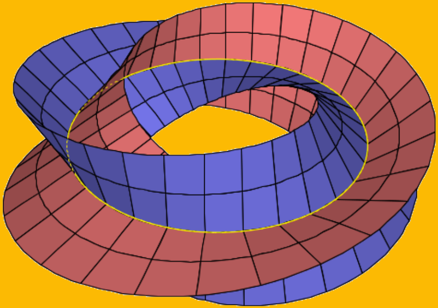
2-Rigs in Topology and Representation Theory



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Mathematicians are slowly categorifying all basic concepts from algebra.

Grothendieck's student Hoàng Xuân Sính categorified *groups* in the late 1960s and got Gr-categories. These are now called '2-groups'. Later people went further and studied n-groups and ∞-groups.

The theory of categorified *rings* is growing more slowly, because there are many different directions to generalize. I'll talk about one approach especially suited to topology and representation theory.

How to categorify rings — or rigs

Often it's best to categorify *rigs* — rings without negatives. The initial ring is $\mathbb{Z}$. The initial rig is $\mathbb{N}$. $\mathbb{N}$ is the set of isomorphism classes of **FinSet**, which is some sort of categorified rig with coproduct as $+$ and product as $\times$.

Sometimes it's good to focus on *commutative* rigs. I'll go down that road and use 'rig' and 'ring' to mean one where the multiplication is commutative. Similarly, I'll only talk about categorified rigs where both the addition and multiplication define *symmetric* monoidal structures.

A **symmetric rig category** **R** is a category with symmetric monoidal structures $(\oplus, 0)$ and $(\otimes, I)$, with natural isomorphisms

$$r \otimes (s \oplus t) \xrightarrow{\sim} (r \otimes s) \oplus (r \otimes t)$$

$$(s \oplus t) \otimes r \xrightarrow{\sim} (s \otimes r) \oplus (t \otimes r)$$

$$r \otimes 0 \xrightarrow{\sim} 0 \qquad\qquad 0 \otimes r \xrightarrow{\sim} 0$$

obeying approximately 19 coherence laws discovered by Laplaza.

Theorem. The initial symmetric rig category is the groupoid of finite sets, with disjoint union as $+$ and cartesian product as $\times$.

Conjectured in 2010, this was proved by Elgueta in 2020. See also Comfort, Delpeuch and Hedges, who gave a string diagram argument for this result, and Yau, who proved a similar result.

Problem. Find a more conceptual definition of rig category and prove it is equivalent to Laplaza's definition. Use this to find a shorter proof of the above theorem.

We want a more streamlined approach to categorified rigs, adapted to these examples:

- The category of (finite-dimensional) representations of a group with its usual $\oplus$ and $\otimes$.
- The category of (finite-dimensional) vector bundles over a space with its usual $\oplus$ and $\otimes$.

These categories are enriched over $\mathbf{Vect}_k$ for some field k — either $\mathbb{R}$ or $\mathbb{C}$ in the second example.

In these categories $\oplus$ is coproduct, but $\otimes$ is not product.

These categories do not have all colimits — and the first doesn't even have coequalizers. It does however have coequalizers of

$$x \underset{1}{\overset{p}{\rightrightarrows}} x$$

where p is idempotent: $p^2 = p$.

2-rigs

A **2-rig** is a symmetric monoidal $\mathbf{Vect}_k$-enriched category that is Cauchy complete.

Or, more slowly:

A **linear category** is a $\mathbf{Vect}_k$-enriched category for some chosen field k. In other words: its hom-sets are vector spaces over k, and composition $(f, g) \mapsto f \circ g$ is linear in each argument.

A **linear functor** between linear categories is a functor that is linear on hom-sets.

A **symmetric monoidal linear category** is a linear category that is symmetric monoidal, such that the tensor product of morphisms $(f, g) \mapsto f \otimes g$ is linear in each argument.

A linear category **C** is **Cauchy complete** if has all **absolute** colimits: those automatically preserved by linear functors. In other words:

- **C** has binary coproducts: $\oplus$
- **C** has an initial object: 0
- **C** has coequalizers of diagrams

$$x \underset{1}{\overset{p}{\rightrightarrows}} x$$

  where p is idempotent: $p^2 = p$.

In a linear category:

- binary coproducts automatically become products too: they are 'biproducts', and I'll call them **direct sums**.
- In a linear category, an initial object automatically becomes terminal: it's a **zero object**.

A **2-rig** is a symmetric monoidal linear category that is Cauchy complete. A **map of symmetric 2-rigs** $F\colon \mathbf{R} \to \mathbf{R}'$ is a symmetric monoidal linear functor.

Note that a map of 2-rigs *automatically* preserves absolute colimits. Also the tensor product $\otimes\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *automatically* preserves absolute colimits in each argument. So we get these natural isomorphisms for free:

$$r \otimes (s \oplus t) \xrightarrow{\sim} (r \otimes s) \oplus (r \otimes t)$$

$$(s \oplus t) \otimes r \xrightarrow{\sim} (s \otimes r) \oplus (t \otimes r)$$

$$r \otimes 0 \xrightarrow{\sim} 0 \qquad\qquad 0 \otimes r \xrightarrow{\sim} 0$$

and their coherence laws too. Thus, every 2-rig is a rig category.

Examples of 2-rigs include:

- The category $\mathbf{Rep}_k(G)$ of representations of a group G on finite-dimensional vector spaces over k, with the usual $\otimes$.
- If $k = \mathbb{R}$ or $\mathbb{C}$, the category $\mathbf{Vect}_k(X)$ of real or complex vector bundles on a compact Hausdorff space X, with the usual $\otimes$.

And many more!

A different kind of example: the free 2-rig on one generator, commonly known as the category of **Schur functors**.

Theorem (BMT). Over a field k of characteristic zero, the free 2-rig on one generator is

$$\bigoplus_{n=0}^{\infty} \mathbf{Rep}_k(S_n)$$

with the tensor product

$$\begin{tikzcd} \mathbf{Rep}_k(S_m) \otimes \mathbf{Rep}_k(S_n) \arrow[r, "\sim"] & \mathbf{Rep}_k(S_m \times S_n) \arrow[rr, bend left, "\text{induction}"] \arrow[rr, phantom, "\perp"] & & \mathbf{Rep}_k(S_{m+n}) \arrow[ll, bend left, "\text{restriction}"] \end{tikzcd}$$

Idea of proof.

Prove we have 2-adjunctions between 2-categories:

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathbf{SMCat} \arrow[l, bend left] \arrow[r, bend left, "k(-)"] \arrow[r, phantom, "\perp"] & \mathbf{SMLin} \arrow[l, bend left] \arrow[r, bend left, "\overline{(-)}"] \arrow[r, phantom, "\perp"] & \mathbf{2Rig} \arrow[l, bend left]
\end{tikzcd}$$

Start with the free category on one object — the terminal category — and feed it into this machine! We get the free 2-rig on one object.

1. First form the free symmetric monoidal category on one object x. This is the groupoid of finite sets, $\mathbf{S}$, with disjoint union as symmetric monoidal structure. In $\mathbf{S}$, the endomorphisms of $\{1, \ldots, n\}$ form the symmetric group S_n.
2. Then form the free linear symmetric monoidal category on $\mathbf{S}$. This is called $k\mathbf{S}$. In $k\mathbf{S}$, the endomorphisms of $\{1, \ldots, n\}$ form the group algebra of S_n, often called $k[S_n]$.
3. Then Cauchy complete $k\mathbf{S}$: that is, take direct sums of objects and split idempotents. The result is a 2-rig called $\overline{k\mathbf{S}}$. As a linear category,

$$\overline{k\mathbf{S}} \simeq \bigoplus_{n=0}^{\infty} \mathbf{Rep}(S_n)$$

   Alternatively it's the category of functors $f \colon \mathbf{S} \to \mathbf{Vect}_k$ such that $\sum_n \dim f(\{1, \ldots, n\}) < \infty$, with Day convolution as the tensor product.

Now suppose k has characteristic zero. For any object x in a 2-rig $\mathbf{R}$ and any $n \geq 0$, we define the **nth exterior power** $\Lambda^n x$ to be the coequalizer of

$$1 \colon x^{\otimes n} \to x^{\otimes n}$$

and the **antisymmetrizer**

$$p = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\mathrm{sign}(\sigma)} \sigma \colon x^{\otimes n} \to x^{\otimes n}$$

which is an idempotent.

We say $x \in \mathbf{R}$ has **subdimension n** if $\Lambda^{n+1} x \cong 0$. We say x is a **subline object** if it has subdimension 1.

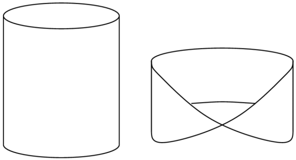
- A vector space has subdimension n iff it has dimension $\leq n$.
- A vector bundle is a subline object iff it is a sub-bundle of a line bundle.

Theorem (BMT). The free 2-rig on a subline object L is the category A of $\mathbb{N}$-graded vector spaces of finite total dimension, with its usual tensor product, and the symmetry with

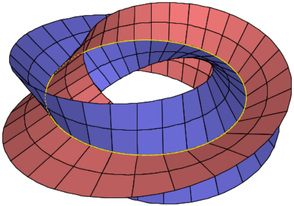
$$S_{L,L} = 1_{L\otimes L}.$$

Conjecture (BMT). The free 2-rig on an object of subdimension n is $\mathbf{Rep}(M(n,k))$: the category of algebraic representations of the monoid of $n \times n$ matrices with entries in k.

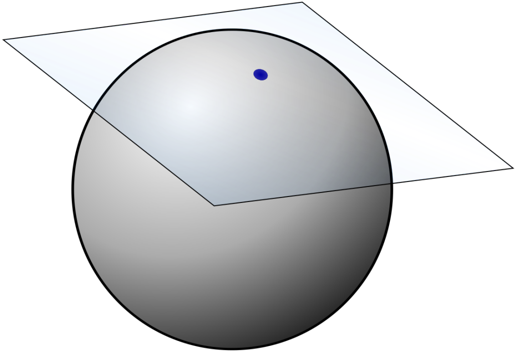
There are two real line bundles over the circle: the trivial line bundle I and the Möbius strip bundle M:



Every real vector bundle over the circle is a direct sum of I and M. Their direct sums obey one relation: $M \oplus M \cong I \oplus I$.



The tangent bundle of the sphere is *not* the direct sum of line bundles.



A Splitting Theorem for Vector Bundles. If $k = \mathbb{R}$ or $\mathbb{C}$ and X is a compact Hausdorff space, then for any $E \in \mathbf{Vect}_k(X)$ there exists a compact Hausdorff space Y and a map $\phi\colon Y \to X$ such that the 2-rig map

$$\phi^*\colon \mathbf{Vect}_k(X) \to \mathbf{Vect}_k(Y)$$

has these properties:

- $\phi^*(E) = L_1 \oplus \cdots \oplus L_n$ for some subline objects L_i.
- ϕ^* is faithful.
- ϕ^* reflects isomorphisms.
- ϕ^* is essentially injective.

Under the same hypotheses it’s also true that

$$K(\phi^*) \colon K(\mathbf{Vect}_k(X)) \to K(\mathbf{Vect}_k(Y))$$

is injective, where K is the **Grothendieck ring** of a 2-rig:

$$\begin{tikzcd}
\mathbf{2Rig} \arrow[r, "\text{decategorification}"] \arrow[rr, bend left=50, "K"] & \text{CommRig} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \text{CommRing} \arrow[l, bend left, "U"]
\end{tikzcd}$$

This is is the traditional statement of the splitting principle!

Conjecture (BMT). If $\mathbf{R}$ is a 2-rig, for any $E \in \mathbf{R}$ of finite subdimension there is a 2-rig map

$$f \colon \mathbf{R} \to \mathbf{R}'$$

with these properties:

- $f(E) = L_1 \oplus \cdots \oplus L_n$ for some subline objects $L_i \in \mathbf{R}'$.
- f is faithful.
- f reflects isomorphisms.
- f is essentially injective.
- $K(f)$ is injective.

The generating object $x \in \overline{k\mathbf{S}}$ doesn't have finite subdimension, so the conjecture does not apply to it.

But the 2-rig $\mathbf{Rep}(M(n,k))$ contains an object x of subdimension n: the obvious representation of $n \times n$ matrices on the vector space k^n.

And we have proved the conjecture in this case!

(We hope $\mathbf{Rep}(M(n,k))$ is the *free* 2-rig on an object of subdimension n, but we don't use this in our proof.)

The category of rings has coproducts, denoted $\otimes$. Similarly the 2-category of 2-rigs has coproducts, denoted $\boxtimes$.

Let $\mathsf{A}^{\boxtimes n}$ be the coproduct of n copies of the free 2-rig on a subline object. $\mathsf{A}^{\boxtimes n}$ is the free 2-rig on n subline objects, say $L_1, \ldots, L_n$.

Theorem (BMT). There is a 2-rig map

$$f \colon \mathbf{Rep}(M(n,k)) \to \mathsf{A}^{\boxtimes n},$$

unique up to isomorphism, with these properties:

- $f(x) = L_1 \oplus \cdots \oplus L_n$.
- f is faithful.
- f reflects isomorphisms.
- f is essentially injective.
- $K(f)$ is injective.

$$K(A^{\boxtimes n}) \cong \mathbb{Z}[x_1, \ldots, x_n]$$

where x_i is the class of the subline object L_i. The image of

$$K(f) \colon K(\mathbf{Rep}(M(n,k))) \hookrightarrow K(\mathsf{A}^{\boxtimes n})$$

consists of **symmetric polynomials** in the variables $x_1, \ldots, x_n$, that is, polynomials that are invariant under all permutations of variables.

Thus, we have categorified the theory of symmetric polynomials, which connect topology and representation theory. The categorified theory emerges naturally from thinking about 2-rigs!

- JB, Joe Moeller and Todd Trimble, Schur functors and categorified plethysm, *Higher Structures*, **8** (2024), 1–53.
- JB, Joe Moeller and Todd Trimble, 2-rig extensions and the splitting principle.