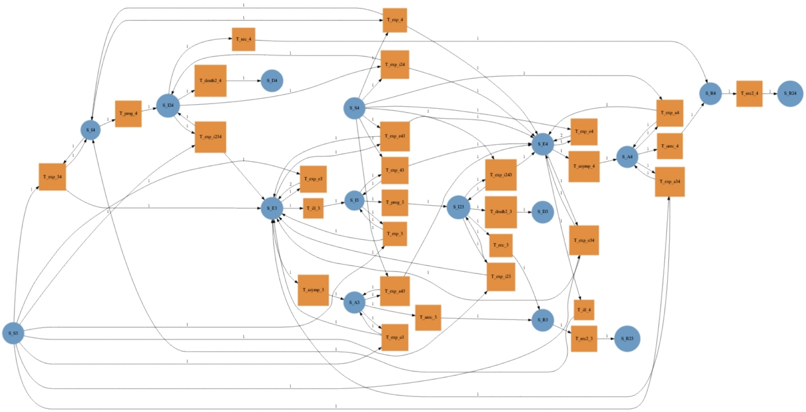
STRUCTURED vs DECORATED COSPANS



John Baez, Kenny Courser, Christina Vasilakopoulou

Categories and Companions, 2021 June 8

It's a good time for category theorists to help save the world.

For example:

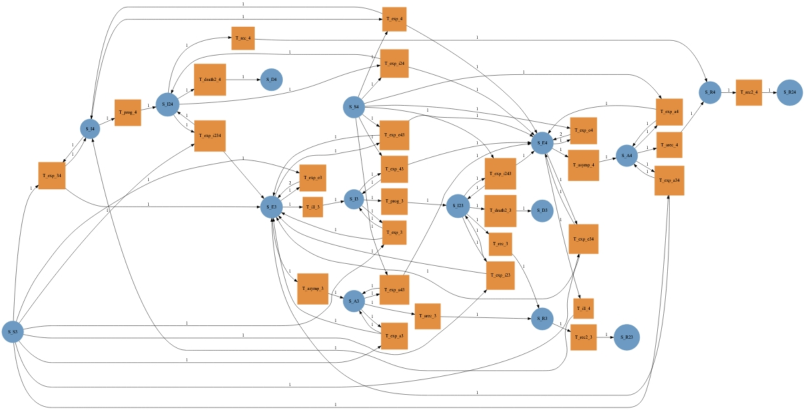
To tackle problems quickly and efficiently, we should learn to quickly assemble models of large, complicated systems from smaller parts. Evan Patterson and Micah Halter showed how to do this for the model of COVID-19 used by the UK government:

- Compositional epidemiological modeling using structured cospans, 17 October 2020.

They did it using ideas from category theory, including:

- Whole-grain Petri nets (Joachim Kock)
- Decorated cospans (Brendan Fong)
- Structured cospans (JB, Kenny Courser)

These ideas let us build a large model from smaller pieces:



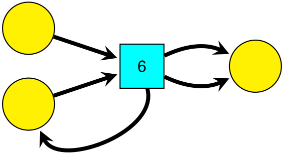
and then get differential equations to solve.

The first step uses ‘structured cospans’, the second uses ‘decorated cospans’.

A **Petri net with rates** is a diagram like this:

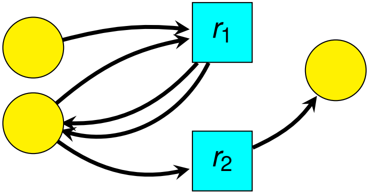
$$(0,\infty) \xleftarrow{\;r\;} T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

where S and T are finite sets, and $\mathbb{N}[S]$ is the set of finite formal sums of elements of S.



We call elements of S **places** ,
elements of T **transitions** ,
and $r(t)$ the **rate constant** of the transition $t \in T$.

A Petri net with rates gives a dynamical system. For example, the **SIR model** of infectious disease:



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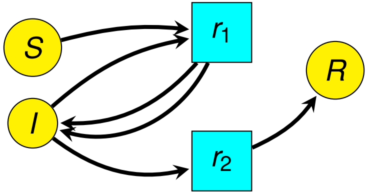
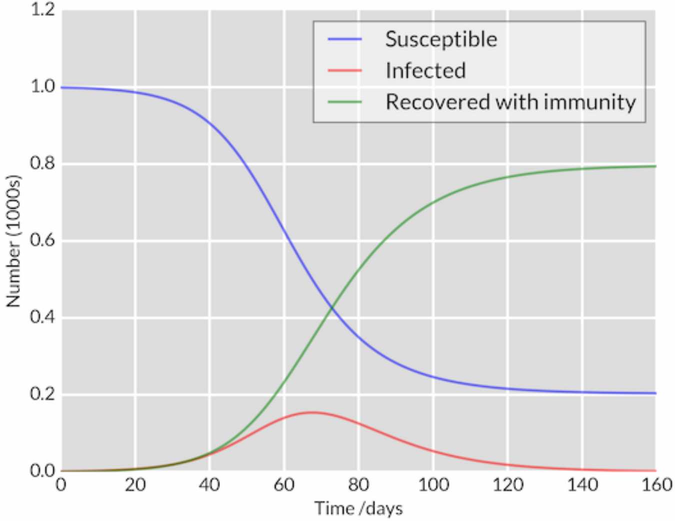


gives this dynamical system:

$$\frac{d}{dt}S(t) = -r_1\, S(t) I(t)$$

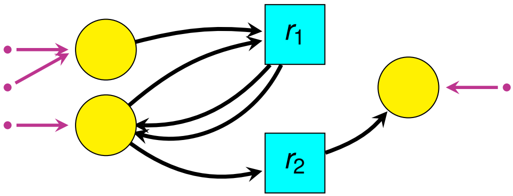
$$\frac{d}{dt}I(t) = r_1\, S(t) I(t) - r_2\, I(t)$$

$$\frac{d}{dt}R(t) = r_2\, I(t)$$



Susceptible
Infected
Recovered with immunity
Number (1000s)
Time /days
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
120
140
160

We can also define *open* Petri nets with rates:



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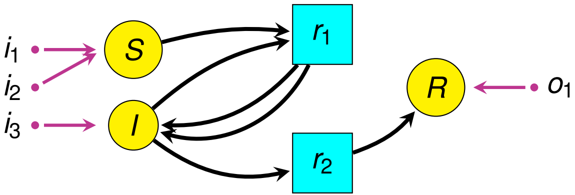


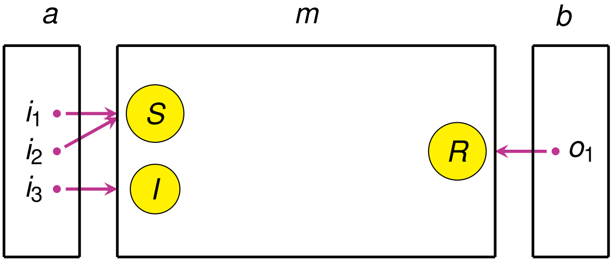
Such a thing gives an *open* dynamical system:

$$\frac{d}{dt}S(t) = -r_1\, S(t)I(t) + i_1(t) + i_2(t)$$

$$\frac{d}{dt}I(t) = r_1\, S(t)I(t) - r_2\, I(t) + i_3(t)$$

$$\frac{d}{dt}R(t) = r_2\, I(t) - o_1(t)$$

An open Petri net with rates is a cospan of finite sets:



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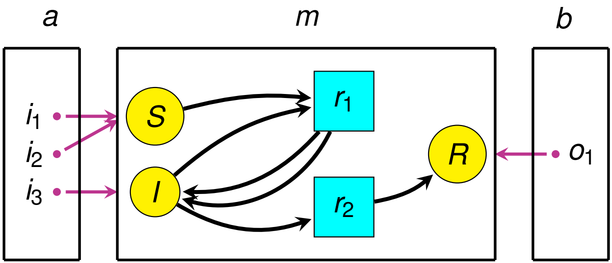

where the apex is equipped with extra data.

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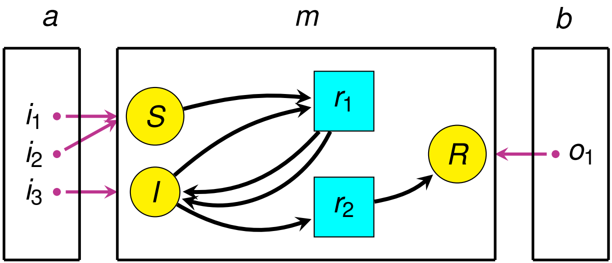


where the apex is equipped with extra data.

An open dynamical system is also a cospan of finite sets where the apex is equipped with extra data!

We use two ways to equip an object of a category A with extra data:

- **"Structuring."** Given a right adjoint $R\colon X \to A$, we can give $a \in A$ extra structure by choosing $x \in X$ with $R(x) = a$.
- **"Decorating."** Given a pseudofunctor $F\colon A \to \mathbf{Cat}$, we can decorate $a \in A$ with an object $d \in F(a)$.

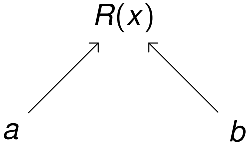
The first is more convenient for open Petri nets. Only the second works for open dynamical systems. So, we want to relate the two approaches!

The key will be the Grothendieck construction: given a pseudofunctor $F\colon A \to \mathbf{Cat}$, we obtain a functor $R\colon \int F \to A$ which *under certain conditions* is a right adjoint.

Given a right adjoint

$$R\colon \mathsf{X} \to \mathsf{A}$$

a structured cospan is a diagram in A of this form:



Think of A as a category of objects with “less structure”, and X as a category of objects with “more structure”.

Given a left adjoint

$$L\colon \mathsf{A} \to \mathsf{X}$$

a **structured cospan** is a diagram in X of this form:

$$\begin{array}{ccc} & x & \\ \nearrow & & \nwarrow \\ L(a) & & L(b) \end{array}$$

Now we can compose structured cospans by doing pushouts in X.

Theorem (Baez–Courser)

Suppose A *and* X *have finite colimits and* $L\colon \mathsf{A} \to \mathsf{X}$ *preserves finite colimits. Then there is a symmetric monoidal double category* $_L\mathbb{C}\mathbf{sp}(\mathbf{X})$ *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is a structured cospan*

$$L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$$

- *a 2-morphism is a commutative diagram*

$$\begin{array}{ccccc} L(a) & \xrightarrow{i} & x & \xleftarrow{o} & L(b) \\ {\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle L(g)} \\ L(a') & \xrightarrow[i']{} & x' & \xleftarrow[o']{} & L(b') \end{array}$$

Horizontal composition is defined using pushouts in X. Composing these:

$$\begin{array}{ccccc} L(a) & \longrightarrow & x & \longleftarrow & L(b) \\ \downarrow & & \downarrow & & \downarrow \\ L(a') & \longrightarrow & x' & \longleftarrow & L(b') \end{array} \qquad \begin{array}{ccccc} L(b) & \longrightarrow & y & \longleftarrow & L(c) \\ \downarrow & & \downarrow & & \downarrow \\ L(b') & \longrightarrow & y' & \longleftarrow & L(c') \end{array}$$

gives this:

$$\begin{array}{ccccc} L(a) & \longrightarrow & x +_{L(b)} y & \longleftarrow & L(c) \\ \downarrow & & \downarrow & & \downarrow \\ L(a') & \longrightarrow & x' +_{L(b')} y' & \longleftarrow & L(c') \end{array}$$

Vertical composition is straightforward.

The symmetric monoidal structure uses binary coproducts in both A and X, and the fact that $L\colon \mathsf{A} \to \mathsf{X}$ preserves these:

$$\begin{array}{ccccc} L(a_1) & \longrightarrow & x_1 & \longleftarrow & L(b_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a_2) & \longrightarrow & x_2 & \longleftarrow & L(b_2) \end{array} \quad \otimes \quad \begin{array}{ccccc} L(a'_1) & \longrightarrow & x'_1 & \longleftarrow & L(b'_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a'_2) & \longrightarrow & x'_2 & \longleftarrow & L(b'_2) \end{array}$$

$$= \quad \begin{array}{ccccc} L(a_1 + a'_1) & \longrightarrow & x_1 + x'_1 & \longleftarrow & L(b_1 + b'_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a_2 + a'_2) & \longrightarrow & x_2 + x'_2 & \longleftarrow & L(b_2 + b'_2) \end{array}$$

Example. There is a category **Petri$_r$** where objects are **Petri nets with rates**:

$$(0,\infty) \xleftarrow{\;r\;} T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

There is a functor $R\colon \mathrm{Petri}_r \to \mathrm{FinSet}$ sending any Petri net with rates to its underlying set of places, S

This has a left adjoint $L\colon \mathrm{FinSet} \to \mathrm{Petri}_r$ sending any set S to the Petri net with that set of places, and no transitions.

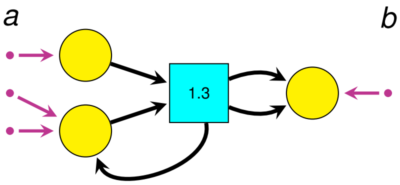
We obtain a symmetric monoidal double category

$$\mathbb{O}\mathbf{pen}(\mathsf{Petri}_r) := {}_L\mathbb{C}\mathbf{sp}(\mathsf{Petri}_r)$$

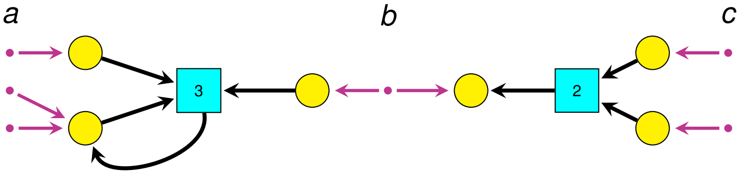
where a horizontal morphism

$$L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$$

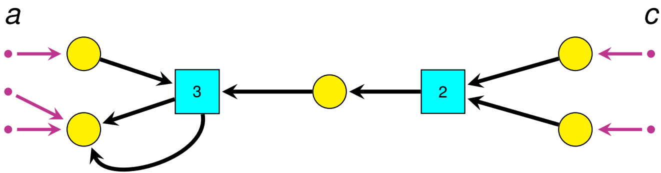
is an open Petri net with rates:



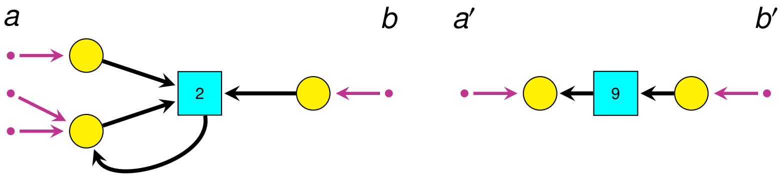
We compose open Petri nets with rates:



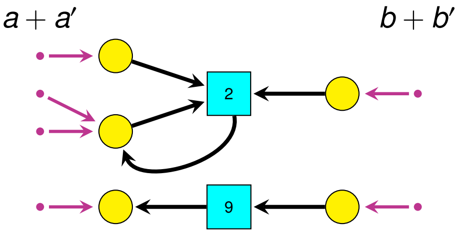
by pushouts in Petri_r:



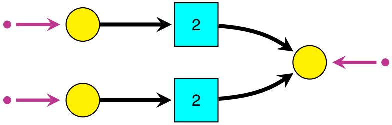
We tensor open Petri nets with rates:



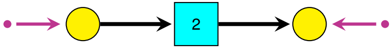
using coproducts in Set and Petri_r:



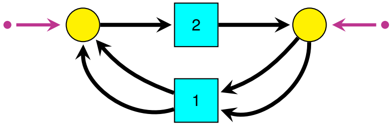
A 2-morphism in $\mathbb{O}\mathbf{pen}(\mathrm{Petri}_r)$ can map this open Petri net:



to this one:



or this one:



For open dynamical systems we need, not structured cospans, but *decorated* cospans.

Given a lax monoidal pseudofunctor $F \colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$, a **decorated cospan** is a diagram in A of this form:

$$a \rightarrow m \leftarrow b$$

together with a **decoration** $d \in F(m)$.

Theorem (Baez–Courser–Vasilakopoulou)

Let A *be a category with finite colimits and* $F\colon (\mathsf{A},+) \to (\mathbf{Cat},\times)$ *a symmetric lax monoidal pseudofunctor. Then there is a symmetric monoidal double category* **$F\mathbb{C}$sp** *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is a decorated cospan:*

$$a \xrightarrow{\;i\;} m \xleftarrow{\;o\;} b \qquad d \in F(m)$$

- *a 2-morphism is a commuting diagram*

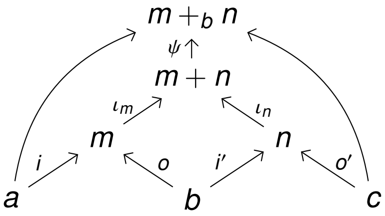
$$\begin{array}{ccccc} a & \xrightarrow{\;i\;} & m & \xleftarrow{\;o\;} & b \\ {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ a' & \xrightarrow{\;i'\;} & m' & \xleftarrow{\;o'\;} & b' \end{array} \qquad \begin{array}{c} d \in F(m) \\ \\ d' \in F(m') \end{array}$$

together with a **decoration morphism** $\tau\colon F(h)(d) \to d'$.

Given decorated cospans

$$M = (a \to m \leftarrow b, d \in F(m)) \qquad N = (b \to n \leftarrow c, e \in F(n))$$

we compose their underlying cospans by pushout:



and give it the decoration that's the image of (d, e) under this composite:

$$(d, e) \in F(m) \times F(n) \xrightarrow{\phi_{m,n}} F(m+n) \xrightarrow{F(\psi)} F(m +_b n)$$

where $\phi_{m,n}$ comes from F being lax monoidal.

Using decorated cospans, we can build a double category $\mathbb{O}\mathbf{pen}(\mathrm{Dynam})$ where horizontal 1-morphisms are open dynamical systems.

Theorem (Baez–Courser–Vasilakopoulou)

There is a symmetric monoidal double functor

$$■ \colon \mathbb{O}\mathbf{pen}(\mathrm{Petri}_r) \to \mathbb{O}\mathbf{pen}(\mathrm{Dynam})$$

sending each open Petri net to its open dynamical system.

Since

$$■(PQ) \cong ■(P)\,■(Q)$$

$$■(P \otimes Q) \cong ■(P) \otimes ■(Q)$$

the process of extracting the open dynamical system from an open Petri net is compositional!

When are decorated copans also structured cospans?

Theorem (Baez–Courser–Vasilakopoulou)

Suppose A *has finite colimits and* $F\colon (\mathsf{A},+) \to (\mathbf{Cat},\times)$ *is a symmetric lax monoidal pseudofunctor. Suppose the corresponding pseudofunctor* $F\colon \mathsf{A} \to \mathbf{SymMonCat}$ *factors through* **Rex***, the 2-category of categories with finite colimits. Then the symmetric monoidal double categories:*

- $F\mathbb{C}\mathbf{sp}$ *of decorated cospans*

and

- ${}_L\mathbb{C}\mathbf{sp}(\int F)$ *of structured cospans*

are isomorphic, where $L\colon \mathsf{A} \to \int F$ *is a left adjoint of the functor* $R\colon \int F \to \mathsf{A}$ *given by the Grothendieck construction.*

What's "the corresponding pseudofunctor
$F\colon \mathsf{A} \to$ **SymMonCat**"?

Theorem (Shulman and Moeller–Vasilakopoulou)

If $(\mathsf{A}, +)$ *has finite coproducts, these three things correspond to each other:*

- *symmetric lax monoidal pseudofunctors* $F \colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$
- *pseudofunctors* $F \colon \mathsf{A} \to \mathbf{SymMonCat}$
- *symmetric monoidal opfibrations* $R \colon (\mathsf{X}, \otimes) \to (\mathsf{A}, +)$.

We build X from F using the Grothendieck construction:

$$X = \int F$$

Things to do:

- Programmers: develop compositional modelling tools using category theory, e.g. building on the work of Fairbanks & Patterson (who use Julia).
- Scientists and engineers: *use* compositional modelling and system design.
- Category theorists: find which operads act on structured/ decorated cospans, e.g. the operad of wiring diagrams. (Talk to Christina Vasilakopoulou.) Unify all network formalisms! (Talk to David Jaz-Myers.) Study new examples: gene regulatory networks, neural networks, etc.

on the arXiv

- Kenny Courser, *Open Systems: A Double Categorical Perspective*.
- John Baez and Kenny Courser, Structured cospans.
- John Baez, Kenny Courser and Christina Vasilakopoluou, Structured versus decorated cospans.
- Michael Shulman, Framed bicategories and monoidal fibrations.
- Joe Moeller and Christina Vasilakopolou, Monoidal Grothendieck construction.

blog article

- Evan Patterson and Micah Halter, Compositional epidemiological modeling using structured cospans, https://www.algebraicjulia.org/blog/.