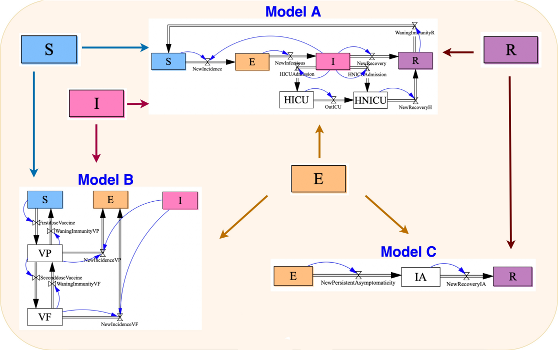
COMPOSITIONAL MODELING WITH DECORATED COSPANS



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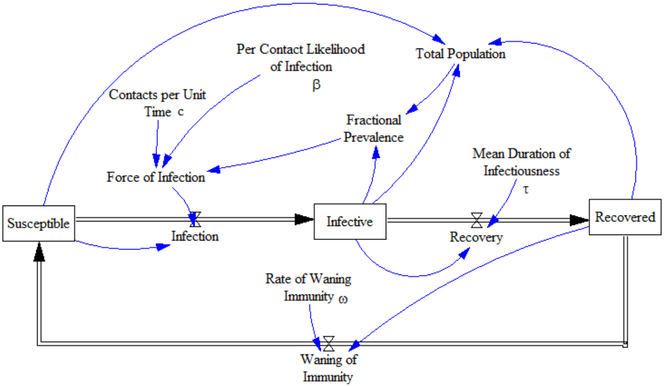
There is a community of epidemiologists who use “stock-flow diagrams” to model the spread of disease. This includes my coauthors Nathaniel Osgood and Xiaoyan Li, who do COVID modeling for the Canadian government.

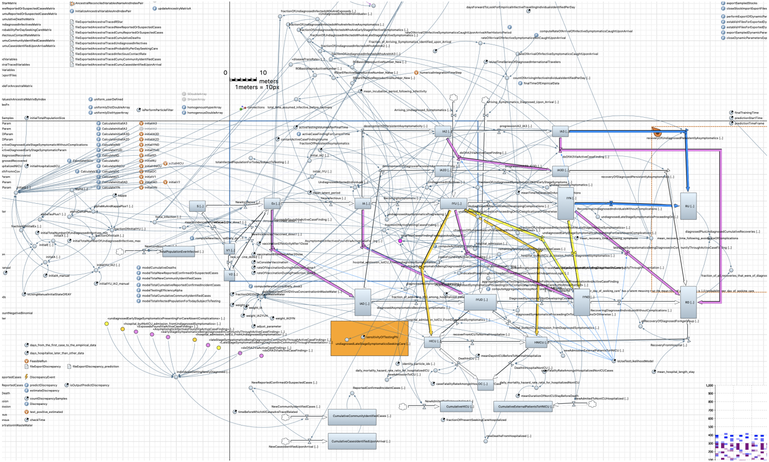


There is a procedure to turn stock-flow diagrams into "dynamical systems" — that is, systems of nonlinear first-order ODE.

If we build a large stock-flow diagram by gluing together smaller ones, the dynamical system for the large diagram can be obtained from those of the smaller open diagrams. This is **compositional modeling**.

But currently most modeling using stock-flow diagrams is done using AnyLogic, a piece of software that does not support compositional modeling.

Here is Osgood and Li’s COVID model in AnyLogic, used by the government of Canada:



This is a problem! We want to let teams of modelers individually build smaller stock-flow diagrams, run them and test them, and then later glue them together to form larger models.

Osgood and Li have now created software that makes this possible — working with Evan Patterson and Sophie Libkind at the Topos Institute, and me.

We used AlgebraicJulia: a framework for high-performance scientific computing using category theory, developed by James Fairbanks, Patterson, Libkind and others.

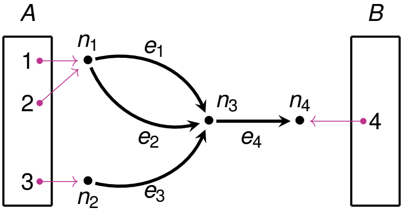
And we used the mathematics of “decorated cospans”.

Decorated cospans have been used to study open systems of many kinds:

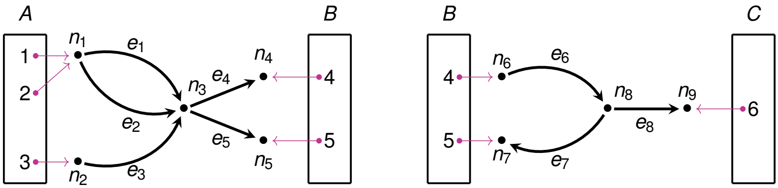
- open graphs
- open Petri nets
- open electrical circuits
- open Markov processes
- open dynamical systems
- open chemical reaction networks
- open stock-flow diagrams

The simplest example: “open graphs”.

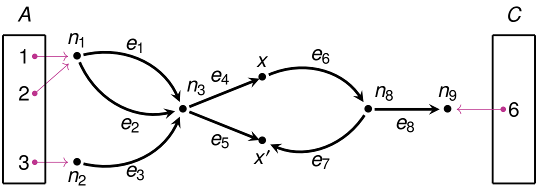
Here is an open graph with inputs A and outputs B:



We can compose open graphs by gluing them end to end:

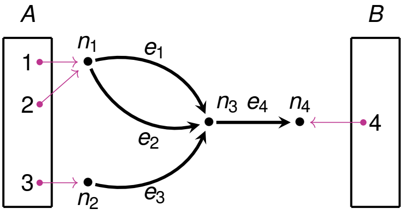


obtaining this:

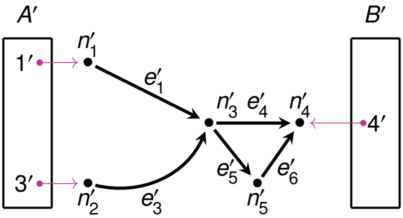


This composition is associative *up to isomorphism*. But what do we mean by ‘isomorphism’ here?

There is a *category* of open graphs. For example there is a morphism from this one:



to this one:



So:

- there are morphisms *between* open graphs, and composition of these morphisms is associative
- open graphs are *themselves* morphisms of some sort: we can compose them by gluing them end to end, and this composition is associative up to isomorphism

The structure that captures both these forms of composition is a “double category”.

A double category has figures like this:

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N"'] & D \end{tikzcd}$$

So, it has:

- **objects** such as A, B, C, D,
- **vertical 1-morphisms** such as f and g,
- **horizontal 1-cells** such as M and N,
- **2-morphisms** such as α.

Vertical 1-morphisms can be composed. Horizontal 1-cells can be composed. 2-morphisms can be composed vertically and horizontally, and the interchange law holds:

$$
\begin{tikzcd}
A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
D \arrow[r, "N"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "M'"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\
E \arrow[r, "N'"'] & F
\end{tikzcd}
$$

$$
\begin{tikzcd}
D \arrow[r, "N"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Downarrow\alpha'"] & E \arrow[d, "g'"] \\
G \arrow[r, "O"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "N'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Downarrow\beta'"] & F \arrow[d, "h'"] \\
H \arrow[r, "P"'] & I
\end{tikzcd}
$$

Vertical composition is strictly associative and unital. Horizontal composition obeys these laws only up to isomorphism.

Now let's see how to build the double category of open graphs using decorated cospans!

The same procedure also works for our other examples:

- open Petri nets
- open electrical circuits
- open Markov processes
- open dynamical systems
- open Petri nets with rates
- open stock-flow diagrams

Given a finite set N, a **graph on N** is a finite set E of **edges** and two functions $s, t\colon E \to N$ giving the **source** and **target** of each edge.

For any finite set N, there is a category $F(N)$ of graphs on N. A morphism from $s, t\colon E \to N$ to $s', t'\colon E' \to N$ is a map of edges $f\colon E \to E'$ preserving source and target:

$$\begin{tikzcd} E \arrow[d, "f"'] \arrow[dr, "s"] & \\ E' \arrow[r, "s'"'] & N \end{tikzcd} \qquad \begin{tikzcd} E \arrow[d, "f"'] \arrow[dr, "t"] & \\ E' \arrow[r, "t'"'] & N \end{tikzcd}$$

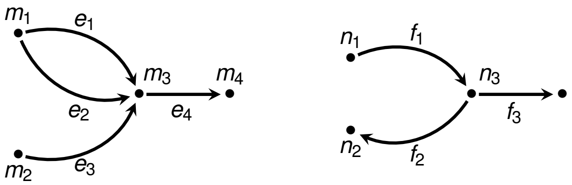
Indeed, there's a lax monoidal pseudofunctor

$$F\colon (\mathrm{FinSet}, +) \to (\mathbf{Cat}, \times)$$

sending each finite set to the category of graphs on that set.

Roughly:

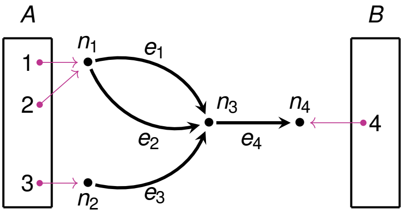
- "pseudofunctor": $F(f) \circ F(g) \cong F(f \circ g)$.
- "lax monoidal": we have $\phi_{M,N}\colon F(M) \times F(N) \to F(M+N)$



An **open graph** is a cospan of finite sets:

$$A \xrightarrow{i} N \xleftarrow{o} B$$

together with a graph on N, say $d \in F(N)$. For example:



More generally, given any lax monoidal pseudofunctor $F\colon (\mathrm{FinSet}, +) \to (\mathbf{Cat}, \times)$, a **decorated cospan** is a cospan of finite sets:

$$A \xrightarrow{i} N \xleftarrow{o} B$$

together with a **decoration** $d \in F(N)$.

So, open graphs are an example of decorated cospans... but so are all the other open structures I mentioned, with different choices of F.

Theorem 1 (Baez–Courser–Vasilakopoulou)

Suppose $F\colon (\mathrm{FinSet}, +) \to (\mathbf{Cat}, \times)$ is a lax monoidal pseudofunctor. Then there is a double category $F\mathbb{C}\mathbf{sp}$ where:

- an object is a finite set.
- a vertical 1-morphism is a function.
- a horizontal 1-cell is a decorated cospan:

$$A \xrightarrow{\;i\;} N \xleftarrow{\;o\;} B \qquad d \in F(N)$$

- a 2-morphism is a commuting diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & N \arrow[d, "h"'] & B \arrow[l, "o"'] \arrow[d, "g"] & d \in F(N) \\ A' \arrow[r, "i'"] & N' & B' \arrow[l, "o'"'] & d' \in F(N') \end{tikzcd}$$

together with a **decoration morphism** $\tau\colon F(h)(d) \to d'$.

Theorem 2 (Baez–Courser–Vasilakopoulou)

Suppose $F, G\colon (\mathrm{FinSet}, +) \to (\mathbf{Cat}, \times)$ are lax monoidal pseudofunctors and $\theta\colon F \Rightarrow G$ is a monoidal natural transformation. Then there is a map of double categories

$$\theta_*\colon F\mathbb{C}\mathbf{sp} \to G\mathbb{C}\mathbf{sp}$$

sending each F-decorated cospan

$$A \xrightarrow{i} N \xleftarrow{o} B \qquad d \in F(N)$$

to the G-decorated cospan

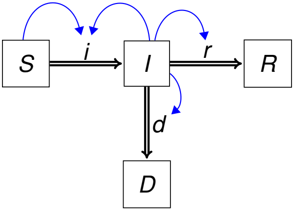
$$A \xrightarrow{i} N \xleftarrow{o} B \qquad \theta_N(d) \in G(N)$$

Now let's apply decorated cospans to stock-flow diagrams!

In its simplest form, a **stock-flow diagram** consists of finite sets and functions:

$$\begin{tikzcd}
\text{Flows} \arrow[rr, shift left, "u"] \arrow[rr, shift right, "d"'] & & \text{Stocks} \\
& \text{Links} \arrow[ul, "t"] \arrow[ur, "s"'] &
\end{tikzcd}$$

together with, for each $f \in$ Flows, a function $\phi_f : \mathbb{R}^{L(f)} \to \mathbb{R}$ where $L(f)$ is the set of links whose target is f.

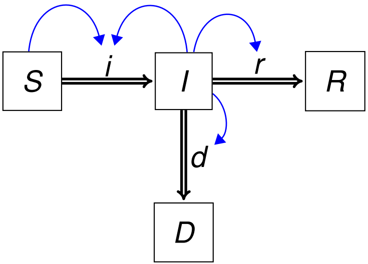


Define a **dynamical system** on a finite set N to be a smooth vector field v on $\mathbb{R}^N$. This gives a differential equation

$$\frac{d}{dt}x(t) = v(x(t))$$

describing how the stocks $x(t) \in \mathbb{R}^N$ change with time.

Each stock-flow diagram with set N of stocks gives a dynamical system on N:



$$\frac{dS}{dt} = -\phi_i(S, I)$$

$$\frac{dI}{dt} = \phi_i(S, I) - \phi_r(I) - \phi_d(I)$$

$$\frac{dR}{dt} = \phi_r(I)$$

$$\frac{dD}{dt} = \phi_d(I)$$

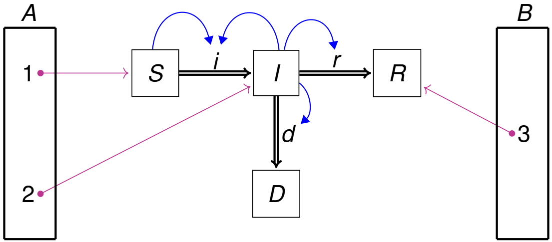
For any finite set N, there is a category $F(N)$ of stock-flow diagrams with N as their set of stocks.

There is a lax monoidal pseudofunctor

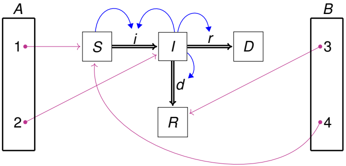
$$F\colon (\mathrm{FinSet}, +) \to (\mathbf{Cat}, \times)$$

sending each finite set N to the category $F(N)$.

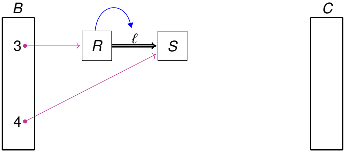
Thus, by Theorem 1, there is a double category $F\mathbb{C}\mathbf{sp}$ of *open* stock-flow diagrams.



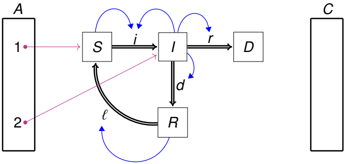
$$A \xrightarrow{F} B$$



$$B \xrightarrow{G} C$$



$$A \xrightarrow{GF} C$$



For any finite set N, there is a category $G(N)$ with dynamical systems on N as objects and only identity morphisms.

There is a lax monoidal pseudofuctor

$$G\colon (\mathrm{FinSet}, +) \to (\mathbf{Cat}, \times)$$

sending each finite set N to the category $G(N)$.

Thus, by Theorem 1, there is a double category $G\mathbb{C}\mathbf{sp}$ of *open* dynamical systems.

Finally, there's a natural transformation

$$\theta \colon F \Rightarrow G$$

such that

$$\theta_N \colon F(N) \to G(N)$$

sends each stock-flow diagram with N as its set of stocks to a dynamical system on N.

Thus, by Theorem 2, there is a map of double categories

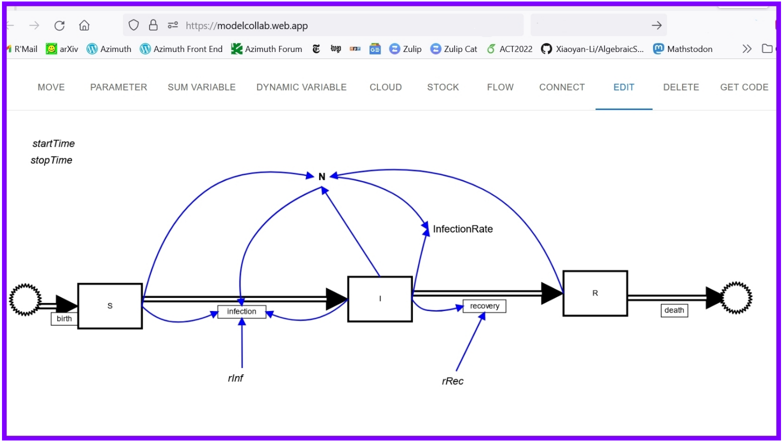
$$\theta_* \colon F\mathbb{C}\mathbf{sp} \to G\mathbb{C}\mathbf{sp}$$

sending open stock-flow diagrams to open dynamical systems!

By implementing these ideas in AlgebraicJulia, my coauthors created a software package called **StockFlow**, now available on GitHub. This lets you:

- build open stock-flow diagrams
- compose them
- turn them into open dynamical systems
- *solve* the differential equations given by these dynamical systems.

Nathaniel Osgood and his students have also made a graphical user interface for StockFlow. It runs in your browser, so teams can collaborate to build stock-flow diagrams.



For more try our papers:

- John Baez, Xiaoyan Li, Sophie Libkind, Nathaniel D. Osgood and Evan Patterson, Compositional modeling with stock and flow diagrams.
- John C. Baez, Xiaoyan Li, Sophie Libkind, Nathaniel D. Osgood, Long Pham and Eric Redekopp, A categorical framework for modeling with stock and flow diagrams.