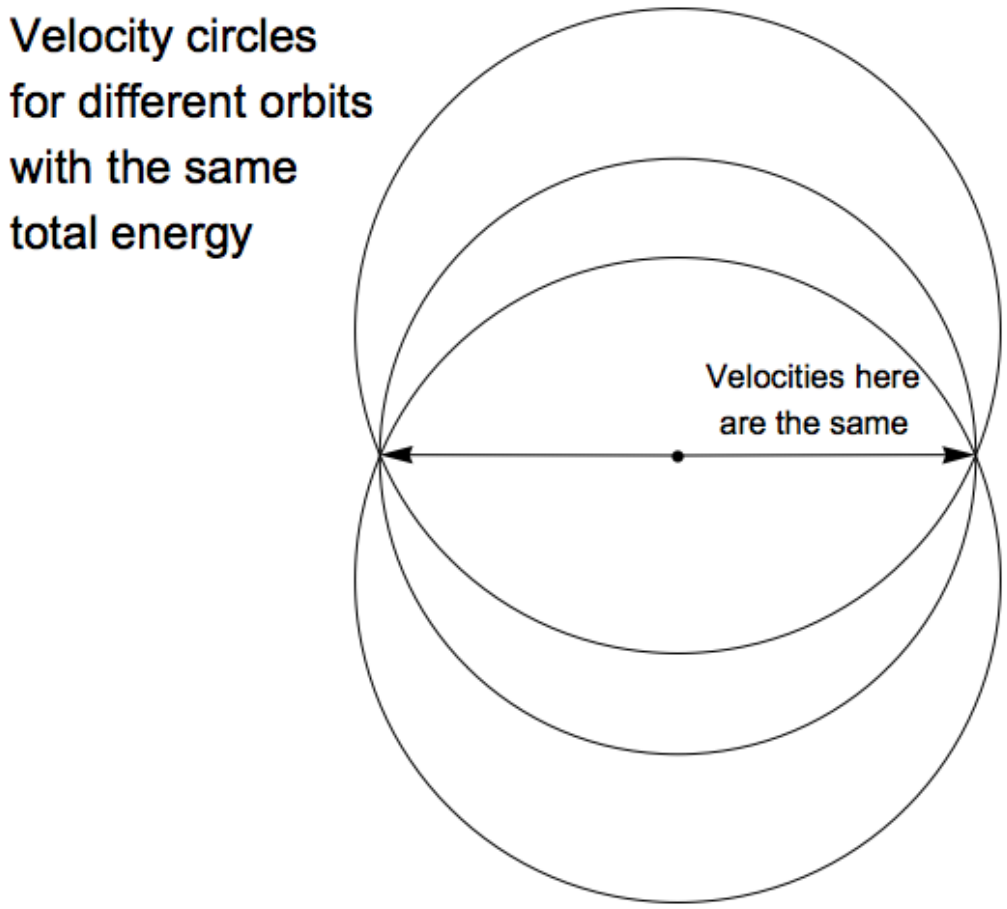
- Greg Egan, The ellipse and the atom.

We can transform any elliptical orbit to others with the same energy by taking the particle at either point where it's travelling parallel to the axis of the ellipse, and swinging it along a circular arc centered on the Sun, while keeping its velocity the same:

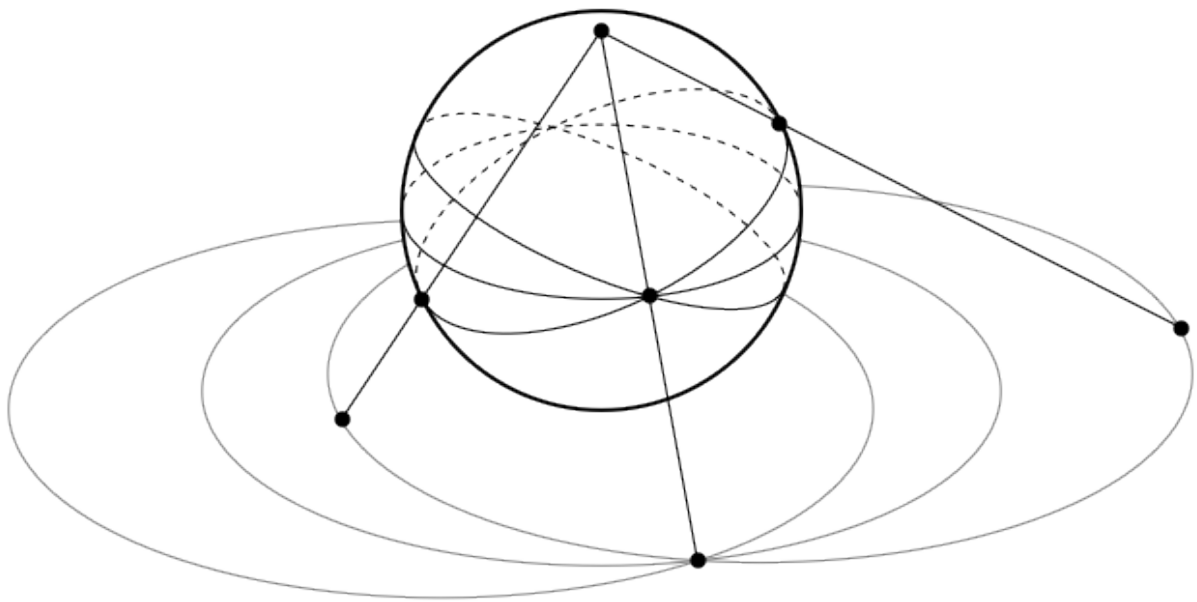


These are some of the hidden symmetries.

In 1846, Hamilton discovered that in the Kepler problem, the velocity of a particle moving in an elliptical orbit always traces out a *circle*. The orbits just discussed give circles like this:



Not every circle in the plane is traced out by the velocity vector of some elliptical orbit. We get precisely those circles that arise from stereographically projecting great circles from some 2-sphere touching the origin:



The hidden symmetries are rotations of this 2-sphere!

All this also works for elliptical orbits for the inverse square force law in 3 dimensions.

The velocity vectors of such elliptical orbits trace out circles in 3d space that are stereographic projections of great circles on a "3-sphere" in 4d space!

The simplest **3-sphere** is

$$S^3 = \{(a, b, c, d) : a^2 + b^2 + c^2 + d^2 = 1\}$$

The hidden symmetries of the Kepler problem come from rotations in 4-dimensional space. These let us rotate the 3-sphere, carrying great circles to great circles.

Quaternions

In 1843, Hamilton discovered a 4-dimensional number system called the **quaternions**:

$$x = a + bi + cj + dk \qquad a, b, c, d \in \mathbb{R}$$

which we can add in the obvious way, and multiply using these rules:

$$i^2 = j^2 = k^2 = -1$$
$$ij = k \qquad jk = i \qquad ki = j$$
$$ji = -k \qquad kj = -i \qquad ik = -j$$

We can define the absolute value of a quaternion by

$$|x| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

Hamilton discovered that quaternion multiplication obeys

$$(xx')x'' = x(x'x'') \qquad |xx'| = |x|\,|x'|$$

Points on the 3-sphere give quaternions with absolute value 1:

$$S^3 = \{(a, b, c, d) : a^2 + b^2 + c^2 + d^2 = 1\}$$

$$\cong \left\{x = a + bi + cj + dk \ : \ |x| = 1\right\}$$

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Since $|xx'| = |x|\,|x'|$, you can multiply two points in S^3 and get another point in S^3.

Any point $x \in S^3$ has an inverse $x^{-1} \in S^3$ with

$$xx^{-1} = x^{-1}x = 1$$

We thus say the 3-sphere forms a **group**. We usually call this group $\mathbf{SU(2)}$.

The Solar System, Revisited

The velocity of a planet in an elliptical orbit gives a point v on the 3-sphere $\mathrm{SU}(2) = S^3$. There are two ways to rotate this point:

- multiply it on the *left* with a quaternion $g \in \mathrm{SU}(2)$, getting $gv \in \mathrm{SU}(2)$.
- multiply it on the *right* with a quaternion $h \in \mathrm{SU}(2)$, getting $vh \in \mathrm{SU}(2)$.

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- multiply it on the *right* with a quaternion $h \in \mathrm{SU}(2)$, getting $vh \in \mathrm{SU}(2)$.

Doing first one and then the other, we get *all* ways to rotate the 3-sphere:

$$v \mapsto gvh$$

That is, we get all rotations in 4-dimensional space from pairs $(g, h) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$.

Remember: the hidden symmetries of the Kepler problem come from 4-dimensional rotations.

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Using just *one* quaternion we can get all rotations in 3 dimensions, by doing this:

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Using just *one* quaternion we can get all rotations in 3 dimensions, by doing this:

$$v \mapsto gvg^{-1}$$

These symmetries give us angular momentum **L** as conserved quantities.

But using *two* we can get all rotations in 4 dimensions

$$v \mapsto gvh$$

These symmetries give us both angular momentum **L** and the eccentricity vector **e**.

The Atom

The electron in a hydrogen atom feels an inverse square force, attracted by the much heavier proton, just as a planet is attracted by the the much heavier Sun!

But to understand the hydrogen atom we need the *quantum* version of the Kepler problem. We'll do the version ignoring the electron's spin and special relativity.

If the energy of a hydrogen atom is ≥ 0, we say it's in a **scattering state**: the electron will shoot away from the proton.

If the energy is < 0, we say the hydrogen atom is in a **bound state**: the electron will orbit the proton.

In suitable units, the allowed energies of bound states are

$$-\frac{1}{n^2} \qquad \text{for } n = 1, 2, 3, 4, \ldots$$

There are n^2 bound states with energy $-1/n^2$. Why?

In quantum mechanics, angular momentum is quantized.

Suppose something has angular momentum at most j along any axis, where $j = 0, \frac{1}{2}, 1, \ldots$. When you measure its angular momentum along some axis, you must get

$$-j, -j+1, \ldots, j-1, j$$

so there are $n = 2j + 1$ states.

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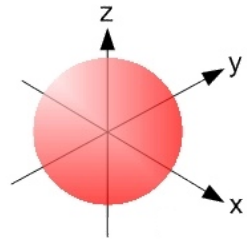
so there are $n = 2j + 1$ states.

But for the hydrogen atom, *both* angular momentum and the eccentricity vector are quantized!

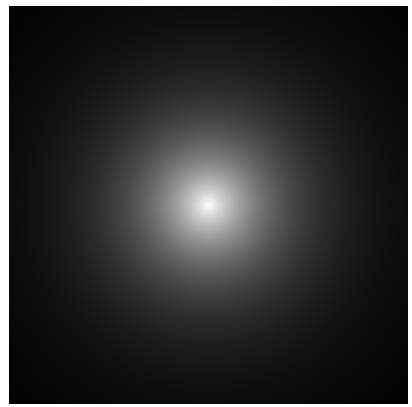
When the atom's energy is $-1/n^2$, *both* can take any value $-j, -j+1, \ldots, j-1, j$ along a chosen axis, so there are n^2 states.

Let's look at examples.

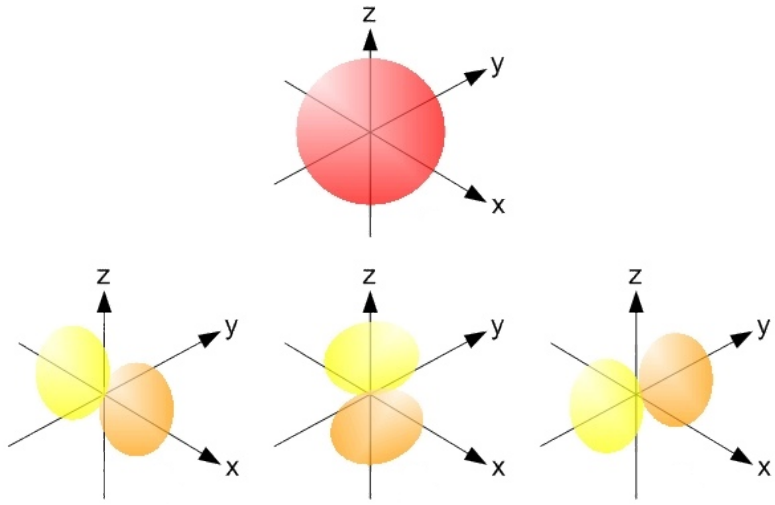
There is just $1^2 = 1$ state of hydrogen with energy $-\frac{1}{1^2}$. It's spherically symmetric:



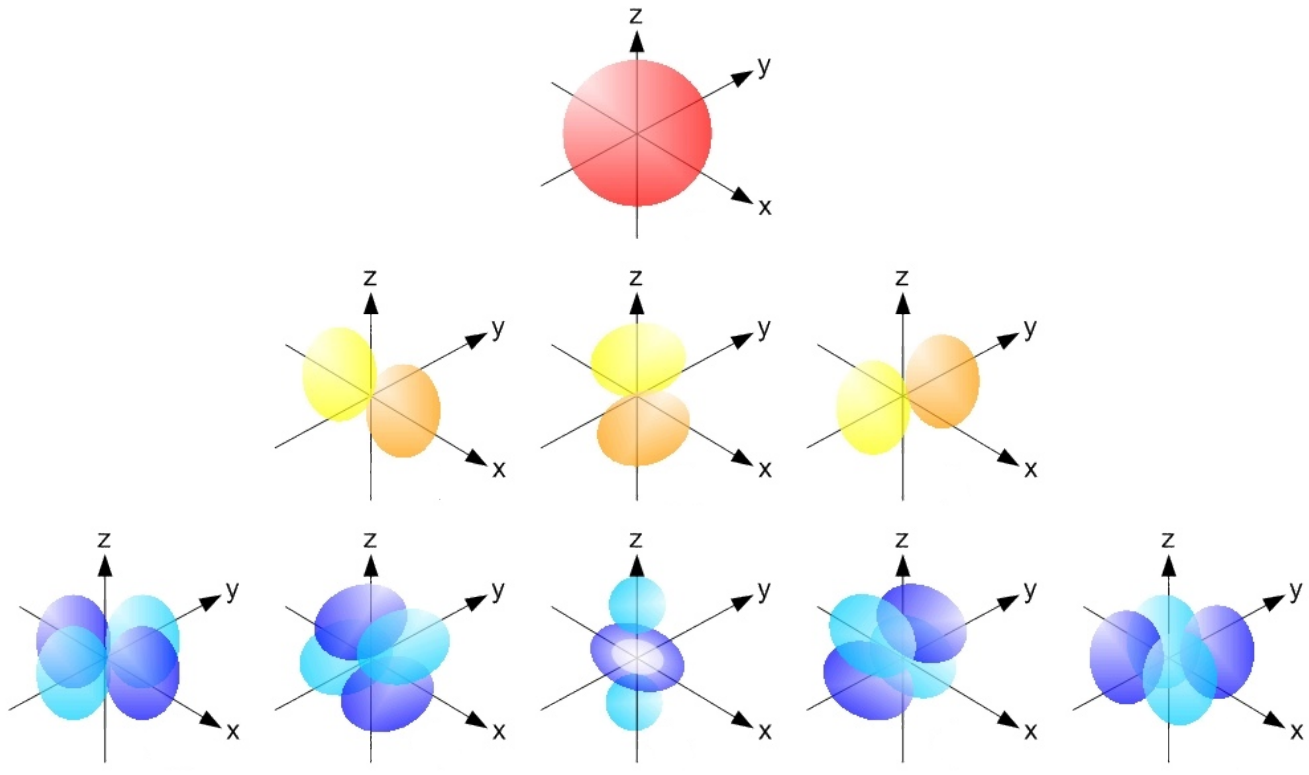
or more accurately:



There are $2^2 = 4$ states of hydrogen with energy $-\frac{1}{2^2}$:



There are $3^2 = 9$ states of hydrogen with energy $-\frac{1}{3^2}$:



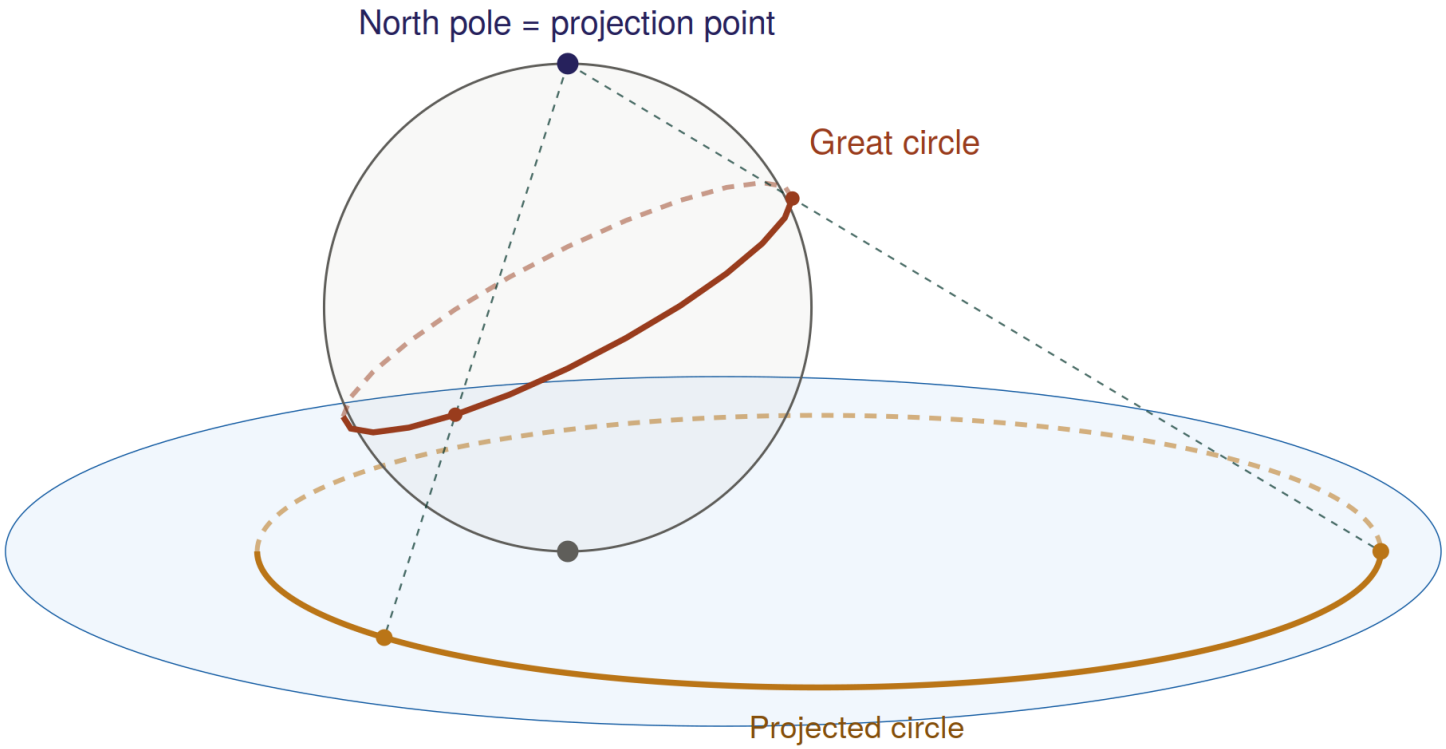
And so on... in general there are

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

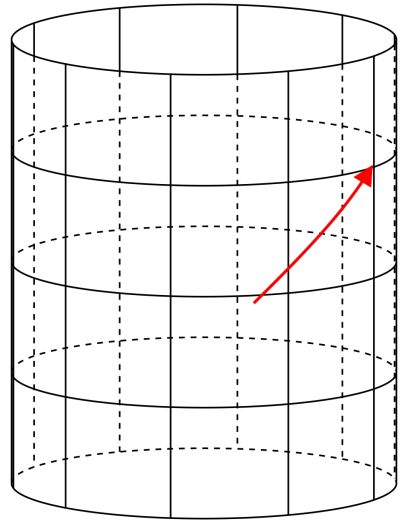
states of energy $-1/n^2$.

The Universe

We saw that the velocity of an elliptical orbit in the Kepler problem, stereographically projected to S^3, traces out a great circle as time passes:



But in 1917, Einstein came up with a theory of the Universe where space is S^3 and time is a line, $\mathbb{R}$.

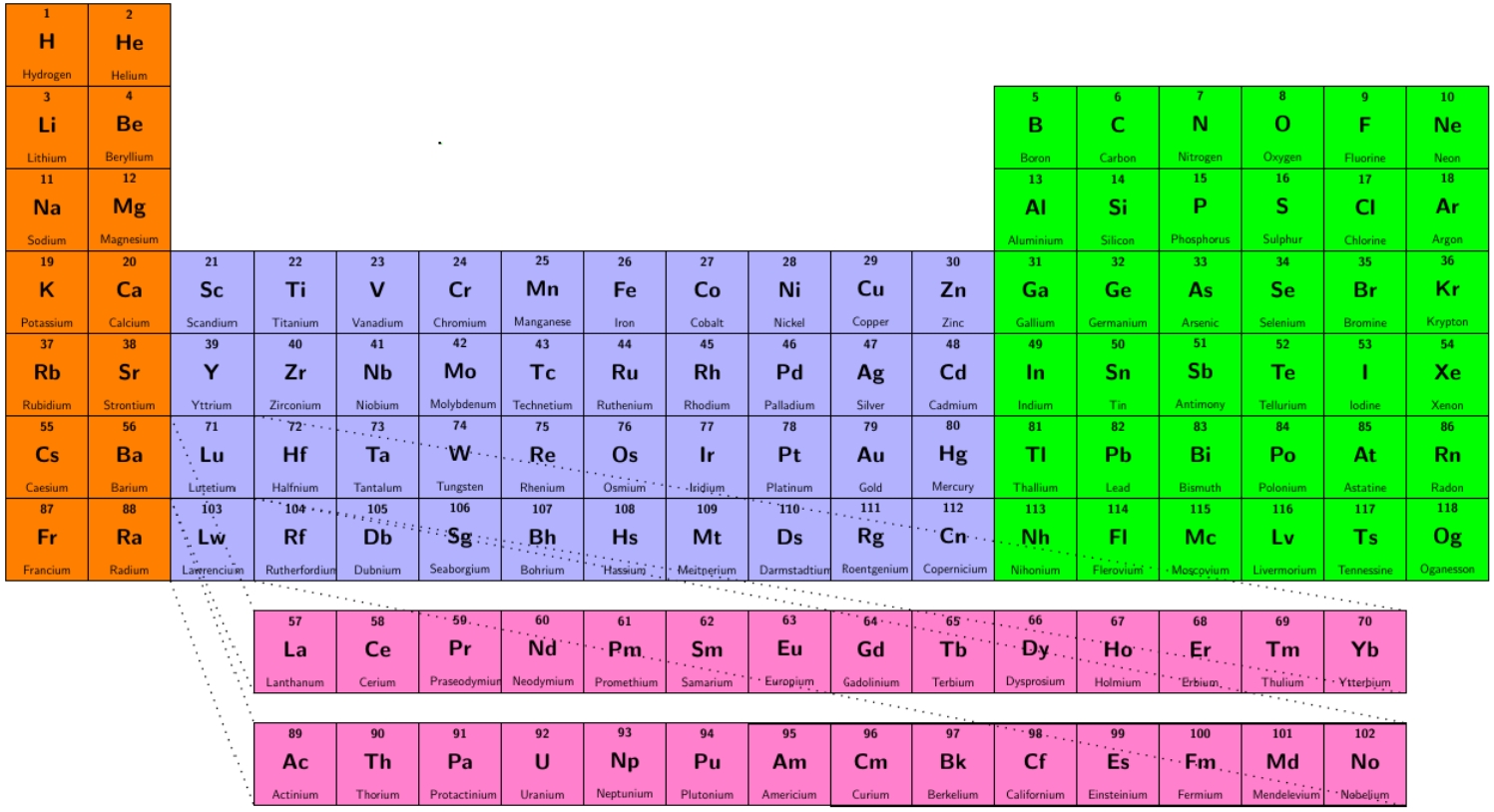


Elliptical orbits in the Kepler problem correspond to particles moving at the speed of light in great circles around the Einstein universe!

And this is true, not only classically, but quantum mechanically!

Quantum field theory on the Einstein universe can describe *collections* of particles moving at the speed of light... which correspond to multi-electron atoms.

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We can even find a quantum field theory that gives a crude version of the periodic table!

There is much more to say, but go here for more:

- John Baez, The Kepler problem, *Azimuth*.

