Loop Groups and Lie 2-Algebras

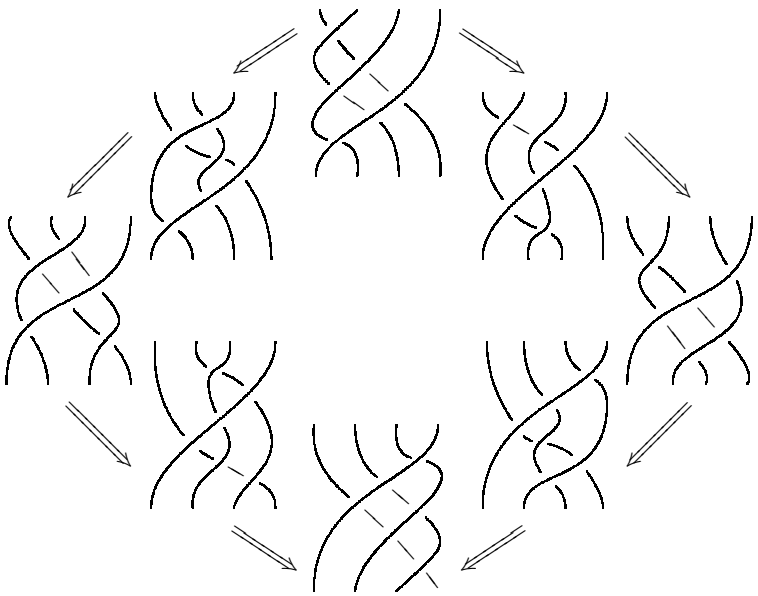
Alissa S. Crans

Joint work with:

John Baez
Urs Schreiber
& Danny Stevenson

in honor of
Ross Street's 60th birthday

July 15, 2005



Lie 2-Algebras

A **2-vector space** L is a category in Vect, the category of vector spaces.

We can also define **linear functors** between 2-vector spaces, and **linear natural transformations** between these, in the obvious way.

Theorem. The 2-category of 2-vector spaces, linear functors and linear natural transformations is equivalent to the 2-category of:

- 2-term chain complexes $C_1 \xrightarrow{d} C_0$,
- chain maps between these,
- chain homotopies between these.

Moral: Homological algebra is secretly categorified linear algebra!

A **Lie 2-algebra** consists of:

- a 2-vector space L

equipped with:

- a functor called the **bracket**:

$$[\cdot,\cdot]\colon L \times L \to L$$

  bilinear and skew-symmetric as a function of objects and morphisms,

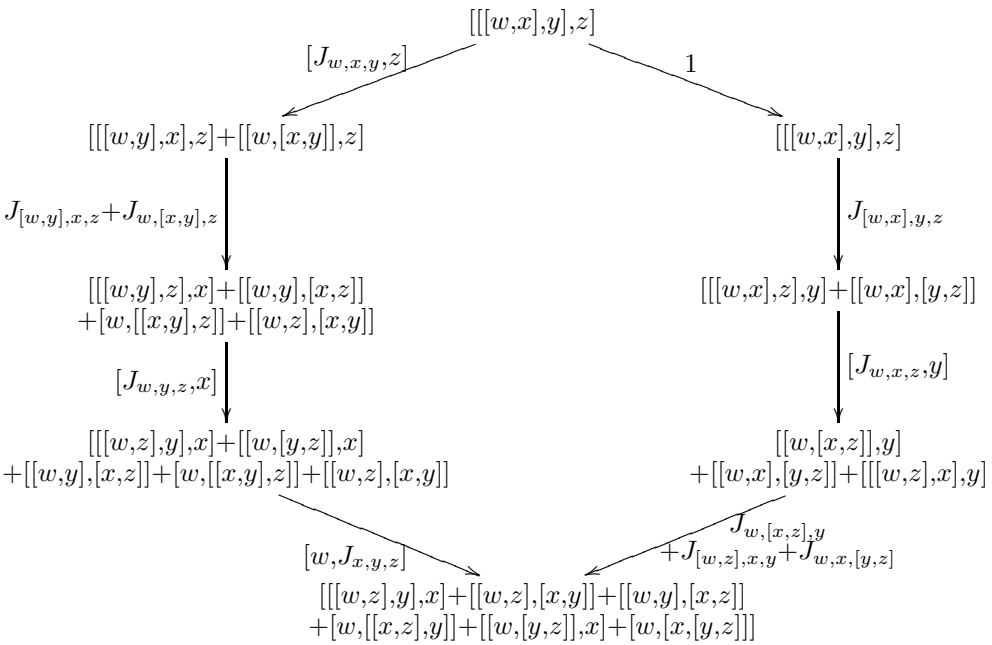
- a natural isomorphism called the **Jacobiator**:

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

  trilinear and antisymmetric as a function of the objects x, y, z,

such that:

- the **Jacobiator identity** holds, meaning the following diagram commutes:



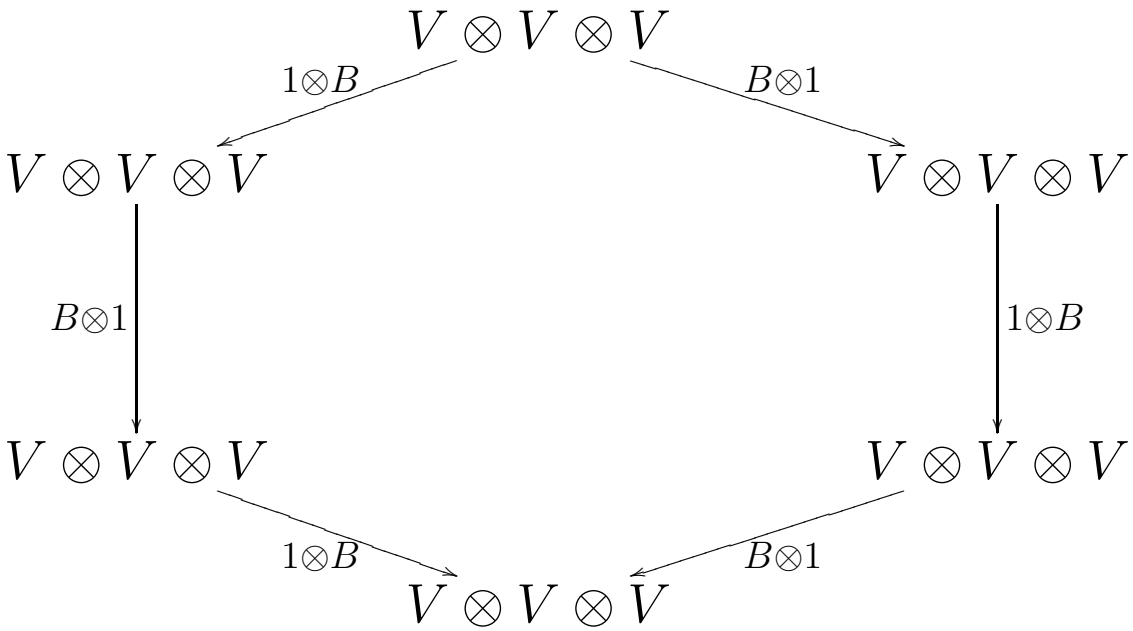
Given a vector space V and an isomorphism

$$B\colon V \otimes V \to V \otimes V,$$

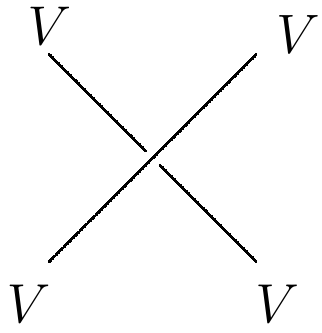
we say B is a **Yang–Baxter operator** if it satisfies the **Yang–Baxter equation**, which says that:

$$(B\otimes 1)(1\otimes B)(B\otimes 1) = (1\otimes B)(B\otimes 1)(1\otimes B),$$

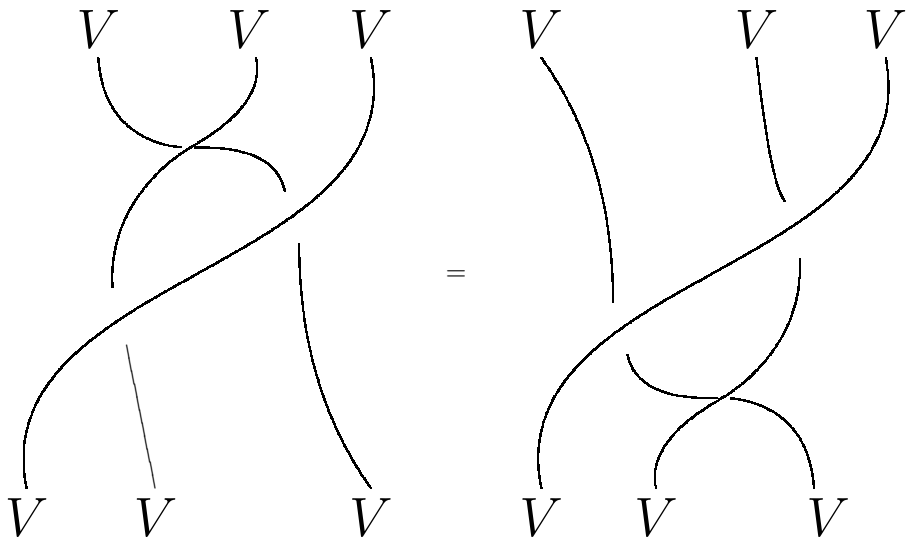
or in other words, that this diagram commutes:



If we draw $B\colon V \otimes V \to V \otimes V$ as a braiding:

$B =$ 

the Yang–Baxter equation says that:



Proposition. Let L be a vector space over k equipped with a skew-symmetric bilinear operation

$$[\cdot,\cdot]\colon L \times L \to L.$$

Let $L' = k \oplus L$ and define the isomorphism

$$B\colon L' \otimes L' \to L' \otimes L' \text{ by}$$

$$B((a,x)\otimes(b,y)) = (b,y)\otimes(a,x)+(1,0)\otimes(0,[x,y]).$$

Then B is a solution of the Yang–Baxter equation if and only if $[\cdot,\cdot]$ satisfies the Jacobi identity.

Zamolodchikov tetrahedron equation

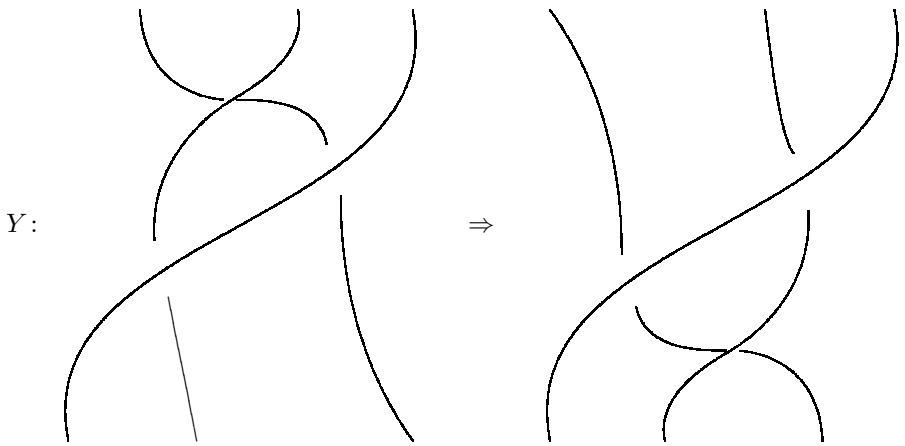
Given a 2-vector space V and an invertible linear functor $B\colon V\otimes V \to V\otimes V$, a linear natural isomorphism

$$Y\colon (B\otimes 1)(1\otimes B)(B\otimes 1) \Rightarrow (1\otimes B)(B\otimes 1)(1\otimes B)$$

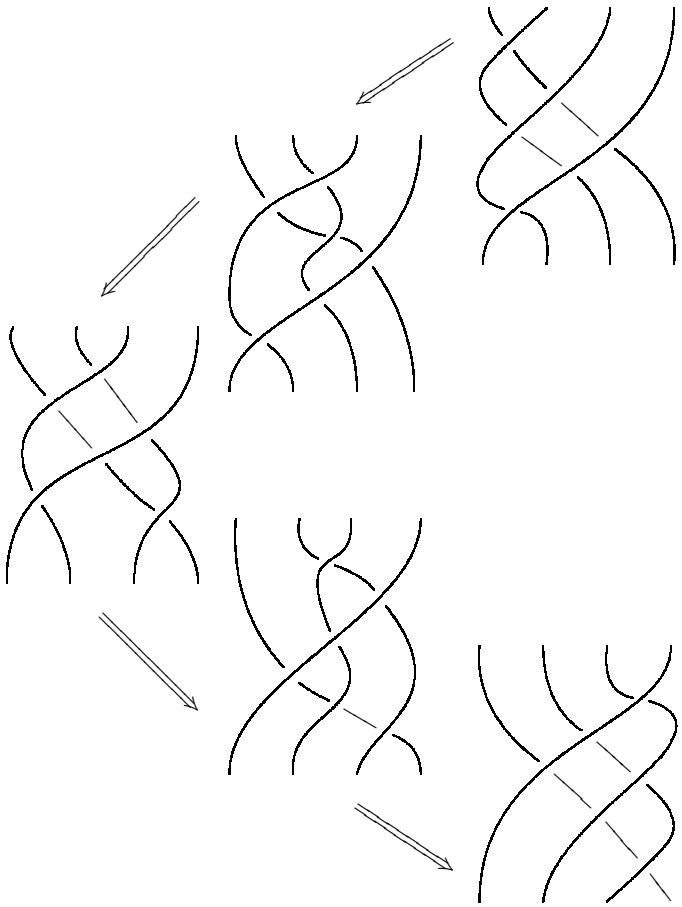
satisfies the **Zamolodchikov tetrahedron equation** if:

$$[Y\circ(1\otimes 1\otimes B)(1\otimes B\otimes 1)(B\otimes 1\otimes 1)][(1\otimes B\otimes 1)(B\otimes 1\otimes 1)\circ Y\circ(B\otimes 1\otimes 1)]$$
$$[(1\otimes B\otimes 1)(1\otimes 1\otimes B)\circ Y\circ(1\otimes 1\otimes B)][Y\circ(B\otimes 1\otimes 1)(1\otimes B\otimes 1)(1\otimes 1\otimes B)]$$
$$=$$
$$[(B\otimes 1\otimes 1)(1\otimes B\otimes 1)(1\otimes 1\otimes B)\circ Y][(B\otimes 1\otimes 1)\circ Y\circ(B\otimes 1\otimes 1)(1\otimes B\otimes 1)]$$
$$[(1\otimes 1\otimes B)\circ Y\circ(1\otimes 1\otimes B)(1\otimes B\otimes 1)][(1\otimes 1\otimes B)(1\otimes B\otimes 1)(B\otimes 1\otimes 1)\circ Y]$$

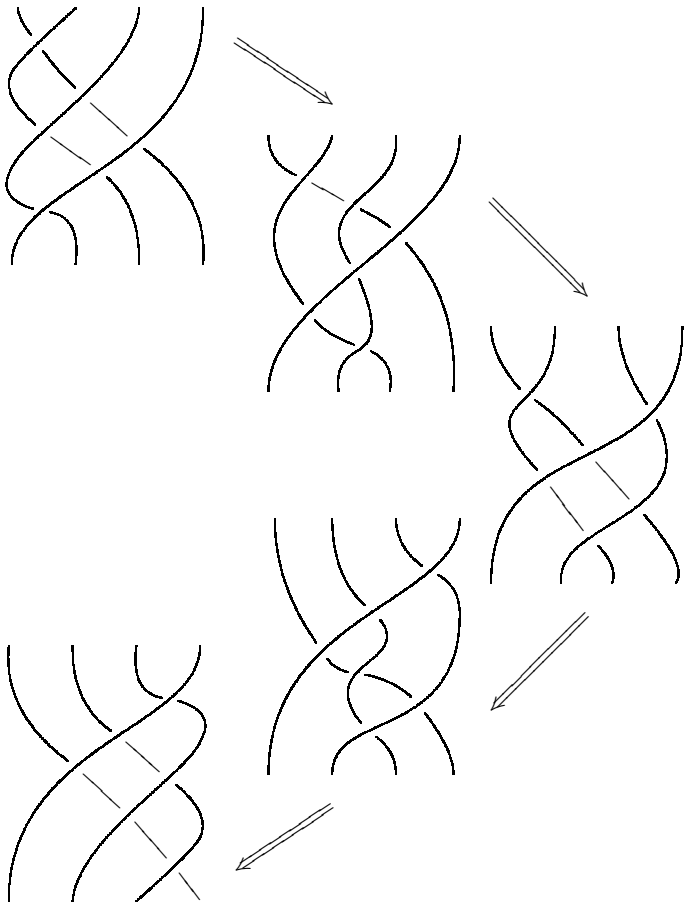
We should think of Y as the surface in 4-space traced out by the *process of performing* the third Reidemeister move:



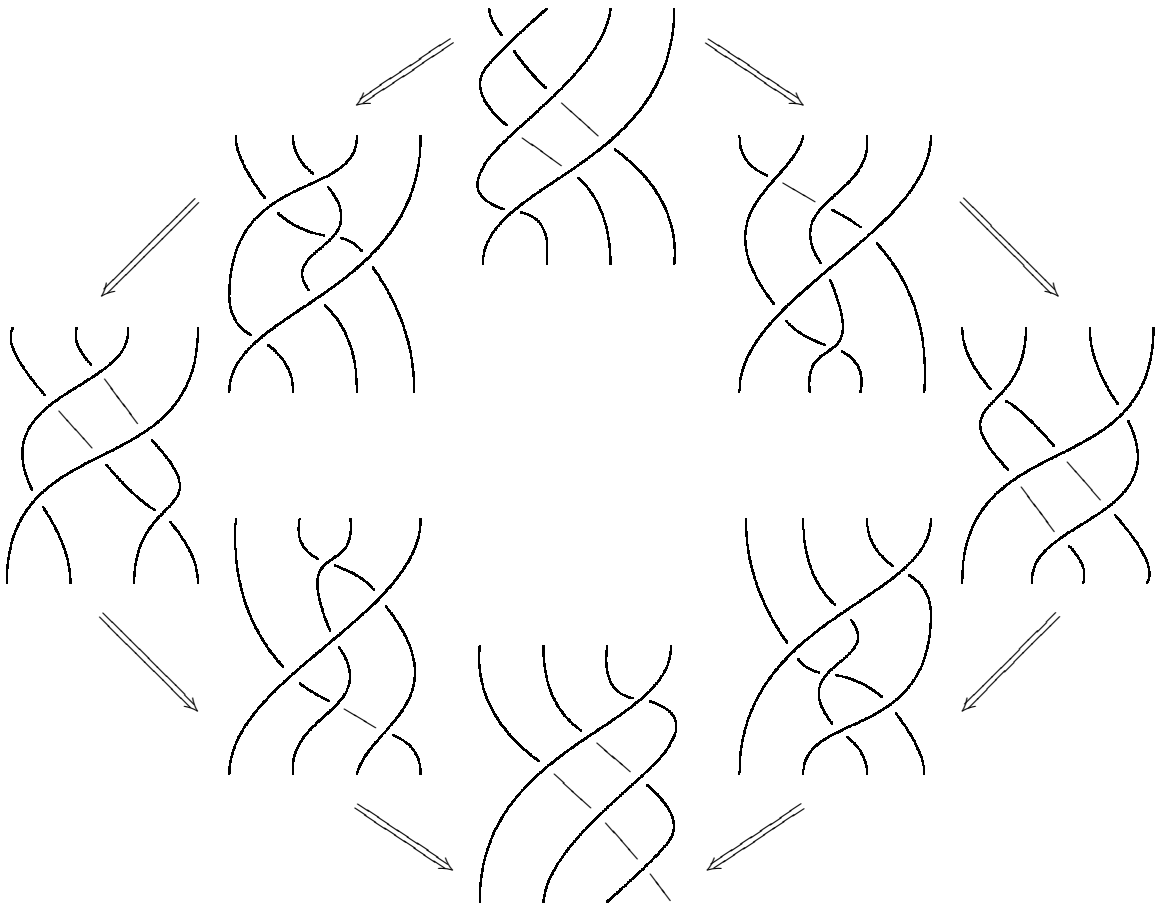
Left side of Zamolodchikov tetrahedron equation:



Right side of Zamolodchikov tetrahedron equation:



In short, the Zamolodchikov tetrahedron equation is a formalization of this commutative octagon:



Theorem. Let L be a 2-vector space, let $[\cdot,\cdot]\colon L \times L \to L$ be a skew-symmetric bilinear functor, and let J be a completely antisymmetric trilinear natural transformation with

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y].$$

Let $L' = K \oplus L$, where K is the categorified ground field.
Let $B\colon L' \otimes L' \to L' \otimes L'$ be defined as follows:

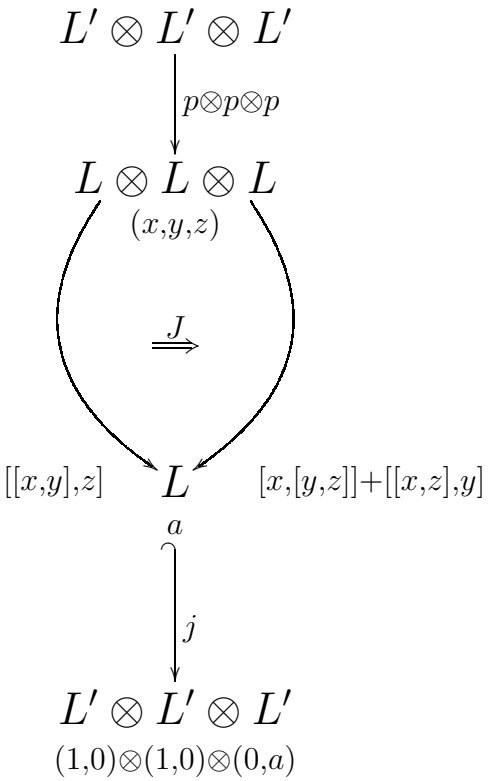
$$B((a,x) \otimes (b,y)) = (b,y) \otimes (a,x) + (1,0) \otimes (0,[x,y])$$

whenever (a,x) and (b,y) are both either objects or morphisms in L'. Finally, let

$$Y\colon (B \otimes 1)(1 \otimes B)(B \otimes 1) \Rightarrow (1 \otimes B)(B \otimes 1)(1 \otimes B)$$

be defined as follows:

$Y =$



where a is either an object or morphism of L. Then Y is a solution of the Zamolodchikov tetrahedron equation if and only if J satisfies the Jacobiator identity.

Hierarchy of Higher Commutativity

Topology	**Algebra**
Crossing	Commutator
Crossing of crossings	Jacobi identity
Crossing of crossing of crossings	Jacobiator identity
⋮	⋮

We can define **homomorphisms** between Lie 2-algebras, and **2-homomorphisms** between these.

Theorem. The 2-category of Lie 2-algebras, homomorphisms and 2-homomorphisms is equivalent to the 2-category of:

- 2-term L_∞-algebras,
- L_∞-homomorphisms between these,
- L_∞-2-homomorphisms between these.

The Lie 2-algebras L and L' are **equivalent** if there are homomorphisms

$$f \colon L \to L' \qquad \bar{f} \colon L' \to L$$

that are inverses up to 2-isomorphism:

$$f\bar{f} \cong 1, \qquad \bar{f}f \cong 1.$$

Theorem. Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra (= vector space) $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

The Lie 2-Algebra $\mathfrak{g}_k$

Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{R}$. To get a Lie 2-algebra having $\mathfrak{g}$ as objects we need:

- a vector space $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

Assume without loss of generality that ρ is irreducible. To get Lie 2-algebras with nontrivial Jacobiator, we need $H^3(\mathfrak{g}, \mathfrak{h}) \neq 0$. By Whitehead's lemma, this only happens when $\mathfrak{h} = \mathbb{R}$ is the trivial representation. Then we have

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

with a nontrivial 3-cocycle given by:

$$\nu(x, y, z) = \langle [x, y], z \rangle.$$

The Lie algebra $\mathfrak{g}$ together with the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ and k times the above 3-cocycle give the Lie 2-algebra $\mathfrak{g}_k$.

The 2-term L_∞-algebra V corresponding to the Lie 2-algebra $\mathfrak{g}_k$ is given by:

- $V_0 =$ the Lie algebra $\mathfrak{g}$,
- $V_1 = \mathbb{R}$,
- $d\colon V_1 \to V_0$ is the zero map,
- $l_2\colon V_0 \times V_0 \to V_0$ given by the bracket in $\mathfrak{g}$:

  $$l_2(x, y) = [x, y],$$

  and $l_2\colon V_0 \times V_1 \to V_1$ given by the trivial representation ρ of $\mathfrak{g}$ on $\mathbb{R}$,
- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ given by:

  $$l_3(x, y, z) = k\langle [x, y], z\rangle$$

  for all $x, y, z \in \mathfrak{g}$.

In summary: *every simple Lie algebra $\mathfrak{g}$ gives a one-parameter family of Lie 2-algebras, $\mathfrak{g}_k$, which reduces to $\mathfrak{g}$ when $k = 0$!*

Puzzle: Does $\mathfrak{g}_k$ come from a Lie 2-group?

Suppose we try to copy the construction of $\mathfrak{g}_k$ for a particularly nice kind of Lie group. Let G be a simply-connected compact simple Lie group whose Lie algebra is $\mathfrak{g}$. We have

$$H^3(G, \mathrm{U}(1)) \xleftarrow{\iota} \mathbb{Z} \hookrightarrow \mathbb{R} \cong H^3(\mathfrak{g}, \mathbb{R})$$

Using the classification of 2-groups, we can build a skeletal 2-group G_k for $k \in \mathbb{Z}$:

- G as its group of objects,
- $\mathrm{U}(1)$ as the group of automorphisms of any object,
- the trivial action of G on $\mathrm{U}(1)$,
- $[a] \in H^3(G, \mathrm{U}(1))$ given by $k\,\iota[\nu]$, which is nontrivial when $k \neq 0$.

Question: Can G_k be made into a Lie 2-group?

Here's the bad news:

(Bad News) Theorem. Unless $k = 0$, there is no way to give the 2-group G_k the structure of a Lie 2-group for which the group G of objects and the group $\mathrm{U}(1)$ of endomorphisms of any object are given their usual topology.

(Good News) Theorem. For any $k \in \mathbb{Z}$, there is a Fréchet Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ is equivalent to $\mathfrak{g}_k$.

An object of $\mathcal{P}_k G$ is a smooth path $f : [0, 2\pi] \to G$ starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, α) consisting of a disk D going from f_1 to f_2 together with $\alpha \in \mathrm{U}(1)$.

For any two such pairs (D_1, α_1) and (D_2, α_2) there is a 3-ball B whose boundary is $D_1 \cup D_2$, and the pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where ν is the left-invariant closed 3-form on G with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that ν gives an integral cohomology class.

The Lie 2-Algebra $\mathcal{P}_k\mathfrak{g}$

$\mathcal{P}_kG$ is a particularly nice kind of Lie 2-group: a *strict* one! Thus, its Lie 2-algebra is easy to compute.

The 2-term L_∞-algebra V corresponding to the Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ is given by:

- $V_0 = P_0\mathfrak{g}$
- $V_1 = \widehat{\Omega_k\mathfrak{g}} \cong \Omega\mathfrak{g} \oplus \mathbb{R}$,
- $d\colon V_1 \to V_0$ equal to the composite
$$\widehat{\Omega_k\mathfrak{g}} \to \Omega\mathfrak{g} \hookrightarrow P_0\mathfrak{g}\,,$$
- $l_2\colon V_0 \times V_0 \to V_0$ given by the bracket in $P_0\mathfrak{g}$:
$$l_2(p_1, p_2) = [p_1, p_2],$$

  and $l_2\colon V_0 \times V_1 \to V_1$ given by the action $d\alpha$ of $P_0\mathfrak{g}$ on $\widehat{\Omega_k\mathfrak{g}}$, or explicitly:
$$l_2(p, (\ell, c)) = \left([p, \ell],\ 2k \int_0^{2\pi} \langle p(\theta), \ell'(\theta)\rangle\, d\theta\ \right)$$
  for all $p \in P_0\mathfrak{g}$, $\ell \in \Omega G$ and $c \in \mathbb{R}$,

- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ equal to zero.

The 2-term L_∞-algebra V corresponding to the Lie 2-algebra $\mathfrak{g}_k$ is given by:

- $V_0 =$ the Lie algebra $\mathfrak{g}$,
- $V_1 = \mathbb{R}$,
- $d\colon V_1 \to V_0$ is the zero map,
- $l_2\colon V_0 \times V_0 \to V_0$ given by the bracket in $\mathfrak{g}$:

  $$l_2(x, y) = [x, y],$$

  and $l_2\colon V_0 \times V_1 \to V_1$ given by the trivial representation ρ of $\mathfrak{g}$ on $\mathbb{R}$,
- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ given by:

  $$l_3(x, y, z) = k\langle [x, y], z\rangle$$

  for all $x, y, z \in \mathfrak{g}$.

The Equivalence $\mathcal{P}_k\mathfrak{g} \simeq \mathfrak{g}_k$

We describe the two Lie 2-algebra homomorphisms forming our equivalence in terms of their corresponding L_∞-algebra homomorphisms:

- $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$ has:

$$\begin{aligned} \phi_0(p) &= p(2\pi) \\ \phi_1(\ell, c) &= c \end{aligned}$$

  where $p \in P_0\mathfrak{g}$, $\ell \in \Omega\mathfrak{g}$, and $c \in \mathbb{R}$.

- $\psi\colon \mathfrak{g}_k \to \mathcal{P}_k\mathfrak{g}$ has:

$$\begin{aligned} \psi_0(x) &= xf \\ \psi_1(c) &= (0, c) \end{aligned}$$

  where $x \in \mathfrak{g}$, $c \in \mathbb{R}$, and $f\colon [0, 2\pi] \to \mathbb{R}$ is a smooth function with $f(0) = 0$ and $f(2\pi) = 1$.

Theorem. With the above definitions we have:

- $\phi \circ \psi$ is the identity Lie 2-algebra homomorphism on $\mathfrak{g}_k$, and
- $\psi \circ \phi$ is isomorphic, as a Lie 2-algebra homomorphism, to the identity on $\mathcal{P}_k\mathfrak{g}$.

What's Next?

We know how to get Lie n-algebras from Lie algebra cohomology! We should:

- Classify their representations
- Find their corresponding Lie n-groups
- Understand their relation to higher braid theory

Moreover, many other questions remain:

- Weak n-categories in Vect?
- Weakening laws governing addition and scalar multiplication?
- Weakening the antisymmetry of the bracket in the definition of Lie 2-algebra?
- What's a free Lie 2-algebra on a 2-vector space?
- Lie 2-algebra cohomology? L_∞-algebra cohomology?
- Deformations of Lie 2-algebras?