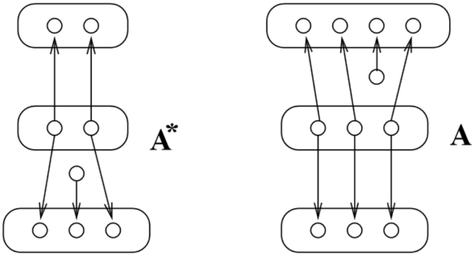
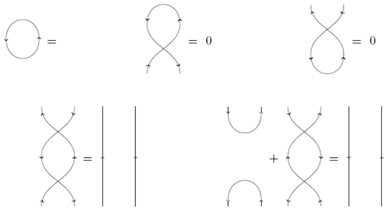
# Spans and the Categorified Heisenberg Algebra

John Baez




for more, see:
http://math.ucr.edu/home/baez/spans/

In 2010, Mikhail Khovanov defined a 'categorified Heisenberg algebra' obeying some relations that look strange at first:



He motivated these relations using the representation theory of symmetric groups.

Later Jeffrey Morton and Jamie Vicary showed they arise from simple ideas about balls in boxes! This sheds new light on the combinatorics of quantum field theory.

What does the canonical commutation relation

$$aa^\dagger = a^\dagger a + 1$$

*really mean?*

There is 1 more way to

create a quantum and then annihilate one,

than to

annihilate a quantum and then create one.

The reason: if you create one *first*, there's 1 more choice of which quantum to annihilate.

How can make this idea precise? We use the groupoid of finite sets, S:

- an object of S is a finite set $s$
- a morphism in S is a bijection $\alpha \colon s \to t$.

There is a functor

$$\begin{array}{rccc} +1\colon & \mathrm{S} & \to & \mathrm{S} \\ & s & \mapsto & s+1 \end{array}$$

sending each finite set $s$ to its disjoint union with a chosen one-element set, called 1.

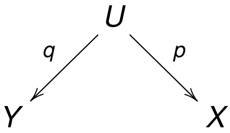
This is the real idea behind the creation operator: it adds one element to a finite set of ‘quanta’.

But what about the annihilation operator? There is no functor $f\colon S \to S$ that takes a finite set and *removes* an element.

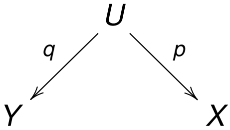
There are $n$ different ways to remove an element from an $n$-element set. There is no functorial way to choose one. But we don't want to choose one. We want to consider *all possible ways*.

We can do this using spans of groupoids.

A **span** is a diagram shaped like this:




In a **span of sets**, $p\colon U \to X$ and $q\colon U \to Y$ are functions between sets.



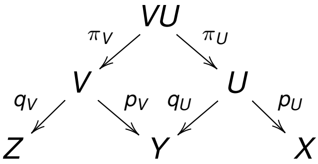

A span of sets gives a matrix of sets:

$$U_{ji} = \{u\colon \ q(u) = j,\ p(u) = i\} \qquad i \in X, j \in Y$$

In physics, $U_{ji}$ is the set of ways for a physical system to go from state $i$ to state $j$. Physicists call these ways **paths** or **histories**.

Spans are closely connected to Heisenberg's **matrix mechanics**, where $U_{ji}$ is a matrix of *numbers* describing the 'amplitude' for the system to go from $i$ to $j$.

We compose spans of sets by taking a **pullback**:



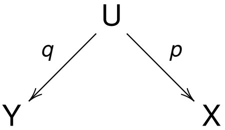

where

$$VU = \{(v, u) \in V \times U : \; p_V(v) = q_U(u)\}$$

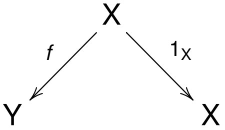
Here we are doing matrix multiplication:

$$(VU)_{ki} = \sum_{j \in Y} V_{kj} \times U_{ji}$$

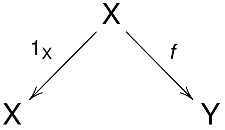
where $\sum$ is disjoint union and $\times$ is cartesian product. This is a baby 'path integral' where we sum over paths from $i$ to some intermediate state $j$ and then to $k$.



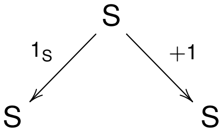

In a **span of groupoids**, $p\colon U \to X$ and $q\colon U \to Y$ are functors between groupoids. Any functor $f\colon X \to Y$ gives a span from X to Y:



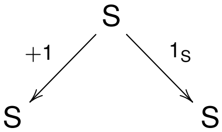

but we can also turn it around and get a span from Y back to X:



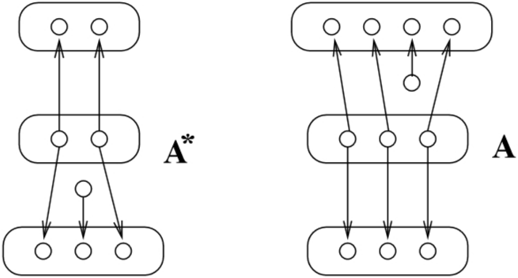
So, if S is the groupoid of finite sets, we have spans called the **annihilation operator** $A$:



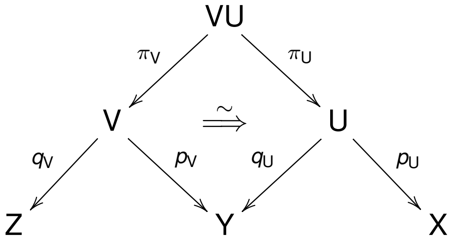

and **creation operator** $A^\dagger$:



From an earlier 2006 paper by Jeffrey Morton:



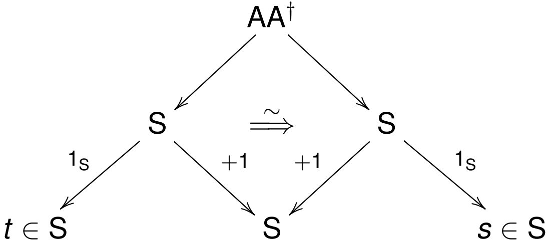
We compose spans of groupoids by taking a **homotopy pullback**:




Now *VU* is the groupoid whose objects are triples

$$\big(v \in \mathsf{V},\ u \in \mathsf{U},\ \alpha \colon p_{\mathsf{V}}(v) \stackrel{\sim}{\longrightarrow} q_{\mathsf{U}}(u)\big)$$

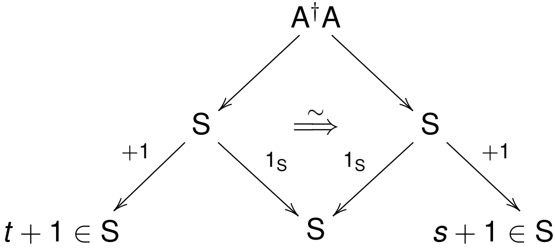
and the diamond commutes *up to natural isomorphism*.




An object of the homotopy pullback $AA^\dagger$ is a way to first add one element to a finite set and then remove one:

$$\big(t \in S,\ s \in S,\ \alpha\colon s + 1 \stackrel{\sim}{\longrightarrow} t + 1\,\big)$$

We can think of this as a little 'history' starting at $s$ and ending at $t$.

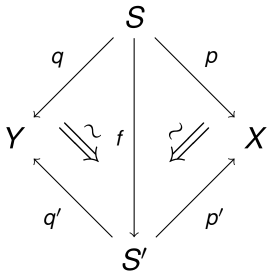



An object of the homotopy pullback $A^\dagger A$ is a way to remove one element from a finite set and then add one:

$$\big(\ t \in S,\ s \in S,\ \alpha \colon s \xrightarrow{\sim} t\ \big)$$

How can we relate $AA^\dagger$ and $A^\dagger A$?

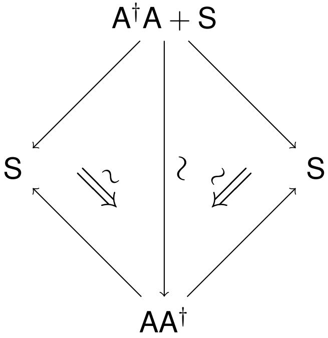
A **map of spans** is roughly a diagram of groupoids and functors like this:




where the triangles commute up to chosen natural isomorphisms.

An **equivalence of spans** is a map of spans where $f$ is an equivalence of groupoids.

Since there's 1 more way to *add an element to a finite set and then remove one* than to *remove an element and then add one*, we get an equivalence of spans:




Here $A^\dagger A + S$ is the disjoint union or **coproduct** of the groupoids $A^\dagger A$ and $S$.

For any groupoid X there is an **identity span** given by

$$\begin{array}{ccccc} & & X & & \\ & {}^{1_X}\swarrow & & \searrow^{1_X} & \\ X & & & & X \end{array}$$

We can also **add** two spans

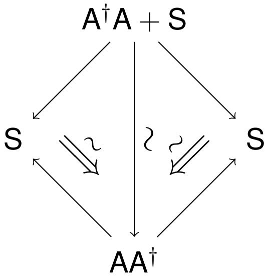
$$\begin{array}{ccccc} & & U & & \\ & \swarrow & & \searrow & \\ Y & & & & X \end{array} \qquad \begin{array}{ccccc} & & V & & \\ & \swarrow & & \searrow & \\ Y & & & & X \end{array}$$

getting a span

$$\begin{array}{ccccc} & & U+V & & \\ & \swarrow & & \searrow & \\ Y & & & & X \end{array}$$

where $U+V$ is the coproduct of the groupoids U and V.

Using these ideas, our isomorphism of spans




can be written as

$$f\colon A^\dagger A + 1_S \overset{\sim}{\Longrightarrow} AA^\dagger$$

where the double arrow denotes a map of spans and $1_S$ is the identity span of S.

This isomorphism of spans

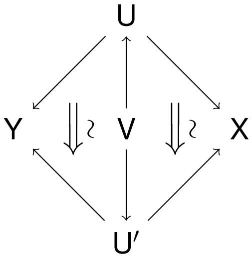
$$f\colon A^\dagger A + 1_S \stackrel{\sim}{\Longrightarrow} AA^\dagger$$

is the categorified version of the canonical commutation relation.

James Dolan and I noticed this in 2000, and used it to describe the combinatorics of Feynman diagrams using spans of groupoids. Jeffrey Morton developed this further, showing how to include complex numbers. But this is just the beginning!

Morton and Vicary showed $f$ obeys certain nontrivial *equations* — which are precisely the relations in Khovanov's categorified Heisenberg algebra!

To see these relations, we need to go beyond maps of spans. We need **spans of spans**:




Just as any functor gives a span of groupoids, any map of spans gives a span of spans.

But a span of spans $V\colon \mathrm{U} \Rightarrow \mathrm{U}'$ can be 'flipped' to give a span of spans $\mathrm{V}^{\dagger}\colon \mathrm{U}' \Rightarrow \mathrm{U}$, just by turning the diagram upside down.

So, the isomorphism of spans

$$f\colon \mathsf{A}^\dagger\mathsf{A} + 1_{\mathsf{S}} \stackrel{\sim}{\Longrightarrow} \mathsf{A}\mathsf{A}^\dagger$$

gives two ‘inclusions’

$$i\colon \mathsf{A}^\dagger\mathsf{A} \Longrightarrow \mathsf{A}\mathsf{A}^\dagger$$

$$j\colon 1_{\mathsf{S}} \Longrightarrow \mathsf{A}\mathsf{A}^\dagger$$

and we can flip these to get ‘projections’

$$i^\dagger\colon \mathsf{A}\mathsf{A}^\dagger \Longrightarrow \mathsf{A}^\dagger\mathsf{A}$$

$$j^\dagger\colon \mathsf{A}\mathsf{A}^\dagger \Longrightarrow 1_{\mathsf{S}}$$

which are spans of spans.

Just as a span of groupoids

$$\begin{array}{ccc} & \mathsf{U} & \\ {}^{q}\swarrow & & \searrow^{p} \\ \mathsf{S} & & \mathsf{S} \end{array}$$

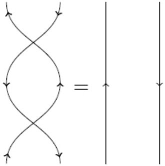
is a collection of ‘histories’ $u \in \mathsf{U}$ going from some finite set $p(x)$ to some finite set $q(x)$...

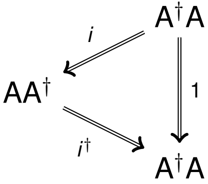
... a span of spans

$$\begin{array}{ccccc} & & \mathsf{U} & & \\ & \swarrow & \uparrow f & \searrow & \\ \mathsf{S} & \Downarrow \wr & \mathsf{V} & \Downarrow \wr & \mathsf{S} \\ & \nwarrow & \downarrow g & \nearrow & \\ & & \mathsf{U}' & & \end{array}$$

is a collection of ‘histories of histories’ $v \in \mathsf{V}$ going from some history $f(v)$ to some history $g(v)$.

Here is one of Khovanov's relations, as he drew it and as a commutative triangle where the double arrows are spans of spans:




What does this mean? We begin with a way to remove one element $x$ from a finite set and then add one element $y$.
$i$ relates this to a way to add $y$ and then remove $x$.
$i^\dagger$ relates this back to a way where we remove $x$ and add $y$.
We come back to the same 'history', so $i^\dagger i = 1$.

To connect their theory of annihilation and creation operators to the work of Khovanov, they use representation theory.

A Kapranov–Voevodsky **2-vector space** is a $\mathbb{C}$-linear abelian category which is **semisimple**, meaning that every object is a finite direct sum of **simple** objects: objects that do not have any nontrivial subobjects.

### Example

The category FinRep($G$) of finite-dimensional (complex) representations of a group is $\mathbb{C}$-linear and abelian. The simple objects are the irreducible representations. If $G$ is finite, FinRep($G$) is a 2-vector space.

We can generalize this example to groupoids.

There is a category Vect of vector spaces and linear operators. A **representation** of a groupoid G is a functor $F\colon \mathrm{G} \to \mathrm{Vect}$. A **morphism** of representations is a natural transformation $\alpha\colon F \Rightarrow F'$ between such functors.

## Example

A groupoid G with one object can be seen as a group. A representation of G is the same as a representation of this group. Morphisms between representations are also the same as usual.

Say a representation of a groupoid G is **finite** if $F(x)$ is finite-dimensional for all objects $x \in$ G, and zero-dimensional except for $x$ in finitely many isomorphism classes.

### Example

A groupoid G with one object can be seen as a group; then a finite representation of G is a finite-dimensional representation of this group.

Let FinRep(G) be the category of finite representations of the groupoid G.

## Example

Let S be the groupoid of finite sets and bijections. S is equivalent to the coproduct

$$\sum_{n=0}^{\infty} S_n$$

where $S_n$ is the symmetric group on $n$ letters, seen as a one-object groupoid.

Thus, a representation $F\colon \mathrm{S} \to \mathrm{Vect}$ is the same as a representation $F_n$ of $S_n$ for each $n \geq 0$. $F$ is finite and only if each $F_n$ is finite dimensional and only finitely many are nonzero.

It follows that FinRep(S) is a 2-vector space.

More generally, say a groupoid is **locally finite** if all its homsets are finite. G is locally finite if and only if it is equivalent to a coproduct of finite groups. In this case FinRep(G) is a 2-vector space.

### Conjecture (Morton, Vicary)

*There is a bicategory* **Span**(**FinGpd**) *with:*

- *locally finite groupoids as objects*
- *spans as morphisms*
- *(equivalence classes of) spans of spans as 2-morphisms*

*and a 2-functor*

$$\mathrm{FinRep}\colon \mathbf{Span}(\mathbf{FinGpd}) \to \mathbf{2Vect}$$

*sending any locally finite groupoid G to its category of finite representations.*

A few remarks on how the proof should go:

**2Vect** is a bicategory (in fact a 2-category) with:

- 2-vector spaces as objects
- exact $\mathbb{C}$-linear functors as morphisms
- natural transformations as 2-morphisms.

How does

$$\mathrm{FinRep}\colon \mathbf{Span}(\mathbf{FinGpd}) \to \mathbf{2Vect}$$

send a span of groupoids

$$Y \xleftarrow{\;q\;} U \xrightarrow{\;p\;} X$$

to an exact functor from FinRep(X) to FinRep(Y)?

This was developed in a 2008 paper by Morton.

Given a functor $p\colon \mathrm{X} \to \mathrm{Y}$ between locally finite groupoids, we get

$$\begin{array}{rcl} p^*\colon \mathrm{FinRep}(\mathrm{Y}) & \to & \mathrm{FinRep}(\mathrm{X}) \\ F & \mapsto & F \circ p \end{array}$$

which has a left (and right!) adjoint

$$p_*\colon \mathrm{FinRep}(\mathrm{X}) \to \mathrm{FinRep}(\mathrm{Y})$$

Both these are exact.

In group theory, $p^*$ and $p_*$ are called **restricting** and **inducing** representations along the homomorphism $p$. The fact that $p_*$ is both left and right adjoint to $p^*$ is called **Frobenius reciprocity**.

So, given a span of groupoids

$$Y \xleftarrow{\;q\;} U \xrightarrow{\;p\;} X$$

we get an exact functor

$$\mathrm{FinRep}(Y) \xleftarrow{\;q_*\;} \mathrm{FinRep}(U) \xleftarrow{\;p^*\;} \mathrm{FinRep}(X)$$

and Morton showed this sends composite spans to composite functors (up to natural isomorphism).

Using

$$\mathrm{FinRep}\colon \mathbf{Span}(\mathbf{FinGpd}) \to \mathbf{2Vect}$$

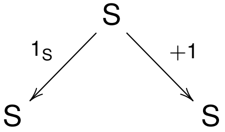
we can map the whole theory of annihilation and creation operators into **2Vect**!

In particular,

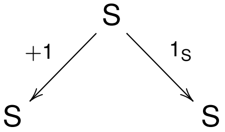
$$\mathrm{Schur} = \mathrm{FinRep}(\mathsf{S})$$

is called the category of **Schur functors**. It has one simple object for each Young diagram.

The annihilation operator A:




and the creation operator $A^\dagger$:

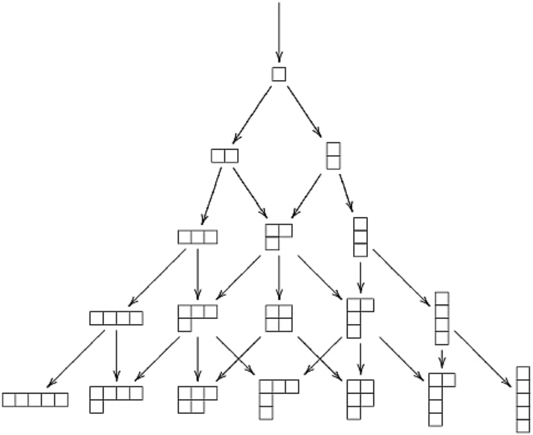



are spans from S to itself, so they should give exact functors

$$\mathbf{A} = \mathrm{FinRep}(A) \colon \mathrm{Schur} \to \mathrm{Schur}$$

$$\mathbf{A}^\dagger = \mathrm{FinRep}(A^\dagger) \colon \mathrm{Schur} \to \mathrm{Schur}$$

$\mathbf{A}^\dagger: \text{Schur} \to \text{Schur}$ has this effect on simple objects — that is, Young diagrams:



Remember that the canonical commutation relation

$$\mathsf{A}^\dagger \mathsf{A} + 1_\mathsf{S} \stackrel{\sim}{\Longrightarrow} \mathsf{A}\mathsf{A}^\dagger$$

gives two ‘inclusions’

$$i\colon \mathsf{A}^\dagger \mathsf{A} \Rightarrow \mathsf{A}\mathsf{A}^\dagger \qquad j\colon 1_\mathsf{S} \Rightarrow \mathsf{A}\mathsf{A}^\dagger$$

which together with their flipped versions $i^\dagger, j^\dagger$ are spans of spans obeying the relations in Khovanov's categorified Heisenberg algebra.

FinRep sends all these to exact functors from Schur to itself:

$$\mathbf{i} = \mathrm{FinRep}(i) \qquad \mathbf{j} = \mathrm{FinRep}(j)$$

$$\mathbf{i}^\dagger = \mathrm{FinRep}(i^\dagger) \qquad \mathbf{j}^\dagger = \mathrm{FinRep}(j^\dagger)$$

which should obey all the same relations.

Putting this all together, we get:

## Theorem (Morton and Vicary)

*The 2-vector space of Schur functors is equipped with exact functors*

$$\mathbf{A}, \mathbf{A}^\dagger \colon \mathrm{Schur} \to \mathrm{Schur}$$

*and natural transformations*

$$\mathbf{i} \colon \mathbf{A}^\dagger\mathbf{A} \Rightarrow \mathbf{A}\mathbf{A}^\dagger \qquad \mathbf{j} \colon 1_{\mathrm{Schur}} \Rightarrow \mathbf{A}\mathbf{A}^\dagger$$

$$\mathbf{i}^\dagger \colon \mathbf{A}\mathbf{A}^\dagger \Rightarrow \mathbf{A}^\dagger\mathbf{A} \qquad \mathbf{j}^\dagger \colon \mathbf{A}\mathbf{A}^\dagger \Rightarrow 1_{\mathrm{Schur}}$$

*obeying the relations in Khovanov's categorified Heisenberg algebra.*

This would follow instantly from Morton and Vicary's calculations if we knew we had a 2-functor

$$\mathrm{FinRep}\colon \mathbf{Span}(\mathbf{FinGpd}) \to \mathbf{2Vect}$$

that preserves addition (both for spans and span of spans).

Surely this is true, but I haven't seen a rigorous proof. Luckily it can also be checked directly!

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**Moral:** in constructing Khovanov's categorified Heisenberg algebra, the use of linear algebra is just the icing on the cake. The real structures involve combinatorics captured by groupoid of finite sets.