# A categorified supergroup for string theory

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A **weak 2-group** is a category $\mathcal{G}$ with invertible morphisms, equipped with a functor:

$$\otimes \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$

satisfying the group axioms up to specified natural isomorphisms—the **associator**:

$$a_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$$

and the **left- and right-unitors**:

$$l_x \colon 1 \otimes x \to x, \quad r_x \colon x \otimes 1 \to x,$$

such that every object $x$ has a specified weak inverse $\overline{x}$:

$$x \otimes \overline{x} \cong 1 \cong \overline{x} \otimes x.$$

### Theorem (Joyal–Street)

*Every weak 2-group is equivalent to a skeletal weak 2-group with:*

- *A group $G$ of objects.*
- *A group $G \ltimes H$ of morphisms.*
- *The left- and right-unitors trivial.*
- *The associator*

$$a_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z),$$

  *given by a normalized $H$-valued group 3-cocycle $a \colon G^3 \to H$.*

We call a 2-group of this form a **slim 2-group**, $\mathrm{String}_a(G, H)$.

A **2-vector space** is a category in vector spaces. A **semistrict Lie 2-algebra** is a 2-vector space $L$ equipped with an antisymmetric, bilinear functor:

$$[-,-] \colon L \times L \to L,$$

satisfying the Jacobi identity up to a natural isomorphism, the **Jacobiator**:

$$J_{x,y,z} \colon [[x,y],z] \to [x,[y,z]] + [[x,z],y].$$

### Theorem (Baez–Crans)

*Every Lie 2-algebra is equivalent to a skeletal Lie 2-algebra with:*

- *A Lie algebra $\mathfrak{g}$ of objects.*
- *A Lie algebra $\mathfrak{g} \ltimes \mathfrak{h}$ of morphisms.*
- *The Jacobiator*

$$J_{x,y,z} \colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

*given by an $\mathfrak{h}$-valued Lie algebra cocycle $J \colon \Lambda^3 \mathfrak{g} \to \mathfrak{h}$.*

We call a Lie 2-algebra of this form a **slim Lie 2-algebra**, $\mathfrak{string}_J(\mathfrak{g}, \mathfrak{h})$.

## Example

The **string Lie 2-algebra**, where:

- $\mathfrak{g} = \mathfrak{so}(n)$.
- $\mathfrak{h} = \mathbb{R}$.
- $J(x, y, z) = \mathrm{tr}(x, [y, z])$ is the canonical 3-cocycle.

In this case, we just write $\mathfrak{string}(n)$ for $\mathfrak{string}_J(\mathfrak{g}, \mathfrak{h})$.

When $G$ and $H$ are Lie groups and $a$ is smooth, we call $\mathrm{String}_a(G, H)$ a **slim Lie 2-group.** There's a naive scheme to integrate slim Lie 2-algebras to slim Lie 2-groups:

- Integrate $\mathfrak{g}$ to $G$.
- Integrate $\mathfrak{h}$ to $H$.
- Find a cocycle $a \colon G^3 \to H$ which somehow integrates $J \colon \Lambda^3 \mathfrak{g} \to \mathfrak{h}$.

Unfortunately, this fails even for $\mathfrak{string}(n)$:

- We can integrate $\mathfrak{g} = \mathfrak{so}(n)$ to $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$.
- We can integrate $\mathfrak{h} = \mathbb{R}$ to $\mathbb{R}$ or $\mathrm{U}(1)$.
- But compact Lie groups like $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ admit no globally smooth, nontrivial cocycles. So we have no hope of integrating $J$!

For $\mathfrak{string}(n)$, there are alternative approaches like those of Baez–Crans–Schreiber–Stevenson and Schommer-Pries.

Here, I'll describe a special case where the naive scheme *does* work:

### Theorem

*For spacetimes of dimension* $n = 3, 4, 6$ *and 10, there is a skeletal Lie 2-superalgebra where:*

- *The objects are*

$$\mathfrak{siso}(n-1,1) = \mathfrak{so}(n-1,1) \ltimes (V \oplus S),$$

  *the Poincaré superalgebra.*
- *The morphisms are* $\mathfrak{siso}(n-1,1) \times \mathbb{R}$.
- *The Jacobiator is* $J(v, \psi, \phi) = g(v, [\psi, \phi])$ *when* $v \in V$ *and* $\psi, \phi \in S$*, and zero otherwise.*

We call this $\mathfrak{superstring}(n-1,1)$. We'll integrate it to $\mathrm{Superstring}(n-1,1)$.

- $V = \mathbb{R}^{n-1,1}$ has a nondegenerate quadratic form.
- **Spinor representations** of $\mathfrak{so}(n-1,1)$ are representations arising from left-modules of $\mathrm{Cliff}(V) = \frac{TV}{v^2 = ||v||^2}$, since

$$\mathfrak{so}(n-1,1) \hookrightarrow \mathrm{Cliff}(V).$$

- Let $S$ be such a representation.

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- Let $S$ be such a representation.
- For $n = 3, 4, 6$ and 10, there is a symmetric map:

$$[-,-]\colon \mathrm{Sym}^2 S \to V.$$

- There is thus a Lie superalgebra $\mathfrak{siso}(n-1,1)$ where:

$$\mathfrak{siso}(n-1,1)_0 = \mathfrak{so}(n-1,1) \ltimes V, \quad \mathfrak{siso}(n-1,1)_1 = S.$$

called the **Poincaré superalgebra**.

In the physics literature, the classical superstring require a certain spinor identity to hold:

Superstring In dimensions 3, 4, 6 and 10, and *only these dimensions*, we have:

$$g([\psi, \psi], [\psi, \psi]) = 0$$

for all spinors $\psi \in S$.

But this is exactly a cocycle condition for the Poincaré superalgebra, for the cocycle defined by:

$$\alpha(v, \psi, \phi) = g(v, [\psi, \phi]),$$

when $v \in V$ and $\psi, \phi \in S$, and vanishing otherwise.

A **2-supervector space** is a category in supervector spaces. A **semistrict Lie 2-superalgebra** is a 2-supervector space $L$ equipped with a graded-antisymmetric, bilinear functor:

$$[-,-]\colon L \times L \to L,$$

satisfying the Jacobi identity up to a natural isomorphism, the **Jacobiator**:

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + (-1)^{|y||z|}[[x,z],y].$$

## Theorem (Baez–Crans–Huerta)

*There is a Lie 2-algebra with:*

- *A Lie superalgebra* $\mathfrak{g}$ *of objects.*
- *A Lie superalgebra* $\mathfrak{g} \ltimes \mathfrak{h}$ *of morphisms.*
- *The Jacobiator*

$$J_{x,y,z} \colon [[x,y],z] \to [x,[y,z]] + (-1)^{|y||z|}[[x,z],y],$$

*given by an* $\mathfrak{h}$*-valued Lie superalgebra cocycle* $J \colon \Lambda^3\mathfrak{g} \to \mathfrak{h}$.

So: $\mathfrak{superstring}(n-1,1)$ is a Lie 2-superalgebra!

Now we would like to integrate $\mathfrak{superstring}(n-1,1)$ to $\mathrm{Superstring}(n-1,1)$, a "skeletal Lie 2-supergroup" with:

- The Lie supergroup $\mathrm{SISO}(n-1,1)$ as objects.
- The Lie supergroup $\mathrm{SISO}(n-1,1) \times \mathbb{R}$ as morphisms.
- Associator given by the Lie supergroup 3-cocycle $\int \alpha \colon (SISO(n-1,1))^3 \to \mathbb{R}$.

A **Lie 2-supergroup** is a weak 2-group in the category of supermanifolds.

## Theorem

*There is a skeletal Lie 2-supergroup with:*

- *A Lie supergroup $G$ of objects.*
- *A Lie supergroup $G \ltimes H$ of morphisms.*
- *Trivial left- and right- unitors.*
- *The associator given by a normalized $H$-valued Lie supergroup 3-cocycle $a\colon G^3 \to H$.*

So: constructing $\mathrm{Superstring}(n-1,1)$ boils down to constructing $\int \alpha$.

This is possible because $\alpha$ is supported on the Lie subalgebra $\mathcal{T} = V \oplus S$, a *nilpotent Lie superalgebra*.

We can integrate $\mathbb{R}$-valued cocycles on any nilpotent Lie *algebra* $\mathfrak{n}$ to cocycles on the corresponding group $N$, using a technique due to Houard.

- Lie algebra cochains $\omega \colon \Lambda^p \mathfrak{n} \to \mathbb{R}$ can be identified with left-invariant differential forms on $N$.
- We can define **left-invariant simplices** in $N$ to be simplices:

$$[n_0, \cdots, n_p] \colon \Delta^p \to N,$$

with the property:

$$n[n_0, \cdots, n_p] = [nn_0, \cdots, nn_p].$$

- We integrate to get Lie group cochains on $N$:

$$\textstyle\int \omega(n_1, \cdots, n_p) = \displaystyle\int_{[1, n_1, n_1 n_2, \cdots, n_1 \cdots n_p]} \omega.$$

- This defines a cochain map by Stokes' theorem!

Now, we superize using the functor of points:

**Theorem (Balduzzi–Carmeli–Fioresi)**

*There is a full and faithful functor:*

$$h\colon \mathrm{SuperManifolds} \to \mathrm{Hom}(\mathrm{SuperWeilAlg}, \mathcal{A}_0\text{-Manifolds}).$$

So: for any supermanifold $M$ and supercommutative Weil algebra $A$, we get a *manifold* $h(M)(A) = M_A$, the ***A*-points** of $M$.

The $A$-points of $\mathcal{T}$ are $\mathcal{T}_A = A_0 \otimes \mathcal{T}_0 \oplus A_1 \otimes \mathcal{T}_1$.

- $\mathcal{T}$ a nilpotent Lie superalgebra $\Rightarrow \mathcal{T}_A$ a nilpotent Lie algebra.
- $\alpha$ a 3-cocycle on $\mathcal{T} \Rightarrow \alpha_A$ a 3-cocycle on $\mathcal{T}_A$.
- $\mathcal{T}_A$ has a group structure, $T_A \Rightarrow \mathcal{T}$ has a supergroup structure $T$.

So: we integrate to get $\int \alpha_A$ and transfer this back to $T$, defining $\int \alpha$ on $T$.

## Theorem

*$\int \alpha$ defines a Lie supergroup 3-cocycle on $T$, which extends to a Lie supergroup 3-cocycle on* $\mathrm{SISO}(n-1,1)$.

## Corollary

*There is a Lie 2-supergroup* $\mathrm{Superstring}(n-1,1)$.

Final thoughts:

- We want to do higher gauge theory with $\mathrm{Superstring}(n-1,1)$. This should be related to string theory, and to the work of Sati–Schreiber–Stasheff.
- There is also a Lie 3-supergroup $2\text{-}\mathrm{Brane}(n,1)$ associated with super-2-branes. The higher gauge theory should be related to M-theory.