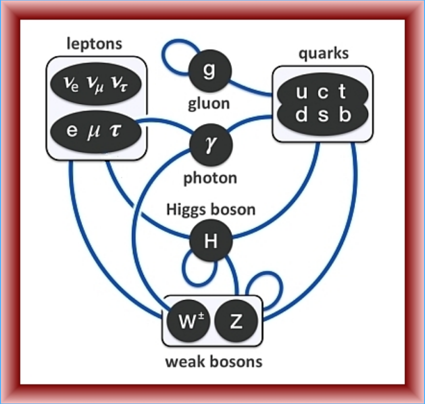
CAN WE UNDERSTAND THE STANDARD MODEL USING OCTONIONS?



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Octonions and the Standard Model

5 April 2021

Can we derive the Standard Model — or something close — from reasonable principles?

The internal degrees of freedom — hypercharge, isospin, color — seem to be described by algebras of observables connected to representations of $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$. Why this particular group, and these representations?

Connes and others have tried to answer this using noncommutative geometry, for example:

- Ali Chamseddine and Alain Connes, Why the Standard Model?

I'll present some much more tentative thoughts involving octonions and Jordan algebras.

Jordan algebras are a framework for dealing with observables in quantum physics. The exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ plays a unique role. It's the algebra of observables of an "octonionic qutrit".

Following ideas of Dubois-Violette and Todorov, we'll see that the true gauge group of the Standard Model, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$, consists of the symmetries of an octonionic qutrit that

1. preserve all the structure arising from a choice of unit imaginary octonion $i \in \mathbb{O}$

and

2. restrict to give symmetries of an octonionic qubit.

But let's start at the beginning: what can we do with observables?

For example, suppose "observables" are self-adjoint complex matrices, $A \in \mathfrak{h}_n(\mathbb{C})$.

We can take real-linear combinations of them.

The product of two self-adjoint matrices is not self-adjoint, but the *square* of a self-adjoint matrix is self-adjoint. From squaring and linear combinations we can define the **Jordan product**

$$a \circ b = \frac{1}{2}((a+b)^2 - a^2 - b^2) = \frac{1}{2}(ab + ba).$$

This product is commutative. It is not associative, but it is **power-associative**: any way of parenthesizing a product of copies of the same observable gives the same result.

Jordan, von Neumann and Wigner turned these ideas into a definition:

A **Euclidean Jordan algebra** is a real vector space with a bilinear, commutative and power-associative product satisfying

$$a_1^2 + \cdots + a_n^2 = 0 \quad \Longrightarrow \quad a_1 = \cdots = a_n = 0$$

for all n.

Jordan, von Neumann and Wigner proved:

Theorem. Every finite-dimensional Euclidean Jordan algebra is isomorphic to a direct sum of ones on this list:

- $\mathfrak{h}_n(\mathbb{R})$: $n \times n$ self-adjoint real matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{C})$: $n \times n$ self-adjoint complex matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{H})$: $n \times n$ self-adjoint quaternionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{O})$: $n \times n$ self-adjoint octonionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$, where $n \leq 3$.
- The **spin factor** $\mathbb{R} \oplus \mathbb{R}^n$, with

$$(t, \vec{x}) \circ (t', \vec{x}') = (tt' + \vec{x} \cdot \vec{x}', t\vec{x}' + t'\vec{x}).$$

What about the spin factors?

Every Euclidean Jordan algebra J comes with a cone of **nonnegative** elements:

$$J_+ = \{a_1^2 + \cdots + a_n^2 : \ a_i \in J\}$$

For the spin factor $\mathbb{R} \oplus \mathbb{R}^n$ this cone is isomorphic to the future cone in $(n+1)$-dimensional Minkowski spacetime!

Every Euclidean Jordan algebra automatically comes with a **determinant** function $\det: J \to \mathbb{R}$ that vanishes on the boundary of J_+. For the spin factor this is the Minkowski metric!

$$\det(t, \vec{x}) = t^2 - \vec{x} \cdot \vec{x}$$

So, spin factors are not only algebras of observables. They are also Minkowski spacetimes!

Jordan algebras of 2×2 self-adjoint matrices are isomorphic to spin factors:

$$\begin{array}{rcccr} \mathfrak{h}_2(\mathbb{R}) & \cong & \mathbb{R} \oplus \mathbb{R}^2 & \cong & \text{3d Minkowski spacetime} \\ \mathfrak{h}_2(\mathbb{C}) & \cong & \mathbb{R} \oplus \mathbb{R}^3 & \cong & \text{4d Minkowski spacetime} \\ \mathfrak{h}_2(\mathbb{H}) & \cong & \mathbb{R} \oplus \mathbb{R}^5 & \cong & \text{6d Minkowski spacetime} \\ \mathfrak{h}_2(\mathbb{O}) & \cong & \mathbb{R} \oplus \mathbb{R}^9 & \cong & \text{10d Minkowski spacetime} \end{array}$$

$$\det \begin{pmatrix} t + x & y \\ y^* & t - x \end{pmatrix} = t^2 - x^2 - |y|^2$$

How can we understand this?

A Euclidean Jordan algebra does not merely describe observables. It also describes states.

Any Euclidean Jordan algebra automatically comes with a **trace** $\mathrm{tr}\colon J \to \mathbb{R}$. An element $s \in J_+$ with $\mathrm{tr}(s) = 1$ is called a **state**.

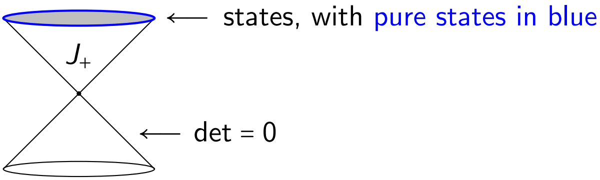
Given a state s and an observable a, the **expected value** of a in the state s is $\mathrm{tr}(s \circ a)$.

A **projection** $p \in J$ is an element with $p^2 = p$. A projection p with $\mathrm{tr}(p) = 1$ is a state called a **pure state**.

For $J = \mathfrak{h}_n(\mathbb{C})$, all this is familiar. Here a state is just a density matrix: a non-negative self-adjoint matrix with trace 1.

- The space of pure states for $\mathfrak{h}_n(\mathbb{R})$ is $\mathbb{RP}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{C})$ is $\mathbb{CP}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{H})$ is $\mathbb{HP}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{O})$ is $\mathbb{OP}^{n-1}$ (for $n \leq 3$).
- The space of pure states for $\mathbb{R} \oplus \mathbb{R}^n$ is S^{n-1}.

A picture of the spin factor $\mathbb{R} \oplus \mathbb{R}^n$ for $n = 2$:



So:

- $\mathfrak{h}_2(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}^2$ has $\mathbb{R}\mathrm{P}^1 \cong S^1$ as its set of pure states.
- $\mathfrak{h}_2(\mathbb{C}) \cong \mathbb{R} \oplus \mathbb{R}^3$ has $\mathbb{C}\mathrm{P}^1 \cong S^2$ as its set of pure states.
- $\mathfrak{h}_2(\mathbb{H}) \cong \mathbb{R} \oplus \mathbb{R}^5$ has $\mathbb{H}\mathrm{P}^1 \cong S^4$ as its set of pure states.
- $\mathfrak{h}_2(\mathbb{O}) \cong \mathbb{R} \oplus \mathbb{R}^9$ has $\mathbb{O}\mathrm{P}^1 \cong S^8$ as its set of pure states.

A chiral spinor in 3, 4, 6 or 10-dimensional spacetime is described by a real, complex, quaternionic or octonionic qubit.

In physics, *observables should generate symmetries*.

Of the Euclidean Jordan algebras on the list, only $\mathfrak{h}_n(\mathbb{C})$ can be made into a Lie algebra that acts nontrivially as derivations of the Jordan product:

$$a, b \in \mathfrak{h}_n(\mathbb{C}) \implies \{a, b\} := i(ab - ba) \in \mathfrak{h}_n(\mathbb{C})$$

$$\{a, b \circ c\} = \{a, b\} \circ c + b \circ \{a, c\}$$

- John Baez, Getting to the bottom of Noether's theorem.

Thus $\mathfrak{h}_n(\mathbb{C})$ is favored. But $\mathfrak{h}_n(\mathbb{R})$ and $\mathfrak{h}_n(\mathbb{H})$ actually *do* play a role in ordinary quantum mechanics:

- John Baez, Division algebras and quantum theory.

WHAT ABOUT $\mathfrak{h}_2(\mathbb{O})$ AND $\mathfrak{h}_3(\mathbb{O})$?

Amazingly, these Jordan algebras are connected to the Standard Model:

- Michel Dubois-Violette, Exceptional quantum geometry and particle physics.
- Ivan Todorov and Michel Dubois-Violette, Exceptional quantum geometry and particle physics II.
- Ivan Todorov and Michel Dubois-Violette, Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra.

Remember from my last talk: choosing a unit imaginary octonion $i \in \mathbb{O}$ gives an inclusion

$$\mathbb{C} \hookrightarrow \mathbb{O}$$

and thus a splitting

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^\perp$$

and a complex structure on $\mathbb{C}^\perp$, from left multiplication by i.

It also gives an inclusion

$$\mathfrak{h}_2(\mathbb{C}) \hookrightarrow \mathfrak{h}_2(\mathbb{O})$$

and a splitting

$$\underbrace{\mathfrak{h}_2(\mathbb{O})}_{\text{10d}} \quad = \quad \underbrace{\mathfrak{h}_2(\mathbb{C})}_{\text{4d}} \quad \oplus \quad \underbrace{\mathbb{C}^\perp}_{\text{6d}}$$

$\mathfrak{h}_2(\mathbb{O})$ naturally has the structure of *both* a 10d Minkowski spacetime:

$$\det\begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

and a 10d Euclidean space:

$$\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \circ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix}\right) = t^2 + x^2 + |y|^2$$

det and tr can both be defined in terms of $\circ$. Thus, the automorphism group of the Jordan algebra $\mathfrak{h}_2(\mathbb{O})$ must be contained in

$$\mathrm{O}(9,1) \cap \mathrm{O}(10) = \mathrm{O}(9)$$

This resolves the "Euclidean or Minkowskian?" puzzle from last time.

More simply, since $\mathfrak{h}_2(\mathbb{O}) \cong \mathbb{R} \oplus \mathbb{R}^9$ with Jordan product

$$(t, \vec{x}) \circ (t', \vec{x}') = (tt' + \vec{x} \cdot \vec{x}', t\vec{x}' + t'\vec{x})$$

the automorphism group of $\mathfrak{h}_2(\mathbb{O})$ is exactly $\mathrm{O}(9)$.

The double cover of the identity component of $\mathrm{O}(9)$ is $\mathrm{Spin}(9)$.

The subgroup of $\mathrm{Spin}(9)$ preserving $\mathfrak{h}_2(\mathbb{C}) \subset \mathfrak{h}_2(\mathbb{O})$ is

$$(\mathrm{Spin}(3) \times \mathrm{Spin}(6))/\mathbb{Z}_2 \quad \cong \quad (\mathrm{SU}(2) \times \mathrm{SU}(4))/\mathbb{Z}_2$$

This contains a copy of the true gauge group of the Standard Model!

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Remember from last time: there is a 3-1 homomorphism

$$\begin{array}{ccc} \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) & \stackrel{\phi}{\to} & \mathrm{SU}(2) \times \mathrm{SU}(4) \\ (\alpha, g, h) & \mapsto & \left(g, \begin{pmatrix} \alpha h & 0 \\ 0 & \alpha^{-3} \end{pmatrix} \right) \end{array}$$

The double cover of the identity component of $\mathrm{O}(9)$ is $\mathrm{Spin}(9)$.

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This contains a copy of the true gauge group of the Standard Model!

There is thus a 6-1 homomorphism

$$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \to (\mathrm{SU}(2) \times \mathrm{SU}(4))/\mathbb{Z}_2$$

The double cover of the identity component of $\mathrm{O}(9)$ is $\mathrm{Spin}(9)$.

The subgroup of $\mathrm{Spin}(9)$ preserving $\mathfrak{h}_2(\mathbb{C}) \subset \mathfrak{h}_2(\mathbb{O})$ is

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There is thus an inclusion

$$\frac{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)}{\mathbb{Z}_6} \hookrightarrow (\mathrm{SU}(2) \times \mathrm{SU}(4))/\mathbb{Z}_2$$

The double cover of the identity component of $\mathrm{O}(9)$ is $\mathrm{Spin}(9)$.

The subgroup of $\mathrm{Spin}(9)$ preserving $\mathfrak{h}_2(\mathbb{C}) \subset \mathfrak{h}_2(\mathbb{O})$ is

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This contains a copy of the true gauge group of the Standard Model!

This gives an inclusion

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \hookrightarrow (\mathrm{SU}(2) \times \mathrm{SU}(4))/\mathbb{Z}_2$$

So: $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ acts as Jordan algebra automorphisms of $\mathfrak{h}_2(\mathbb{O})$ preserving $\mathfrak{h}_2(\mathbb{C})$.

Put more dramatically: the true gauge group of the Standard Model acts as symmetries of an octonionic qubit, and preserves the subalgebra of observables of a complex qubit.

That sounds impressive, but it leaves open two big questions:

A. While $\mathrm{Spin}(9)$ acts on $\mathfrak{h}_2(\mathbb{O})$, the automorphism group of $\mathfrak{h}_2(\mathbb{O})$ is actually $\mathrm{O}(9)$. Why work with $\mathrm{Spin}(9)$?

B. $(\mathrm{Spin}(3) \times \mathrm{Spin}(6))/\mathbb{Z}_2$ is the subgroup of $\mathrm{Spin}(9)$ preserving $\mathfrak{h}_2(\mathbb{C})$. What picks out the smaller subgroup $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$?

Both questions can be answered with the help of $\mathfrak{h}_3(\mathbb{O})$.

$\mathfrak{h}_3(\mathbb{O})$ is the Jordan algebra of observables of an “octonionic qutrit”:

$$\mathfrak{h}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \ \alpha, \beta, \gamma \in \mathbb{R}, \ x, y, z \in \mathbb{O} \right\}$$

The automorphism group of $\mathfrak{h}_3(\mathbb{O})$ is the 52-dimensional compact Lie group F_4.

F_4 cannot act on $\mathbb{O}^3$ in any nontrivial way: its smallest nontrivial representation is 26-dimensional. There is thus no “Hilbert space” picture of the octonionic qutrit.

Pick any copy of $\mathfrak{h}_2(\mathbb{O})$ sitting inside $\mathfrak{h}_3(\mathbb{O})$ as a Jordan subalgebra, e.g.:

$$\mathfrak{h}_2(\mathbb{O}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & x \\ 0 & x^* & \gamma \end{pmatrix} : \ \beta, \gamma \in \mathbb{R}, \ x \in \mathbb{O} \right\}$$

The subgroup of F_4 preserving this is $\mathrm{Spin}(9)$.

This answers question A: "why $\mathrm{Spin}(9)$ instead of $\mathrm{O}(9)$?"

Don't work with automorphisms of $\mathfrak{h}_2(\mathbb{O})$, which form the group $\mathrm{O}(9)$. Work with automorphisms of $\mathfrak{h}_3(\mathbb{O})$ that map $\mathfrak{h}_2(\mathbb{O}) \subset \mathfrak{h}_3(\mathbb{O})$ to itself. These form the group $\mathrm{Spin}(9)$.

Why do we get $\mathrm{Spin}(9)$?

As representations of $\mathrm{Spin}(9)$ we have

$$\mathfrak{h}_3(\mathbb{O}) \cong \mathbb{R} \oplus \mathfrak{h}_2(\mathbb{O}) \oplus \mathbb{O}^2$$

$$\left(\begin{array}{c|cc} \alpha & z & y^* \\ \hline z^* & \beta & x \\ y & x^* & \gamma \end{array}\right) = \left(\begin{array}{cc} \alpha & \psi^\dagger \\ \psi & v \end{array}\right) \longmapsto (\alpha, v, \psi)$$

Here $\mathrm{Spin}(9)$ acts on $\mathbb{R}$ trivially, on $\mathbb{O}^2$ via the real spinor representation, and on $\mathfrak{h}_2(\mathbb{O})$ as before: it's (9+1)d spacetime, or 10d space.

The Jordan product on $\mathfrak{h}_3(\mathbb{O})$ can be described using $\mathrm{Spin}(9)$-invariant operations on $\mathbb{R}$, $\mathbb{O}^2$ and $\mathfrak{h}_2(\mathbb{O})$. Only $\mathrm{Spin}(9)$ preserves all these operations.

Now we can answer question B: "what picks out the Standard Model gauge group as a subgroup of $\mathrm{Spin}(9)$?"

- First, choose a copy of $\mathfrak{h}_2(\mathbb{O})$ in $\mathfrak{h}_3(\mathbb{O})$. The subgroup of F_4 preserving this is $\mathrm{Spin}(9)$.
- Next, choose a unit imaginary octonion $i \in \mathbb{O}$. The subgroup of F_4 preserving all the structure this puts on $\mathfrak{h}_3(\mathbb{O})$ is

$$\frac{\mathrm{SU}(3) \times \mathrm{SU}(3)}{\mathbb{Z}_3}$$

- The subgroup of F_4 preserving *all* the above structure is the true gauge group of the Standard Model:

$$\frac{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(3)}{\mathbb{Z}_6} = \frac{\mathrm{SU}(3) \times \mathrm{SU}(3)}{\mathbb{Z}_3} \cap \mathrm{Spin}(9)$$

In short, the true gauge group of the Standard Model consists of precisely the symmetries of an octonionic qutrit that

1. preserve all the structure arising from a choice of unit imaginary octonion $i \in \mathbb{O}$

and

2. restrict to give symmetries of an octonionic qubit.

But let's see how this works in more detail.

If we choose a unit imaginary octonion $i \in \mathbb{O}$, we get an inclusion $\mathbb{C} \hookrightarrow \mathbb{O}$ and thus an inclusion

$$\mathfrak{h}_3(\mathbb{C}) \hookrightarrow \mathfrak{h}_3(\mathbb{O})$$

and a splitting

$$\mathfrak{h}_3(\mathbb{O}) \quad = \quad \mathfrak{h}_3(\mathbb{C}) \quad \oplus \quad \mathfrak{h}_3(\mathbb{C})^\perp$$

where

$$\begin{aligned} \mathfrak{h}_3(\mathbb{C})^\perp &= \{a \in \mathfrak{h}_3(\mathbb{O}) : \mathrm{tr}(a \circ x) = 0 \text{ for all } x \in \mathfrak{h}_3(\mathbb{C})\} \\ &= \left\{ \begin{pmatrix} 0 & z & y^* \\ z^* & 0 & x \\ y & x^* & 0 \end{pmatrix} : \ x, y, z \in \mathbb{C}^\perp \subset \mathbb{O} \right\} \end{aligned}$$

gets a complex structure from left multiplication by i.

Theorem. For any choice of unit imaginary octonion $i \in \mathbb{O}$, the subgroup of F_4 that preserves the resulting splitting

$$\mathfrak{h}_3(\mathbb{O}) \quad = \quad \mathfrak{h}_3(\mathbb{C}) \quad \oplus \quad \mathfrak{h}_3(\mathbb{C})^\perp$$

and complex structure on $\mathfrak{h}_3(\mathbb{C})^\perp$ is isomorphic to

$$\frac{\mathrm{SU}(3) \times \mathrm{SU}(3)}{\mathbb{Z}_3}$$

Proof. This follows, with some work, from Theorem 2.12.2 in

- Ichiro Yokota, Exceptional Lie groups.

But let's see how $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ acts.

$\mathbb{C}^\perp \subset \mathbb{O}$ is a 3d complex vector space. Choosing an isomorphism $\mathbb{C}^\perp \cong \mathbb{C}^3$ we get

$$\begin{aligned} \mathfrak{h}_3(\mathbb{C})^\perp &= \left\{ \begin{pmatrix} 0 & z & y^* \\ z^* & 0 & x \\ y & x^* & 0 \end{pmatrix} : \; x, y, z \in \mathbb{C}^\perp \subset \mathbb{O} \right\} \\ &\cong \{(x, y, z) : \; x, y, z \in \mathbb{C}^\perp\} \\ &\cong \mathrm{M}_3(\mathbb{C}) \end{aligned}$$

where $\mathrm{M}_3(\mathbb{C})$ is the space of 3×3 complex matrices.

We thus get an isomorphism

$$\begin{aligned} \mathfrak{h}_3(\mathbb{O}) &= \mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{h}_3(\mathbb{C})^\perp \\ &\cong \mathfrak{h}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C}) \end{aligned}$$

We thus can think of an element of $\mathfrak{h}_3(\mathbb{O})$ as a pair

$$(X, M) \in \mathfrak{h}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C})$$

$(g, h) \in \mathrm{SU}(3) \times \mathrm{SU}(3)$ acts on such pairs as follows:

$$(g, h)(X, M) = (gXg^\dagger, hMg^\dagger)$$

$(e^{2\pi i/3}, e^{2\pi i/3}) \in \mathrm{SU}(3) \times \mathrm{SU}(3)$ acts trivially.

We thus get a representation of $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ on $\mathfrak{h}_3(\mathbb{O})$ that preserves:

- the splitting $\mathfrak{h}_3(\mathbb{O}) = \mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{h}_3(\mathbb{C})^\perp$ (obvious)
- the complex structure on $\mathfrak{h}_3(\mathbb{C})^\perp$ (obvious)
- the Jordan product on $\mathfrak{h}_3(\mathbb{O})$ (a calculation: see Yokota).

The two $\mathrm{SU}(3)$'s in $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ act very differently on $\mathfrak{h}_3(\mathbb{O})$.

The second $\mathrm{SU}(3)$ becomes the strong force $\mathrm{SU}(3)$: it acts *separately* on each matrix entry

$$\begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix}$$

as octonion automorphisms that preserve $i \in \mathbb{O}$.

The first $\mathrm{SU}(3)$ acts to *mix up* the matrix entries, and only the electroweak group $(\mathrm{U}(1) \times \mathrm{SU}(2))/\mathbb{Z}_2 \subset \mathrm{SU}(3)$ preserves

$$\mathfrak{h}_2(\mathbb{O}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & x \\ 0 & x^* & \gamma \end{pmatrix} : \; \beta, \gamma \in \mathbb{R}, \; x \in \mathbb{O} \right\} \subset \mathfrak{h}_3(\mathbb{O})$$

Using this idea one can show:

Theorem. Choose a unit imaginary octonion $i \in \mathbb{O}$, giving a Jordan subalgebra

$$\mathfrak{h}_3(\mathbb{C}) \subset \mathfrak{h}_3(\mathbb{O})$$

Also choose a Jordan subalgebra

$$\mathfrak{h}_2(\mathbb{O}) \subset \mathfrak{h}_3(\mathbb{O})$$

The group of automorphisms of the Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ that preserve

- the splitting $\mathfrak{h}_3(\mathbb{O}) = \mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{h}_3(\mathbb{C})^\perp$
- the complex structure on $\mathfrak{h}_3(\mathbb{C})^\perp$
- the Jordan subalgebra $\mathfrak{h}_2(\mathbb{O})$

is isomorphic to the true gauge group of the Standard Model, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$.

Summary and Speculations

The true gauge group of the Standard Model consists of the automorphisms of $\mathfrak{h}_3(\mathbb{O})$ that

1. preserve all the structure coming from a unit imaginary octonion $i \in \mathbb{O}$

and

2. preserve a copy of $\mathfrak{h}_2(\mathbb{O})$ in $\mathfrak{h}_3(\mathbb{O})$.

These symmetries simultaneously act as symmetries of:

- an octonionic qutrit: $\mathfrak{h}_3(\mathbb{O})$
- an octonionic qubit: $\mathfrak{h}_2(\mathbb{O})$
- a complex qutrit: $\mathfrak{h}_3(\mathbb{C})$
- a complex qubit: $\mathfrak{h}_2(\mathbb{C})$.

Maybe this is all just a coincidence. Maybe not!

If an "octonionic qutrit" is relevant to physics, what is it? $\mathfrak{h}_3(\mathbb{O})$ acts as operators on $\mathbb{O}^3$. But F_4 does not act on $\mathbb{O}^3$, only on $\mathfrak{h}_3(\mathbb{O})$ (observables) and $\mathbb{O}\mathrm{P}^2$ (pure states).

The "octonionic qubit" is less mysterious. The Standard Model gauge group

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \quad \subset \quad \mathrm{Spin}(9) \quad \subset \quad \mathrm{F}_4$$

acts on $\mathfrak{h}_3(\mathbb{O})$, but also on $\mathfrak{h}_2(\mathbb{O})$. It also acts on $\mathbb{O}^2$ via the spinor representation of $\mathrm{Spin}(9)$. This is our octonionic qubit.

$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ is precisely the subgroup of $\mathrm{Spin}(9)$ whose action on $\mathbb{O}^2$ commutes with right multiplication by $i \in \mathbb{O}$:

- Kirill Krasnov, SO(9) characterisation of the Standard Model gauge group.

$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ acts on $\mathbb{O}^2$ with this complex structure precisely as it does on the *left-handed* fermions in one generation.

So, the *left-handed* fermions in one generation can be seen as an octonionic qubit with a certain complex structure — but the octonionic qutrit remains mysterious.