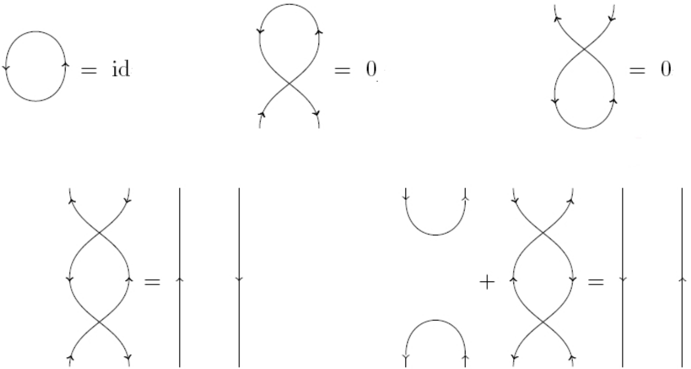
# Spans and the Categorified Heisenberg Algebra

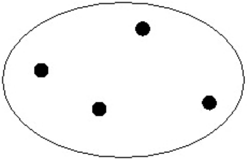
John Baez



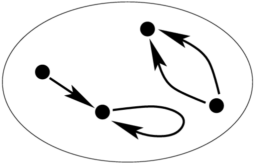

for more, see:

**http://tinyurl.com/baez-spans**

Once upon a time, mathematics was all about *sets*:



In 1945, Eilenberg and Mac Lane introduced *categories*:



Category theory takes *processes* (morphisms):

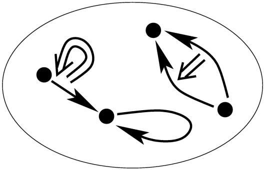
$$\bullet \longrightarrow \bullet$$

just as seriously as *things* (objects):

$$\bullet$$

So, it's obviously good for physics!

In 1967 Bénabou introduced *bicategories*:



These include *processes between processes*, or ‘2-morphisms’:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

This goes on forever... leading to the
**periodic table of *n*-categories.**

# $k$-tuply monoidal $n$-categories

| | $n = \mathbf{0}$ | $n = \mathbf{1}$ | $n = \mathbf{2}$ |
|---|---|---|---|
| $k = \mathbf{0}$ | **sets** | **categories** | **bicategories** |
| $k = \mathbf{1}$ | **monoids** | **monoidal categories** | **monoidal bicategories** |
| $k = \mathbf{2}$ | **commutative monoids** | **braided monoidal categories** | **braided monoidal bicategories** |
| $k = \mathbf{3}$ | ‘’ | **symmetric monoidal categories** | **sylleptic monoidal bicategories** |
| $k = \mathbf{4}$ | ‘’ | ‘’ | **symmetric monoidal bicategories** |
| $k = \mathbf{5}$ | ‘’ | ‘’ | ‘’ |

Let's see what we can do with just bicategories. There's a category $\mathrm{Hilb}$ where:

- objects are (finite-dimensional) Hilbert spaces;
- morphisms are linear operators.

Similarly, there's a bicategory $2\mathrm{Hilb}$ where:

- objects are (finite-dimensional) 2-Hilbert spaces;
- morphisms are linear functors;
- 2-morphisms are natural transformations.

The idea:

$$2\mathrm{Hilb} \text{ is to } \mathrm{Hilb} \text{ as } \mathrm{Hilb} \text{ is to } \mathbb{C}$$

Just as $\mathbb{C}^n$ is a Hilbert space, $\mathrm{Hilb}^n$ is a 2-Hilbert space.

In a 2-Hilbert space, $\oplus$ acts like addition. For example, every object in $\mathrm{Hilb}^n$ is a direct sum of 'basis vectors'

$$e_i = (0, 0, \ldots, 0, \mathbb{C}, 0, \ldots 0, 0)$$

In a 2-Hilbert space, $\hom\colon H^{\mathrm{op}} \times H \to \mathrm{Hilb}$ acts like an inner product. Adjoint functors between 2-Hilbert spaces are like adjoint operators between Hilbert spaces!

More precisely, a **2-Hilbert space** $H$ is a $\mathrm{Hilb}$-enriched abelian $\dagger$-category such that

$$\langle fg, h\rangle = \langle g, f^\dagger h\rangle = \langle f, hg^\dagger\rangle$$

for any triangle of morphisms

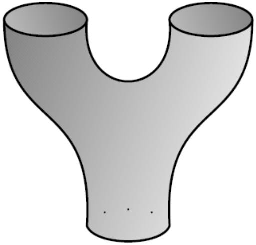
$$\begin{tikzcd} & y \arrow[dr, "f"] & \\ x \arrow[ur, "g"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

A good example is the category of continuous unitary representations of a compact Lie group.

There's a symmetric monoidal category $n\mathrm{Cob}$ where:

- objects are $(n-1)$-dimensional compact oriented smooth manifolds;
- morphisms are cobordisms between these.

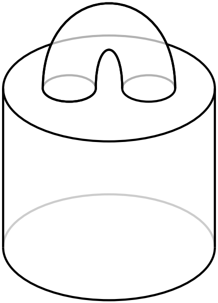
Example of a morphism when $n = 2$:



There's a symmetric monoidal bicategory $n\mathrm{Cob}_2$ where:

- objects are $(n-2)$-dimensional compact oriented smooth manifolds;
- morphisms are cobordisms between these.
- 2-morphisms are cobordisms between those.

Example of a 2-morphism when $n = 3$:



A **TQFT** is a symmetric monoidal functor

$$Z\colon n\mathrm{Cob} \to \mathrm{Hilb}$$

These can be nicely classified when $n = 2$.

A **once extended TQFT** is a symmetric monoidal functor

$$Z\colon n\mathrm{Cob}_2 \to 2\mathrm{Hilb}$$

These can be nicely classified when $n = 3$.

A once extended TQFT in dimension 3 assigns a 2-Hilbert space to the circle

$$Z(S^1) = H$$

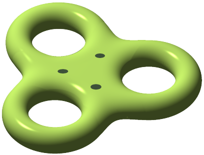
This is the **2-Hilbert space of a particle.**

$H$ is not just a 2-Hilbert space: it's an 'anomaly-free modular tensor category'. This is enough structure for us to completely reconstruct the whole once extended TQFT.

See the forthcoming work of Bartlett, Douglas, Schommer-Pries and Vicary — and previous work by many others.

A simple object $i \in H$ is a type of particle.

An oriented surface $\Sigma$ whose boundary consists of $n$ circles:



is a morphism

$$\Sigma \colon S^1 \cup \cdots \cup S^1 \to \emptyset$$

so it gives a linear functor

$$Z(\Sigma) \colon H^{\otimes n} \to \mathrm{Hilb}$$

For objects $i_1, \ldots i_n \in H$, this functor sends $i_1 \otimes \cdots \otimes i_n$ to the Hilbert space of states for a collection of particles of these types in the space $\Sigma$.

A once extended TQFT for $n = 4$ will, among other things, assign a 2-Hilbert space to the sphere:

$$Z(S^2) = H$$

This is again the **2-Hilbert space of a particle**.

We also get $Z(T^2)$ and other 2-Hilbert spaces for higher genus, which describe 1-dimensional defects ('string networks'), but let us focus on particles.

How do we describe collections of particles? We should categorify Fock space, and the annihilation and creation operators.

Given any Hilbert space $H$, the **Fock space** is the free commutative algebra on $H$:

$$SH = \bigoplus_{n=0}^{\infty} H^{\otimes n}/S_n$$

completed to form a Hilbert space. Here we mod out $H^{\otimes n}$ by the action of the permutation group $S_n$.

For any basis vector $e_i \in H$ there is **creation operator**

$$\begin{array}{rccc} a_i^\dagger : & SH & \to & SH \\ & v_1 \cdots v_n & \mapsto & e_i v_1 \cdots v_n \end{array}$$

and its adjoint, the **annihilation operator** $a_i$. These obey the relations of the **Heisenberg algebra**:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

Given any 2-Hilbert space $H$, we can define a **2-Fock space**

$$SH = \bigoplus_{n=0}^{\infty} H^{\otimes n} // S_n$$

Here we 'weakly mod out' $H^{\otimes n}$ by the action of the permutation group $S_n$, putting in an isomorphism

$$f_\sigma \colon v_1 \otimes \cdots \otimes v_n \longrightarrow v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

for each permutation $\sigma \in S_n$, such that

$$f_\sigma f_{\sigma'} = f_{\sigma\sigma'}$$

The 2-Fock space is an infinite-dimensional 2-Hilbert space.

We can define **creation operators** on the 2-Fock space: for each simple object $i \in H$ there is a linear functor

$$\begin{array}{rccc} a_i^\dagger \colon & SH & \to & SH \\ & v_1 \cdots v_n & \mapsto & e_i \otimes v_1 \otimes \cdots \otimes v_n \end{array}$$

The adjoints of these functors are the **annihilation operators**

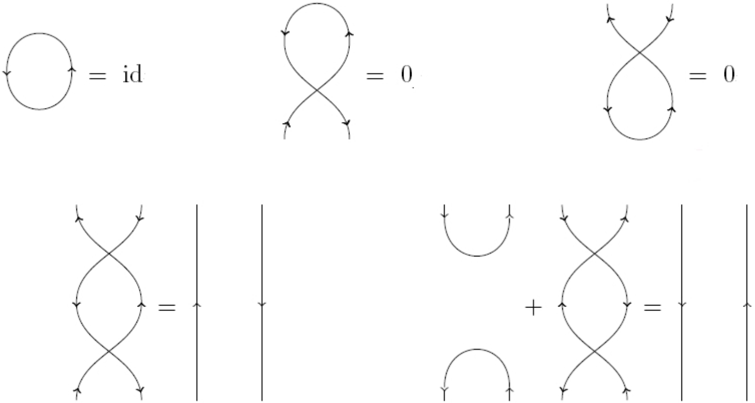
$$a_i \colon SH \to SH$$

These obey the relations of the Heisenberg algebra up to natural isomorphism, e.g.

$$a_i a_j^\dagger \cong a_j^\dagger a_i \oplus \delta_{ij}$$

*But what equations do these isomorphisms obey?*

In 2010, Mikhail Khovanov answered this question. The equations look strange at first sight:




But in 2012, Jeffrey Morton and Jamie Vicary showed they arise from simple ideas about creating and annihilating particles!

The key idea is that $a_i^\dagger$ creates a new particle of type $i$... while $a_i$ is a sum over all ways of *choosing* a particle of type $i$ and annihilating it.

If we consider just one type of particle, then

$$aa^\dagger = a^\dagger a + 1$$

says there is 1 more way to

create a particle and then annihilate one,

than to

annihilate a particle and then create one.

The reason: if you create one *first*, there's 1 more choice of which particle to annihilate!

How can make this idea precise? We use the groupoid of finite sets, **S**:

- an object of **S** is a finite set $s$
- a morphism in **S** is a bijection $\alpha \colon s \to t$.

There is a functor

$$\begin{array}{rccc} +1\colon & \mathbf{S} & \to & \mathbf{S} \\ & s & \mapsto & s+1 \end{array}$$

sending each finite set $s$ to its disjoint union with a chosen one-element set, called 1.

This is the idea behind the creation operator: it adds one element to a finite set of 'particles'.

But what about the annihilation operator? There is no functor $f\colon \mathbf{S} \to \mathbf{S}$ that takes a finite set and *removes* an element.

There are $n$ different ways to remove an element from an $n$-element set. There is no best way to choose one. But we don't want to choose one. We want to consider *all possible choices*.

We can do this using spans of groupoids.

A **span** is a diagram shaped like this:

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

In a span of groupoids, $p\colon S \to X$ and $q\colon S \to Y$ are functors between groupoids.

Any functor $f\colon X \to Y$ gives a span from $X$ to $Y$:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\ Y & & X \end{tikzcd}$$

but we can also turn it around and get a span from $Y$ back to $X$:

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

So, if **S** is the groupoid of finite sets, we have spans called the **annihilation operator** $A$:

$$
\begin{tikzcd}
& \mathbf{S} \arrow[dl, "1_{\mathbf{S}}"'] \arrow[dr, "+1"] & \\
\mathbf{S} & & \mathbf{S}
\end{tikzcd}
$$

and the **creation operator** $A^\dagger$:

$$
\begin{tikzcd}
& \mathbf{S} \arrow[dl, "+1"'] \arrow[dr, "1_{\mathbf{S}}"] & \\
\mathbf{S} & & \mathbf{S}
\end{tikzcd}
$$

We compose spans of groupoids by taking a **weak pullback**:

$$
\begin{tikzcd}
 & & TS \arrow[dl, "\pi_T"'] \arrow[dr, "\pi_S"] & & \\
 & T \arrow[dl, "q_T"'] \arrow[dr, "p_T"] \arrow[rr, phantom, "\overset{\sim}{\Longrightarrow}"] & & S \arrow[dl, "q_S"'] \arrow[dr, "p_S"] & \\
Z & & Y & & X
\end{tikzcd}
$$

$TS$ is the groupoid whose objects are triples

$$\big(t \in T,\ s \in S,\ \alpha\colon p_T(t) \xrightarrow{\sim} q_S(s)\big)$$

The diamond then commutes up to natural isomorphism.

How can we relate $AA^\dagger$ and $A^\dagger A$?

In fact we have an equivalence of spans

$$A^\dagger A + 1 \simeq AA^\dagger$$

But what does this mean, exactly?

First, for any groupoid $X$ there is an **identity span** given by

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & \\ X & & X \end{tikzcd}$$

Second, we can **add** two spans

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}$$

getting a span

$$\begin{tikzcd} & S+T \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}$$

where $S+T$ is the disjoint union of the groupoids $S$ and $T$.

Third, a **map of spans** is a diagram of groupoids and functors like this:

$$
\begin{tikzcd}
& S \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, "f"] & \\
Y \arrow[r, phantom, "\overset{\sim}{\Rightarrow}"] & \phantom{S} \arrow[r, phantom, "\overset{\sim}{\Leftarrow}"] & X \\
& S' \arrow[ul, "q'"] \arrow[ur, "p'"'] &
\end{tikzcd}
$$

where the triangles commute up to chosen natural isomorphisms.

An **equivalence of spans** is a map of spans where $f$ is an equivalence of groupoids.

Since there's 1 more way to *add an element to a finite set and then remove one* than to *remove an element and then add one*, we get an equivalence of spans:

$$f\colon A^\dagger A + 1 \stackrel{\sim}{\Longrightarrow} AA^\dagger$$

where the double arrow denotes a map of spans, and 1 is the identity span from **S** to **S**.

In 2000, James Dolan and I noticed this equivalence of spans

$$f\colon A^\dagger A + 1 \stackrel{\sim}{\Longrightarrow} AA^\dagger$$

and used it to describe the combinatorics of Feynman diagrams using spans of groupoids. Jeffrey Morton developed this further, showing how to include complex numbers.

In 2012, Morton and Vicary showed $f$ obeys certain nontrivial *equations* — precisely the relations in Khovanov's categorified Heisenberg algebra!

To state these relations, we need to go beyond maps of spans. We need **spans of spans**:

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & \\
Y \arrow[r, phantom, "\Downarrow\wr"] & Z \arrow[u] \arrow[d] \arrow[r, phantom, "\Downarrow\wr"] & X \\
 & S' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Just as any functor gives a span of groupoids, any map of spans gives a span of spans.

But a span of spans $Z\colon S \Rightarrow S'$ can be ‘flipped’ to give a span of spans $Z^\dagger\colon S' \Rightarrow S$, just by turning the diagram upside down.

So, the isomorphism of spans

$$f\colon A^\dagger A + 1 \stackrel{\sim}{\Longrightarrow} AA^\dagger$$

gives two ‘inclusions’

$$i\colon A^\dagger A \Longrightarrow AA^\dagger$$

$$j\colon 1 \Longrightarrow AA^\dagger$$

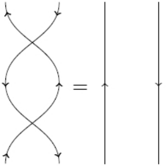
and we can flip these to get ‘projections’

$$i^\dagger\colon AA^\dagger \Longrightarrow A^\dagger A$$

$$j^\dagger\colon AA^\dagger \Longrightarrow 1$$

which are spans of spans. The relations in the categorified Heisenberg algebra involve $i, j, i^\dagger, j^\dagger$.

Here is one of these relations, both as Khovanov drew it and as a commutative triangle where the double arrows are spans of spans:



$$\begin{tikzcd}
& A^\dagger A \arrow[dl, Rightarrow, "i"'] \arrow[dd, Rightarrow, "1_{A^\dagger A}"] \\
AA^\dagger \arrow[dr, Rightarrow, "i^\dagger"'] & \\
& A^\dagger A
\end{tikzcd}$$

What does this mean? We begin with a way to remove one element $x$ from a finite set and then add one element $y$. $i^\dagger$ relates this to a way to add $y$ and then remove $x$. $i$ relates this back to a way where we remove $x$ and add $y$. This gives the identity: we come back to the same ‘history’!

So, we have found a new layer of quantum theory, with:

- 2-Hilbert spaces subsuming Hilbert spaces,
- once extended TQFTs subsuming TQFTs, and
- the categorified Heisenberg algebra subsuming the Heisenberg algebra.

This new layer is just the next of many—presumably infinitely many.

To set this work in a good context, we should prove this:

**Conjecture (Morton and Vicary)**

*There is a symmetric monoidal bicategory with:*

- *groupoids as objects*
- *spans of groupoids as morphisms*
- *spans of spans as 2-morphisms*

Here is a big step toward proving the conjecture:

**Theorem (Alex Hoffnung and Mike Stay)**

*There is a symmetric monoidal bicategory with:*

- *groupoids as objects*
- *spans of groupoids as morphisms*
- *maps of spans as 2-morphisms*

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