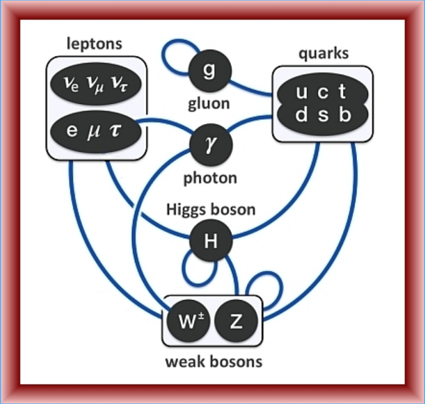
THE STANDARD MODEL GAUGE GROUP FROM OCTONIONS



John Baez – University of Edinburgh
Paul Schwahn – Universidade Estadual de Campinas
May 31, 2026

The gauge group of the Standard Model of particle physics is

$$\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)) \quad := \quad \left\{ x \in \mathrm{SU}(5) : x = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \right\}$$

$$\cong \quad \left(\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)\right)/\mathbb{Z}_6$$

Nobody knows why! What is special about this group?

Todorov and Dubois-Violette tried to answer this using the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$, which consists of 3×3 self-adjoint matrices of octonions:

- Ivan Todorov and Michel Dubois-Violette, Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra.

Euclidean Jordan algebras are a framework for dealing with observables in quantum physics. The exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ first showed up in the classification of these. It's the algebra of observables of an "octonionic qutrit".

Euclidean Jordan algebras are a framework for dealing with observables in quantum physics. The exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ first showed up in the classification of these. It's the algebra of observables of an "octonionic qutrit".

Following ideas of Dubois-Violette and Todorov, we'll see that the gauge group of the Standard Model consists of those symmetries of an octonionic qutrit that preserve a complex qutrit contained within it, and complex qubit contained within that!

That is, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ is the connected component of the group of Jordan algebra automorphisms of $\mathfrak{h}_3(\mathbb{O})$ that preserve

- a Jordan subalgebra $B \subset \mathfrak{h}_3(\mathbb{O})$ with $B \cong \mathfrak{h}_3(\mathbb{C})$ and
- a Jordan subalgebra $X \subset B$ with $X \cong \mathfrak{h}_2(\mathbb{C})$.

But let's start at the beginning: what can we do with observables?

For example, suppose "observables" are self-adjoint complex matrices, $A \in \mathfrak{h}_n(\mathbb{C})$.

We can take real-linear combinations of them.

The product of two self-adjoint matrices is not self-adjoint, but the *square* of a self-adjoint matrix is self-adjoint. From squaring and linear combinations we can define the **Jordan product**

$$a \circ b = \frac{1}{2}((a+b)^2 - a^2 - b^2) = \frac{1}{2}(ab + ba).$$

This product is commutative. It is not associative, but it is **power-associative**: any way of parenthesizing a product of copies of the same observable gives the same result.

In 1934, Jordan, von Neumann and Wigner turned these ideas into a definition:

A **Euclidean Jordan algebra** is a real vector space with a bilinear, commutative and power-associative product satisfying

$$a_1^2 + \cdots + a_n^2 = 0 \quad \Longrightarrow \quad a_1 = \cdots = a_n = 0$$

for all n.

Jordan, von Neumann and Wigner proved:

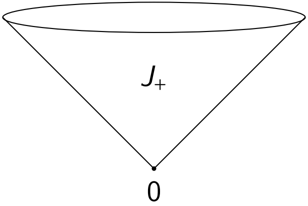
Theorem. Every finite-dimensional Euclidean Jordan algebra is isomorphic to a direct sum of ones on this list:

- $\mathfrak{h}_n(\mathbb{R})$: $n \times n$ self-adjoint real matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{C})$: $n \times n$ self-adjoint complex matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{H})$: $n \times n$ self-adjoint quaternionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{O})$: $n \times n$ self-adjoint octonionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$, where $n \leq 3$.
- The **spin factor** $\mathbb{R} \oplus \mathbb{R}^n$, with

$$(t, \vec{x}) \circ (t', \vec{x}') = (tt' + \vec{x} \cdot \vec{x}', t\vec{x}' + t'\vec{x}).$$

Every Euclidean Jordan algebra J comes with a cone of **nonnegative** elements:

$$J_+ = \{a_1^2 + \cdots + a_n^2 : \; a_i \in J\}$$

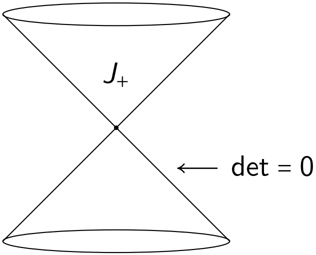


For a spin factor $\mathbb{R} \oplus \mathbb{R}^n$ this cone is isomorphic to the future cone in $(n+1)$-dimensional Minkowski spacetime!

Every Euclidean Jordan algebra also has a **determinant**

$$\det\colon J \to \mathbb{R}$$

an invariant polynomial that vanishes on the boundary of J_+ and is positive in the interior.



For a spin factor this is the Minkowski metric!

$$\det(t, \vec{x}) = t^2 - \vec{x} \cdot \vec{x}$$

So, spin factors are not only algebras of observables. They are also Minkowski spacetimes!

So, spin factors are not only algebras of observables. They are also Minkowski spacetimes!

But beware: any Euclidean Jordan algebra has a **trace**: an invariant linear functional

$$\mathrm{tr}\colon J \to \mathbb{R},$$

that is positive on the cone J_+. This gives the Jordan algebra an invariant *Euclidean* inner product

$$\langle a, b \rangle = \mathrm{tr}(a \circ b).$$

So, spin factors are not only algebras of observables. They are also Minkowski spacetimes!

But beware: any Euclidean Jordan algebra has a **trace**: an invariant linear functional

$$\mathrm{tr}\colon J \to \mathbb{R},$$

that is positive on the cone J_+. This gives the Jordan algebra an invariant *Euclidean* inner product

$$\langle a, b \rangle = \mathrm{tr}(a \circ b).$$

So, any spin factor has the structure of a Minkowski spacetime *and* a Euclidean spacetime!

Jordan algebras of 2×2 self-adjoint matrices are isomorphic to spin factors, and thus both Minkowski spacetimes and Euclidean spaces:

$$\begin{array}{lcccl}
\mathfrak{h}_2(\mathbb{R}) & \cong & \mathbb{R} \oplus \mathbb{R}^2 & \cong & \text{3d Minkowski/Euclidean spacetime} \\
\mathfrak{h}_2(\mathbb{C}) & \cong & \mathbb{R} \oplus \mathbb{R}^3 & \cong & \text{4d Minkowski/Euclidean spacetime} \\
\mathfrak{h}_2(\mathbb{H}) & \cong & \mathbb{R} \oplus \mathbb{R}^5 & \cong & \text{6d Minkowski/Euclidean spacetime} \\
\mathfrak{h}_2(\mathbb{O}) & \cong & \mathbb{R} \oplus \mathbb{R}^9 & \cong & \text{10d Minkowski/Euclidean spacetime}
\end{array}$$

If $t, x \in \mathbb{R}$ and $y \in \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ we have

$$\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

$$\frac{1}{2} \mathrm{tr} \left(\begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \circ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \right) = t^2 + x^2 + |y|^2$$

A Euclidean Jordan algebra does not merely describe observables.
It also describes states.

A Euclidean Jordan algebra does not merely describe observables. It also describes states.

An element $s \in J_+$ with $\mathrm{tr}(s) = 1$ is called a **state**.

Given a state s and an observable a, the **expected value** of a in the state s is $\mathrm{tr}(s \circ a)$.

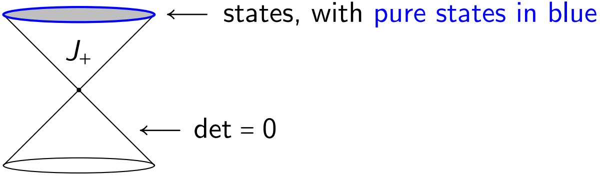
A **projection** $p \in J$ is an element with $p^2 = p$. A projection p with $\mathrm{tr}(p) = 1$ is a state called a **pure state**.

For $J = \mathfrak{h}_n(\mathbb{C})$, all this is familiar. Here a state is just a density matrix: a non-negative self-adjoint matrix with trace 1.

- The space of pure states for $\mathfrak{h}_n(\mathbb{R})$ is $\mathbb{R}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{C})$ is $\mathbb{C}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{H})$ is $\mathbb{H}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{O})$ is $\mathbb{O}\mathrm{P}^{n-1}$ (for $n \le 3$).

- The space of pure states for $\mathfrak{h}_n(\mathbb{R})$ is $\mathbb{R}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{C})$ is $\mathbb{C}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{H})$ is $\mathbb{H}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{O})$ is $\mathbb{O}\mathrm{P}^{n-1}$ (for $n \le 3$).
- The space of pure states for $\mathbb{R} \oplus \mathbb{R}^n$ is S^{n-1}.

A picture of the spin factor $\mathbb{R} \oplus \mathbb{R}^n$ for $n = 2$:



So:

- $\mathfrak{h}_2(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}^2$ has $\mathbb{R}\mathrm{P}^1 \cong S^1$ as its set of pure states.
- $\mathfrak{h}_2(\mathbb{C}) \cong \mathbb{R} \oplus \mathbb{R}^3$ has $\mathbb{C}\mathrm{P}^1 \cong S^2$ as its set of pure states.
- $\mathfrak{h}_2(\mathbb{H}) \cong \mathbb{R} \oplus \mathbb{R}^5$ has $\mathbb{H}\mathrm{P}^1 \cong S^4$ as its set of pure states.
- $\mathfrak{h}_2(\mathbb{O}) \cong \mathbb{R} \oplus \mathbb{R}^9$ has $\mathbb{O}\mathrm{P}^1 \cong S^8$ as its set of pure states.

A chiral spinor in 3, 4, 6 or 10-dimensional spacetime is described by a real, complex, quaternionic or octonionic qubit.

But what about octonionic qutrits?

WHAT ABOUT $\mathfrak{h}_3(\mathbb{O})$?

The **exceptional Jordan algebra**

$$\mathfrak{h}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R},\ x, y, z \in \mathbb{O} \right\}$$

is 27-dimensional, and its automorphism group is the 52-dimensional exceptional Lie group F_4.

Ivan Todorov and Michel Dubois-Violette showed the symmetries of $\mathfrak{h}_3(\mathbb{O})$ are connected to the Standard Model gauge group!

Building on their work we showed this stronger result, with help from David Madore:

Theorem 1. Suppose A, B are Jordan subalgebras of $\mathfrak{h}_3(\mathbb{O})$ such that

$$A \cong h_2(\mathbb{O}) \qquad B \cong \mathfrak{h}_3(\mathbb{C})$$

$$A \cap B \cong \mathfrak{h}_2(\mathbb{C})$$

Then

$$\mathrm{Stab}(A) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

where Stab_0 means the *identity component* of the stabilizer.

For example, since

$$\mathfrak{h}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R},\ x, y, z \in \mathbb{O} \right\}$$

we can take

$$A = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R},\ z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

$$B = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R},\ x, y, z \in \mathbb{C} \right\} \cong \mathfrak{h}_3(\mathbb{C})$$

$$A \cap B = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R},\ z \in \mathbb{C} \right\} \cong \mathfrak{h}_2(\mathbb{C})$$

Dubois-Violette and Todorov showed that *in this example*

$$\mathrm{Stab}(A) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

So they showed that the theorem holds *in this special case*.

Let's sketch an argument for this.

Dubois-Violette and Todorov showed that *in this example*

$$\mathrm{Stab}(A) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

So they showed that the theorem holds *in this special case*.

Let's sketch an argument for this.

The first step is to show that

$$\mathrm{Stab}_0(B) \cong (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$$

when

$$B = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \ \alpha, \beta, \gamma \in \mathbb{R}, \ x, y, z \in \mathbb{C} \right\} \cong \mathfrak{h}_3(\mathbb{C})$$

We can build the quaternions as

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$$

with multiplication given by

$$(a + \vec{a})(b + \vec{b}) = ab - \vec{a} \cdot \vec{b} + a\vec{b} + b\vec{a} + \vec{a} \times \vec{b}$$

Thus $\mathrm{SO}(3)$ acts as automorphisms of the quaternions.

We can build the quaternions as

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$$

with multiplication given by

$$(a + \vec{a})(b + \vec{b}) = ab - \vec{a} \cdot \vec{b} + a\vec{b} + b\vec{a} + \vec{a} \times \vec{b}$$

Thus $\mathrm{SO}(3)$ acts as automorphisms of the quaternions.

Similarly, we can build the octonions as

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$$

with multiplication

$$(a + \vec{a})(b + \vec{b}) = ab - \langle \vec{a}, \vec{b} \rangle + \overline{a}\,\vec{b} + b\vec{a} + \overline{\vec{a} \times \vec{b}}$$

where the bar is componentwise complex conjugation. Thus $\mathrm{SU}(3)$ acts as automorphisms of $\mathbb{O}$ preserving $\mathbb{C} \subset \mathbb{O}$.

Splitting the exceptional Jordan algebra as an orthogonal direct sum of vector spaces:

$$\mathfrak{h}_3(\mathbb{O}) \cong \mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{h}_3(\mathbb{C})^\perp$$

we have

$$\begin{aligned}
\mathfrak{h}_3(\mathbb{C})^\perp &= \left\{ \begin{pmatrix} 0 & z & y^* \\ z^* & 0 & x \\ y & x^* & 0 \end{pmatrix} : \; x, y, z \in \mathbb{C}^3 \subset \mathbb{O} \right\} \\
&\cong \{(x, y, z) : \; x, y, z \in \mathbb{C}^3\} \\
&\cong \mathrm{M}_3(\mathbb{C})
\end{aligned}$$

where $\mathrm{M}_3(\mathbb{C})$ is the space of 3×3 complex matrices.

We thus get a vector space isomorphism

$$\mathfrak{h}_3(\mathbb{O}) \cong \mathfrak{h}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C})$$

which lets us think of an element of $\mathfrak{h}_3(\mathbb{O})$ as a pair

$$(X, M) \in \mathfrak{h}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C})$$

$(g, h) \in \mathrm{SU}(3) \times \mathrm{SU}(3)$ acts on such pairs as follows:

$$(g, h) \colon (X, M) \mapsto (gXg^\dagger, hMg^\dagger)$$

$(e^{2\pi i/3}, e^{2\pi i/3}) \in \mathrm{SU}(3) \times \mathrm{SU}(3)$ acts trivially, so we get an action of $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ on $\mathfrak{h}_3(\mathbb{O})$.

We thus get a representation of $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ on $\mathfrak{h}_3(\mathbb{O})$ that preserves:

- the subalgebra $B \cong \mathfrak{h}_3(\mathbb{C})$ (by construction)
- the Jordan product on $\mathfrak{h}_3(\mathbb{O})$ (an easy calculation)

With more work — e.g. Borel–de Siebenthal theory — one can show this group is the *largest* connected group of automorphisms of $\mathfrak{h}_3(\mathbb{O})$ that preserves B.

Thus

$$\mathrm{Stab}_0(B) \cong (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$$

The two $\mathrm{SU}(3)$'s in $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ act very differently on $\mathfrak{h}_3(\mathbb{O})$.

The first $\mathrm{SU}(3)$ acts to *mix up* the matrix entries:

$$g\colon (X, M) \mapsto (gXg^\dagger, Mg^\dagger)$$

and only its subgroup $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ preserves

$$A = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \ \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

The two $\mathrm{SU}(3)$'s in $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ act very differently on $\mathfrak{h}_3(\mathbb{O})$.

The first $\mathrm{SU}(3)$ acts to *mix up* the matrix entries:

$$g\colon (X, M) \mapsto (gXg^\dagger, Mg^\dagger)$$

and only its subgroup $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ preserves

$$A = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \ \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

This subgroup becomes the electroweak gauge group:

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \cong \big(\mathrm{SU}(2) \times \mathrm{U}(1)\big)/\mathbb{Z}_2$$

The second $\mathrm{SU}(3)$ becomes the strong force gauge group: it acts *separately* on each matrix entry

$$\begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix}$$

as octonion automorphisms that preserve $\mathbb{C} \subset \mathbb{O}$. Thus, this whole copy of $\mathrm{SU}(3)$ preserves

$$A = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \ \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

Therefore

$$\mathrm{Stab}(A) \cap \mathrm{Stab}_0(B) \ \cong \ \big(\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \times \mathrm{SU}(3)\big)/\mathbb{Z}_3$$

But

$$\big(\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \times \mathrm{SU}(3)\big)/\mathbb{Z}_3 \;\cong\; \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

We thus get

$$\mathrm{Stab}(A) \cap \mathrm{Stab}_0(B) \cong \; \big(\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)) \times \mathrm{SU}(3)\big)/\mathbb{Z}_3$$

proving Theorem 1 in this special case.

To prove Theorem 1, it thus suffices to show that Dubois-Violette and Todorov's example is universal, in the following sense:

Proposition. F_4 acts transitively on pairs of Jordan subalgebras $A, B \subset \mathfrak{h}_3(\mathbb{O})$ with $A \cong \mathfrak{h}_2(\mathbb{O})$ and $B \cong \mathfrak{h}_3(\mathbb{C})$ such that $A \cap B \cong \mathfrak{h}_2(\mathbb{C})$.

I'll put the proof at the end.

To prove Theorem 1, it thus suffices to show that Dubois-Violette and Todorov's example is universal, in the following sense:

Proposition. F_4 acts transitively on pairs of Jordan subalgebras $A, B \subset \mathfrak{h}_3(\mathbb{O})$ with $A \cong \mathfrak{h}_2(\mathbb{O})$ and $B \cong \mathfrak{h}_3(\mathbb{C})$ such that $A \cap B \cong \mathfrak{h}_2(\mathbb{C})$.

I'll put the proof at the end. In proving this, we discovered

Lemma. Every Jordan subalgebra of $\mathfrak{h}_3(\mathbb{O})$ isomorphic to $\mathfrak{h}_2(\mathbb{C})$ is contained in a unique Jordan subalgebra isomorphic to $\mathfrak{h}_2(\mathbb{O})$.

This lets us avoid mentioning $\mathfrak{h}_2(\mathbb{O})$ in Theorem 1: B and $A \cap B$ determine A.

We thus get a nicer result:

Theorem 2. Suppose X, B are Jordan subalgebras of $\mathfrak{h}_3(\mathbb{O})$ such that

$$X \cong h_2(\mathbb{C}) \qquad B \cong \mathfrak{h}_3(\mathbb{C})$$

$$X \subset B$$

Then

$$\mathrm{Stab}(X) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

CONCLUSIONS

So, loosely we can say the Standard Model gauge group consists of symmetries of an octonionic qutrit that preserve a complex qutrit contained within it, and complex qubit contained within that:

$$\mathfrak{h}_2(\mathbb{C}) \subset \mathfrak{h}_3(\mathbb{C}) \subset \mathfrak{h}_3(\mathbb{O})$$

More precisely it's the identity component: there's another component that switches i and $-i$.

This might be just a coincidence. Or maybe not.

Finally, let's sketch the proof of this:

Proposition. F_4 acts transitively on pairs of Jordan subalgebras $A, B \subset \mathfrak{h}_3(\mathbb{O})$ with $A \cong \mathfrak{h}_2(\mathbb{O})$ and $B \cong \mathfrak{h}_3(\mathbb{C})$ such that $A \cap B \cong \mathfrak{h}_2(\mathbb{C})$.

Proof. Let $(A, B), (A', B')$ be pairs as above. We need $\tilde{g} \in \mathrm{F}_4$ with $gA = A',\ gB = B'$.

Lemma. F_4 acts transitively on the set of Jordan subalgebras $B \subset \mathfrak{h}_3(\mathbb{O})$ with $B \cong \mathfrak{h}_3(\mathbb{C})$.

Given this we can assume w.l.o.g. that $B = B'$ is the 'standard' copy of $\mathfrak{h}_3(\mathbb{C})$ in $\mathfrak{h}_3(\mathbb{O})$. It then suffices to find $g \in \mathrm{Stab}(\mathfrak{h}_3(\mathbb{C}))$ with $gA = A'$.

By assumption, $A \cap \mathfrak{h}_3(\mathbb{C})$ and $A' \cap \mathfrak{h}_3(\mathbb{C})$ are Jordan subalgebras isomorphic to $\mathfrak{h}_2(\mathbb{C})$. $\mathrm{SU}(3)$ acts on $\mathfrak{h}_3(\mathbb{C})$ by

$$g : X \to gXg^\dagger$$

and it acts transitively on the set of subalgebras of $\mathfrak{h}_3(\mathbb{C})$ isomorphic to $\mathfrak{h}_2(\mathbb{C})$. Thus, we can find $g_0 \in \mathrm{SU}(3)$ moving $A \cap \mathfrak{h}_3(\mathbb{C})$ to $A' \cap \mathfrak{h}_3(\mathbb{C})$.

We have earlier seen $\mathrm{Stab}_0(\mathfrak{h}_3(\mathbb{C})) = (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$, and $g = [g_0, 1] \in \mathrm{Stab}_0(\mathfrak{h}_3(\mathbb{C}))$ acts on $\mathfrak{h}_3(\mathbb{C})$ just as g_0 does. Thus g stabilizes $\mathfrak{h}_3(\mathbb{C})$ and moves $A \cap \mathfrak{h}_3(\mathbb{C})$ to $A' \cap \mathfrak{h}_3(\mathbb{C})$. To conclude, we just need to show $gA = A'$.

Lemma. Any Jordan subalgebra of $\mathfrak{h}_3(\mathbb{O})$ isomorphic to $\mathfrak{h}_2(\mathbb{C})$ is contained in a unique subalgebra isomorphic to $\mathfrak{h}_2(\mathbb{O})$.

gA is a Jordan algebra isomorphic to $\mathfrak{h}_2(\mathbb{O})$ that contains $g(A \cap \mathfrak{h}_3(\mathbb{C})) = A' \cap \mathfrak{h}_3(\mathbb{C}) \cong \mathfrak{h}_2(\mathbb{C})$, but A' is another subalgebra with this property, so by the Lemma $gA = A'$. □