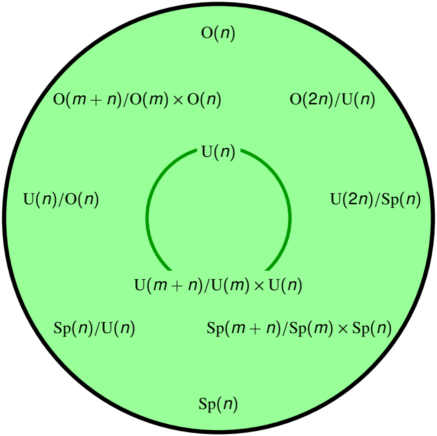
SYMMETRIC SPACES AND THE TENFOLD WAY



John Baez

There are 10 of each of these things:

- Kinds of real and complex Clifford algebras.
- Associative super division algebras.
- Ways that Hamiltonians can get along with time reversal (T) and/or charge conjugation (C) symmetry.

They're all connected!

In 1926, Cartan showed there are 10 of these:

- Infinite families of compact symmetric spaces.

These are also connected! I'll explain how these 10 families arise from Clifford algebras. This is already visible in Milnor's book *Morse Theory*, but I want to expand on it.

What's a symmetric space?

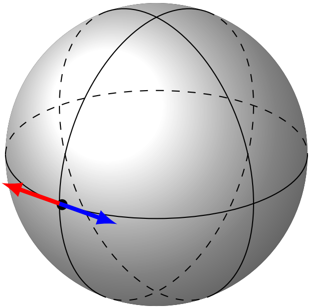
It's a connected Riemannian manifold M such that for each point $p \in M$ there's a metric-preserving smooth map

$$f \colon M \to M$$

called **inversion about *p*** such that

$$f(p) = p \quad \text{and} \quad df_p = -1$$

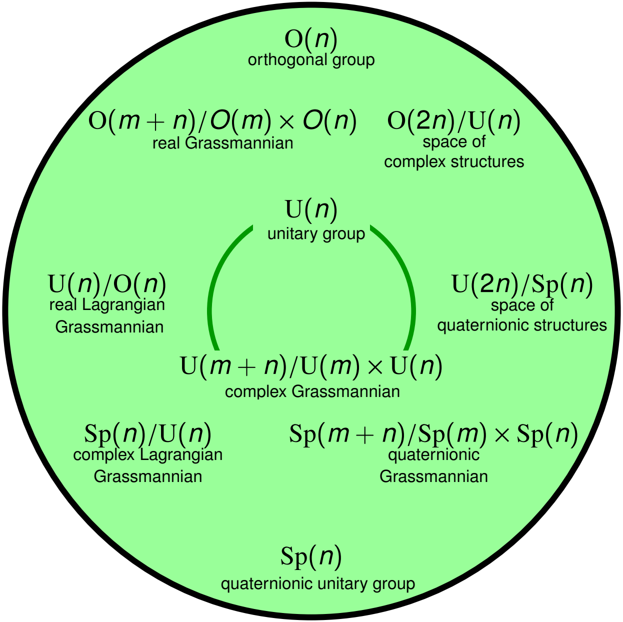
For example: a sphere S^n with its usual metric is a compact symmetric space.



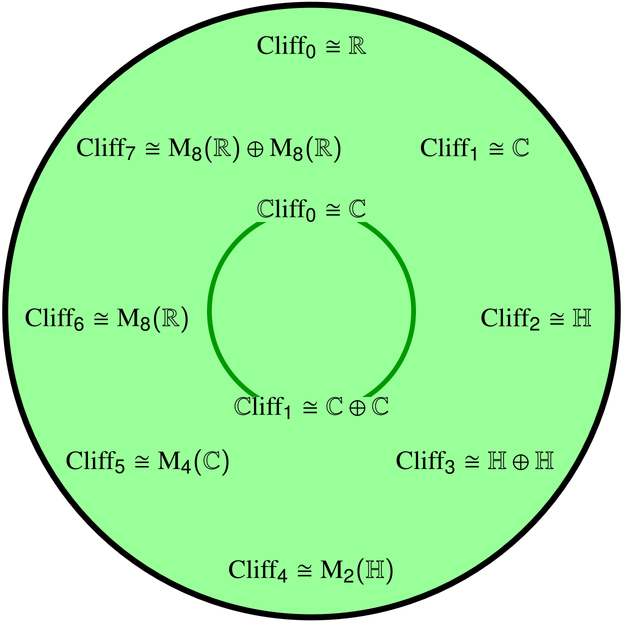
So is the space of all m-dimensional subspaces of $\mathbb{R}^{m+n}$, which is the **real Grassmannian**

$$\mathrm{O}(m+n)/\mathrm{O}(m) \times \mathrm{O}(n)$$

SERIES OF COMPACT SYMMETRIC SPACES



REAL AND COMPLEX CLIFFORD ALGEBRAS



Here:

- The **real Clifford algebra** Cliff_n is the algebra over $\mathbb{R}$ freely generated by n square roots of -1, all of which anticommute.
- The **complex Clifford algebra** $\mathbb{C}\mathrm{liff}_n$ is the algebra over $\mathbb{C}$ freely generated by n square roots of -1, all of which anticommute.

We have **Bott periodicity**:

$$\mathrm{Cliff}_{n+8} \cong \mathrm{M}_{16}(\mathrm{Cliff}_n) \qquad \mathbb{C}\mathrm{liff}_{n+2} \cong \mathrm{M}_2(\mathbb{C}\mathrm{liff}_n)$$

so these Clifford algebras come in 10 different "kinds".

How do we get the 10 infinite families of compact symmetric spaces from these 10 kinds of Clifford algebras?

The key is to notice that Clifford algebras are $*$-algebras, and to look at their $*$-representations.

For real Clifford algebras, the idea is *roughly* this:

We get a compact symmetric space by taking a $*$-representation of Cliff_n and forming the space of all ways of extending this to a $*$-representation of Cliff_{n+1}.

Let's see the details! Complex Clifford algebras work similarly.

In the real case the definitions go like this:

A **$*$-algebra** is a real associative algebra A with unit $1 \in A$ and an operation $*\colon A \to A$ with

$$a^{**} = a, \quad (a+b)^* = a^* + b^*, \quad (\alpha a)^* = \alpha a^*, \quad (ab)^* = b^* a^*$$

for all $a, b \in A$ and $\alpha \in \mathbb{R}$.

A **$*$-representation** of the $*$-algebra A on a finite-dimensional real Hilbert space H is a linear operator $\rho(a) \colon H \to H$ for each $a \in A$, obeying

$$\rho(a+b) = \rho(a) + \rho(b), \quad \rho(\alpha a) = \alpha \rho(a),$$

$$\rho(ab) = \rho(a)\rho(b), \quad \rho(1) = 1, \quad \rho(a^*) = \rho(a)^\dagger$$

for all $a, b \in A$ and $\alpha \in \mathbb{R}$. Here for any linear map $T \colon H \to H'$ between real Hilbert spaces, $T^\dagger \colon H' \to H$ is defined by

$$\langle T^\dagger \psi, \phi \rangle = \langle \psi, T\phi \rangle$$

An **orthogonal equivalence** of $*$-representations of A, say ρ on H and ρ' on H', is a linear operator $T \colon H \to H'$ with

$$T \circ \rho(a) = \rho'(a) \circ T$$

for all $a \in A$, and also

$$T^\dagger T = 1_H, \qquad TT^\dagger = 1_{H'}.$$

Let **Rep(A)** be the category of $*$-representations of the $*$-algebra A and orthogonal equivalences. For example:

- $\mathbb{R}$ is a $*$-algebra with

$$\alpha^* = \alpha \text{ for all } \alpha \in \mathbb{R}$$

  Rep($\mathbb{R}$) is the category of finite-dimensional real Hilbert spaces and **orthogonal** operators: real-linear $T: H \to H'$ with $T^\dagger T = 1_H$ and $TT^\dagger = 1_{H'}$.

- $\mathbb{C}$ is a $*$-algebra with

$$\alpha^* = \overline{\alpha} \text{ for all } \alpha \in \mathbb{C}$$

  Rep($\mathbb{C}$) is the category of finite-dimensional complex Hilbert spaces and unitary operators.

- For any $*$-algebra A,

$$\mathbf{Rep}(A) \simeq \mathbf{Rep}(\mathrm{M}_n(A))$$

Any Clifford algebra Cliff_n becomes a $*$-algebra in a unique way if for the generating square roots of -1, say $e_1, \ldots, e_n$, we set

$$e_i^* = -e_i.$$

So let's look at $\mathbf{Rep}(\mathrm{Cliff}_n)$. In what follows all Hilbert spaces are finite-dimensional:

- $\mathrm{Cliff}_0 \cong \mathbb{R}$, so $\mathbf{Rep}(\mathrm{Cliff}_0) \simeq \mathbf{Rep}(\mathbb{R})$, the category of **real Hilbert spaces**.
- $\mathrm{Cliff}_1 \cong \mathbb{C}$, so $\mathbf{Rep}(\mathrm{Cliff}_1) \simeq \mathbf{Rep}(\mathbb{C})$, the category of **complex Hilbert spaces**.
- $\mathrm{Cliff}_2 \cong \mathbb{H}$, so $\mathbf{Rep}(\mathrm{Cliff}_2) \simeq \mathbf{Rep}(\mathbb{H})$, the category of **quaternionic Hilbert spaces**.
- $\mathrm{Cliff}_3 \cong \mathbb{H} \oplus \mathbb{H}$, so $\mathbf{Rep}(\mathrm{Cliff}_3) \simeq \mathbf{Rep}(\mathbb{H} \oplus \mathbb{H})$, the category of **split quaternionic Hilbert spaces**: quaternionic Hilbert spaces H with a direct sum decomposition $H \cong H' \oplus H''$.

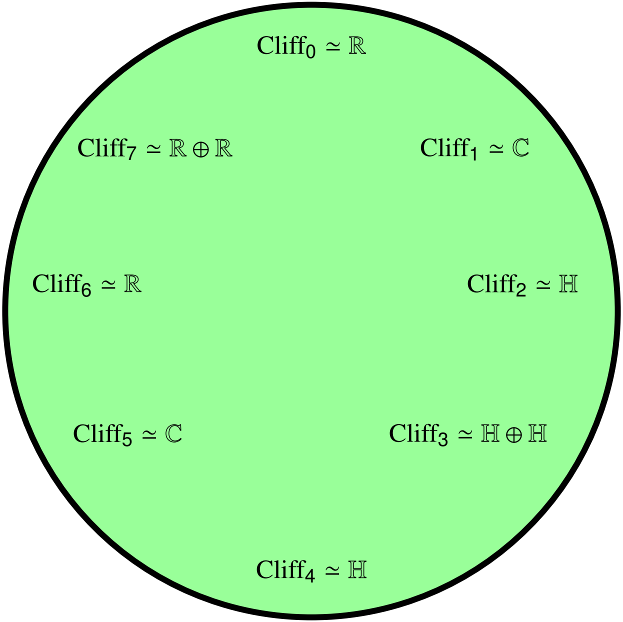
- $\mathrm{Cliff}_4 \cong \mathrm{M}_2(\mathbb{H})$, so $\mathbf{Rep}(\mathrm{Cliff}_4) \simeq \mathbf{Rep}(\mathbb{H})$, the category of **quaternionic Hilbert spaces**.
- $\mathrm{Cliff}_5 \cong \mathrm{M}_4(\mathbb{C})$, so $\mathbf{Rep}(\mathrm{Cliff}_5) \simeq \mathbf{Rep}(\mathbb{C})$, the category of **complex Hilbert spaces**.
- $\mathrm{Cliff}_6 \cong \mathrm{M}_8(\mathbb{R})$, so $\mathbf{Rep}(\mathrm{Cliff}_6) \simeq \mathbf{Rep}(\mathbb{R})$, the category of **real Hilbert spaces**.
- $\mathrm{Cliff}_7 \cong \mathrm{M}_8(\mathbb{R}) \oplus \mathrm{M}_8(\mathbb{R})$, so $\mathbf{Rep}(\mathrm{Cliff}_7) \simeq \mathbf{Rep}(\mathbb{R} \oplus \mathbb{R})$, the category of **split real Hilbert spaces**: real Hilbert spaces H with a direct sum decomposition $H \cong H' \oplus H''$.

After that we get

$$\mathbf{Rep}(\mathrm{Cliff}_{n+8}) \simeq \mathbf{Rep}(\mathrm{Cliff}_n)$$

If two $*$-algebras A and B have equivalent categories of $*$-representations we call them **Morita equivalent** and write $A \simeq B$.

REAL CLIFFORD ALGEBRAS



$*$-REPRESENTATIONS OF REAL CLIFFORD ALGEBRAS

$\mathbf{Rep}(\mathrm{Cliff}_0) \simeq$
real Hilbert spaces

$\mathbf{Rep}(\mathrm{Cliff}_1) \simeq$
complex Hilbert spaces

$\mathbf{Rep}(\mathrm{Cliff}_2) \simeq$
quaternionic Hilbert spaces

$\mathbf{Rep}(\mathrm{Cliff}_3) \simeq$
split quaternionic Hilbert spaces

$\mathbf{Rep}(\mathrm{Cliff}_4) \simeq$
quaternionic Hilbert spaces

$\mathbf{Rep}(\mathrm{Cliff}_5) \simeq$
complex Hilbert spaces

$\mathbf{Rep}(\mathrm{Cliff}_6) \simeq$
real Hilbert spaces

$\mathbf{Rep}(\mathrm{Cliff}_7) \simeq$
split real Hilbert spaces

Any $*$-representation of Cliff_{n+1} restricts to a $*$-representation of Cliff_n, so we get a “forgetful functor”

$$F\colon \mathbf{Rep}(\mathrm{Cliff}_{n+1}) \to \mathbf{Rep}(\mathrm{Cliff}_n)$$

For any object $H \in \mathbf{Rep}(\mathrm{Cliff}_n)$ there is a set of ways it can come from some object of $\mathbf{Rep}(\mathrm{Cliff}_{n+1})$. Let’s call this $F^{-1}(H)$. This is either a compact symmetric space, or a finite union of compact symmetric spaces!

This idea is a bit subtle. How *exactly* should we define the set of ways H comes from some object in $\mathbf{Rep}(\mathrm{Cliff}_{n+1})$?

Luckily category theorists know all about this stuff. But let’s do some examples!

- $\mathrm{Cliff}_0 \cong \mathbb{R}$ and **Rep**(Cliff_0) is the category of real Hilbert spaces.
- $\mathrm{Cliff}_1 \cong \mathbb{C}$ and **Rep**(Cliff_1) is the category of complex Hilbert spaces.

The functor

$$F \colon \mathbf{Rep}(\mathrm{Cliff}_1) \to \mathbf{Rep}(\mathrm{Cliff}_0)$$

takes a complex Hilbert space and gives the underlying real Hilbert space.

If we take a real Hilbert space H it can be made into a complex Hilbert space X by choosing a **complex structure**: a real-linear $J \colon H \to H$ with

$$J^2 = -1, \qquad JJ^\dagger = J^\dagger J = 1$$

We then have $F(X) = H$.

The set of complex structures on a real Hilbert space H is

$$F^{-1}(H) := \left\{ J\colon H \to H \;\middle|\; J^2 = -1, JJ^\dagger = J^\dagger J = 1 \right\}$$

If H is odd-dimensional, $F^{-1}(H)$ is empty.

If H is even-dimensional, it's isomorphic to $\mathbb{R}^{2n}$ with its usual real Hilbert space structure. The group $\mathrm{O}(2n)$ acts on $F^{-1}(\mathbb{R}^{2n})$ by

$$J \mapsto gJg^{-1}$$

It acts transitively, and the subgroup that fixes the "standard" complex structure on $\mathbb{R}^{2n}$ is $\mathrm{U}(n)$, so

$$F^{-1}(\mathbb{R}^{2n}) = \mathrm{O}(2n)/\mathrm{U}(n)$$

This is a compact symmetric space!

- $\mathrm{Cliff}_1 \cong \mathbb{C}$ and $\mathbf{Rep}(\mathrm{Cliff}_1)$ is the category of complex Hilbert spaces.
- $\mathrm{Cliff}_2 \cong \mathbb{H}$ and $\mathbf{Rep}(\mathrm{Cliff}_2)$ is the category of quaternionic Hilbert spaces.

Now the functor

$$F\colon \mathbf{Rep}(\mathrm{Cliff}_2) \to \mathbf{Rep}(\mathrm{Cliff}_1)$$

takes a quaternionic Hilbert space and gives the underlying complex Hilbert space.

This case works like the last one: now $F^{-1}(H)$ is the set of quaternionic structures on the complex Hilbert space H, and

$$F^{-1}(\mathbb{C}^{2n}) \cong \mathrm{U}(2n)/\mathrm{Sp}(n)$$

is a compact symmetric space.

- $\mathrm{Cliff}_2 \cong \mathbb{H}$ and $\mathbf{Rep}(\mathrm{Cliff}_2)$ is the category of quaternionic Hilbert spaces.
- $\mathrm{Cliff}_3 \cong \mathbb{H} \oplus \mathbb{H}$ and $\mathbf{Rep}(\mathrm{Cliff}_3)$ is the category of split quaternionic Hilbert spaces.

Now the functor

$$F\colon \mathbf{Rep}(\mathrm{Cliff}_3) \to \mathbf{Rep}(\mathrm{Cliff}_2)$$

takes a split quaternionic vector space and gives the underlying quaternionic Hilbert space.

Now $F^{-1}(H)$ is the set of splittings of H as an orthogonal direct sum of two subspaces, so $F^{-1}(\mathbb{H}^n)$ is a disjoint union of quaternionic Grassmannians:

$$F^{-1}(\mathbb{H}^n) \cong \bigsqcup_{d=0}^{n} \mathrm{Sp}(n)/\mathrm{Sp}(d) \times \mathrm{Sp}(n-d)$$

Each component is a compact symmetric space!

Theorem. For each n we have a functor

$$F\colon \mathbf{Rep}(\mathrm{Cliff}_n) \to \mathbf{Rep}(\mathrm{Cliff}_{n-1})$$

and for each $H \in \mathbf{Rep}(\mathrm{Cliff}_{n-1})$, the set $F^{-1}(H)$ naturally has the structure of a compact symmetric space. Similarly we have a functor

$$F\colon \mathbf{Rep}(\mathbb{C}\mathrm{liff}_n) \to \mathbf{Rep}(\mathbb{C}\mathrm{liff}_{n-1})$$

and for each $H \in \mathbf{Rep}(\mathbb{C}\mathrm{liff}_{n-1})$, the set $F^{-1}(H)$ naturally has the structure of a compact symmetric space.

Moreover, all the compact symmetric spaces in Cartan's 10 infinite series arise this way.

$*$-REPRESENTATIONS OF REAL CLIFFORD ALGEBRAS

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Rep$(\mathrm{Cliff}_5) \simeq$
complex Hilbert spaces

Rep$(\mathrm{Cliff}_6) \simeq$
real Hilbert spaces

Rep$(\mathrm{Cliff}_7) \simeq$
split real Hilbert spaces

$$
\begin{tikzcd}
\mathbf{Rep}(\mathrm{Cliff}_7) \simeq \text{split real Hilbert spaces} \simeq \mathbf{Rep}(\mathrm{Cliff}_7) \arrow[d, shift right=3, "\text{direct sum}"'] \\
\mathbf{Rep}(\mathrm{Cliff}_6) \simeq \text{real Hilbert spaces} \simeq \mathbf{Rep}(\mathrm{Cliff}_0) \arrow[u, shift right=3, "\text{doubling}"'] \arrow[d, shift right=3, "\text{complexification}"'] \\
\mathbf{Rep}(\mathrm{Cliff}_5) \simeq \text{complex Hilbert spaces} \simeq \mathbf{Rep}(\mathrm{Cliff}_1) \arrow[u, shift right=3, "\text{underlying real space}"'] \arrow[d, shift right=3, "\text{quaternionification}"'] \\
\mathbf{Rep}(\mathrm{Cliff}_4) \simeq \text{quaternionic Hilbert spaces} \simeq \mathbf{Rep}(\mathrm{Cliff}_2) \arrow[u, shift right=3, "\text{underlying complex space}"'] \arrow[d, shift right=3, "\text{direct sum}"'] \\
\mathbf{Rep}(\mathrm{Cliff}_3) \simeq \text{split quaternionic Hilbert spaces} \simeq \mathbf{Rep}(\mathrm{Cliff}_3) \arrow[u, shift right=3, "\text{doubling}"']
\end{tikzcd}
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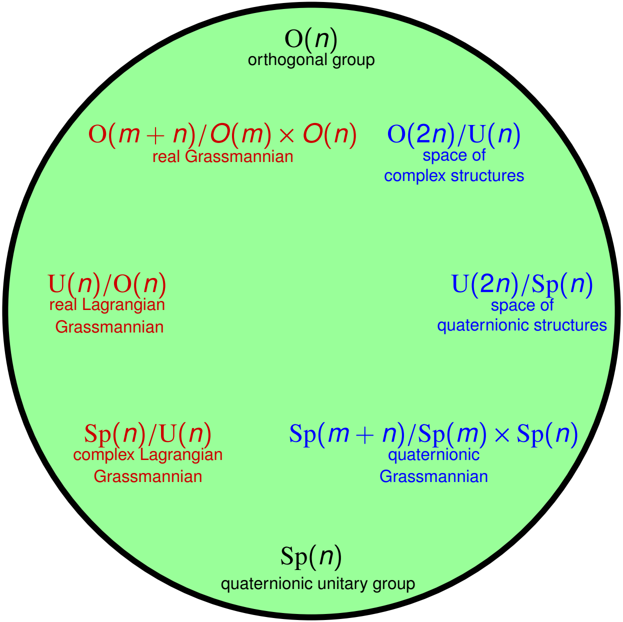
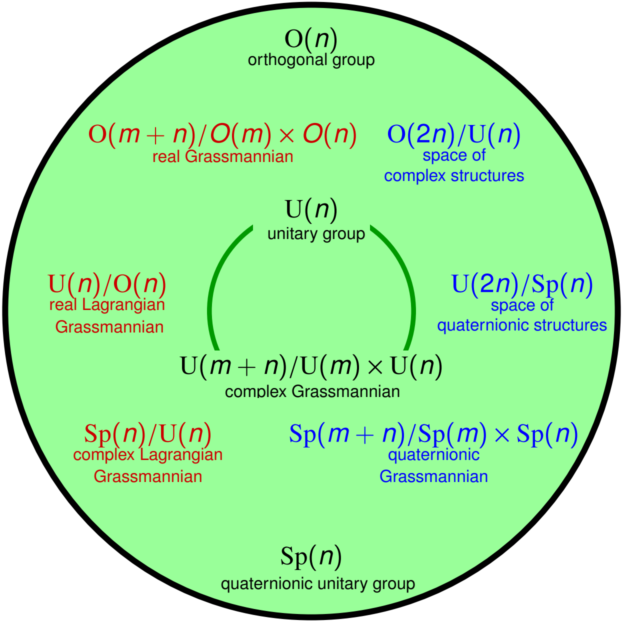
SERIES OF COMPACT SYMMETRIC SPACES



SERIES OF COMPACT SYMMETRIC SPACES



Some last remarks

- The correct way to define $F^{-1}(H)$ is subtle. We often have $X \in \mathbf{Rep}(\mathrm{Cliff}_n)$ with $F(X) \neq H$ but $F(X) \cong H$. This should be good enough. But we usually have many X with $F(X) \cong H$. When do we count two as different elements of $F^{-1}(H)$? Luckily category theorists have figured this out: we should let $F^{-1}(H)$ be the "essential fiber" of F over X.
- There is a simple, unified proof that $F^{-1}(H)$ is always a disjoint union of compact symmetric spaces.
- What's the best way to show that *all* infinite series of compact symmetric spaces arise from Clifford algebras? Currently we just check the list and notice that they do!

Details: what is the essential fiber and how does it work?

Given any functor $F\colon \mathbf{X} \to \mathbf{H}$ and any object $H \in \mathbf{H}$, the **essential fiber over** H is a category $F^{-1}(H)$ where:

- an object is a pair (X, α) consisting of an object $X \in \mathbf{X}$ and an isomorphism $\alpha\colon F(X) \xrightarrow{\sim} H$.
- a morphism from (X, α) to (X', α') is a morphism $\phi\colon F(X) \to F(X')$ with $\alpha = \alpha' \circ \phi$.

$$\begin{tikzcd} F(X) \arrow[rr, "\phi"] \arrow[dr, "\alpha"'] & & F(X') \arrow[dl, "\alpha'"] \\ & H & \end{tikzcd}$$

But in our examples this category is "basically just a set". What does that mean, and why is it true?

We start with some easy lemmas.

Lemma 1. Suppose a functor $F\colon \mathbf{X} \to \mathbf{H}$ is **faithful**: if $f, g\colon X \to X'$ are morphisms in $\mathbf{X}$ with $F(f) = F(g)$, then $f = g$.

Then for any object $H \in \mathbf{H}$, the essential fiber $F^{-1}(H)$ is a **preorder**: a category where any two morphisms with the same source and target are equal.

Lemma 2. Suppose a functor $F\colon \mathbf{X} \to \mathbf{H}$ is **conservative**: for every morphism f in $\mathbf{X}$, if $F(f)$ is an isomorphism then f is an isomorphism.

Then for any object $H \in \mathbf{H}$, the essential fiber $F^{-1}(H)$ is a **groupoid**: a category where every morphism is an isomorphism.

Lemma 3. Suppose a functor $F: \mathbf{X} \to \mathbf{H}$ is faithful and conservative. Then for any object $H \in \mathbf{H}$, the essential fiber $F^{-1}(H)$ is equivalent to the **discrete category** on some set S: that is, the category with S as its set of objects and only identity morphisms.

This follows straight from Lemmas 1 and 2, since any category that is both a preorder and a groupoid is equivalent to the discrete category on its set of isomorphism classes of objects.

In this situation we can treat $F^{-1}(H)$ as a set. We do this from now on.

Lemma 4. Suppose any functor $F\colon \mathbf{X} \to \mathbf{H}$ is faithful and conservative. Then for any object $H \in \mathbf{H}$,

$$F^{-1}(H) \;\cong\; \bigsqcup_{X} \frac{\mathrm{Aut}(H)}{\mathrm{Aut}(X)}$$

where the disjoint union is taken over objects X, one from each isomorphism class of objects with $F(X) \cong H$.

Why? The automorphism group $\mathrm{Aut}(H)$ acts on $F^{-1}(H)$, with $\beta\colon H \xrightarrow{\sim} H$ sending $\alpha\colon F(X) \xrightarrow{\sim} H$ to $\beta \circ \alpha\colon F(X) \xrightarrow{\sim} H$. Whenever a group G acts on a set S we have

$$S \;\cong\; \bigsqcup_{x} \frac{G}{\mathrm{Stab}_x}$$

where $\mathrm{Stab}_x \subseteq G$ is the subgroup fixing $x \in S$, and the disjoint union is taken over points $x \in S$, one from each orbit.

All the conditions in Lemma 4 hold for the forgetful functor

$$F\colon \mathbf{Rep}(\mathrm{Cliff}_n) \to \mathbf{Rep}(\mathrm{Cliff}_{n-1})$$

so:

Theorem 1. For any object $H \in \mathbf{Rep}(\mathrm{Cliff}_{n-1})$,

$$F^{-1}(H) \;\cong\; \coprod_X \frac{\mathrm{Aut}(H)}{\mathrm{Aut}(X)}$$

where the disjoint union is taken over objects $X \in \mathbf{Rep}(\mathrm{Cliff}_n)$, one in each isomorphism class of objects with $F(X) \cong H$.

The analogous result is also true for

$$F\colon \mathbf{Rep}(\mathbb{C}\mathrm{liff}_{n+1}) \to \mathbf{Rep}(\mathbb{C}\mathrm{liff}_n)$$

Example 1. For the forgetful functor from complex Hilbert spaces to real Hilbert spaces

$$F\colon \mathbf{Rep}(\mathrm{Cliff}_1) \to \mathbf{Rep}(\mathrm{Cliff}_0)$$

every complex Hilbert space X with $F(X) \cong H := \mathbb{R}^{2n}$ has $X \cong \mathbb{C}^n$ so Theorem 2 gives

$$F^{-1}(\mathbb{R}^{2n}) \cong \frac{\mathrm{Aut}(H)}{\mathrm{Aut}(X)} \cong \frac{\mathrm{O}(2n)}{\mathrm{U}(n)}$$

as we've seen before.

Example 2. For the forgetful functor from split real Hilbert spaces to real Hilbert spaces

$$F\colon \mathbf{Rep}(\mathrm{Cliff}_7) \to \mathbf{Rep}(\mathrm{Cliff}_6)$$

there are different nonisomorphic choices of split real Hilbert spaces X with $F(X) \cong H := \mathbb{R}^n$.

Indeed, any $X_d = \mathbb{R}^d \oplus \mathbb{R}^{n-d}$ has $F(X_d) \cong \mathbb{R}^n$. These are all the choices, up to isomorphism, so Theorem 2 gives

$$F^{-1}(\mathbb{R}^n) \cong \bigsqcup_{d=0}^{n} \frac{\mathrm{Aut}(H)}{\mathrm{Aut}(X_d)} \cong \bigsqcup_{d=0}^{n} \frac{\mathrm{O}(n)}{\mathrm{O}(d) \times \mathrm{O}(n-d)}$$

a disjoint union of real Grassmannians.

This disjoint union is not connected, but its components, the Grassmannians, are compact symmetric spaces!

Details: why do we get symmetric spaces?

It's a known fact that we get a symmetric space from any compact simple Lie group G with an **involution**: that is, a homomorphism $\sigma\colon G \to G$ with $\sigma^2 = 1$. We can then define a subgroup

$$K = \left\{ g \in G \,\middle|\, \sigma(g) = g \right\}$$

and G/K is a compact symmetric space.

Let

$$F\colon \mathbf{Rep}(\mathrm{Cliff}_n) \to \mathbf{Rep}(\mathrm{Cliff}_{n-1})$$

be the forgetful functor. Suppose $F(X) = H$. We've seen $\mathrm{Aut}(X) \subseteq \mathrm{Aut}(H)$. I claim there's an involution $\sigma\colon \mathrm{Aut}(H) \to \mathrm{Aut}(H)$ with

$$\mathrm{Aut}(X) = \left\{g \in \mathrm{Aut}(H) \,\middle|\, \sigma(g) = g\right\}$$

Given this, we get:

Theorem 2. For any $H \in \mathbf{Rep}(\mathrm{Cliff}_{n-1})$,

$$F^{-1}(H) \;\cong\; \bigsqcup_X \frac{\mathrm{Aut}(H)}{\mathrm{Aut}(X)}$$

where the disjoint union is taken over objects $X \in \mathbf{Rep}(\mathrm{Cliff}_n)$, one in each isomorphism class of objects with $F(X) \cong H$, and component $\mathrm{Aut}(X)/\mathrm{Aut}(H)$ is a compact symmetric space.

What is the involution σ?

Cliff_n has a representation ρ on X. H is X seen as a representation of the subalgebra Cliff_{n-1}. Thus an automorphism $g\colon H \to H$ is an automorphism of X if it also commutes with the last square root of -1 generating Cliff_n. So, if $i = \rho(e_n)$:

$$\mathrm{Aut}(X) = \left\{ g \in \mathrm{Aut}(H) \,\middle|\, i\,g = g\,i \right\}$$

Thus if we define $\sigma\colon \mathrm{Aut}(H) \to \mathrm{Aut}(H)$ by

$$\sigma(g) = i\,g\,i^{-1}$$

then σ is an involution and

$$\mathrm{Aut}(X) = \left\{ g \in \mathrm{Aut}(H) \,\middle|\, \sigma(g) = g \right\}$$