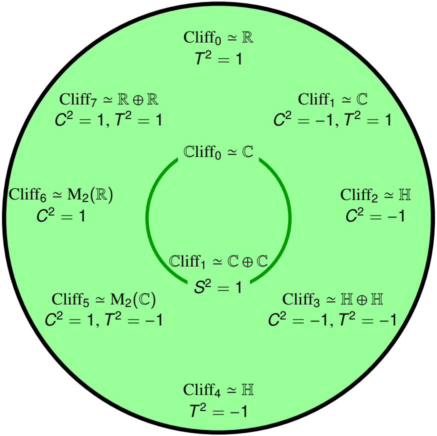
THE TENFOLD WAY



John Baez

There are ten of each of these things:

- Ways that Hamiltonians can get along with time reversal (T) and charge conjugation (C) symmetry.
- Associative real super division algebras.
- Morita equivalence classes of real and complex Clifford algebras.
- Classical families of compact symmetric spaces.

They're all connected! This is the **tenfold way**.

Let's start from the beginning: the threefold way.

Given two unit vectors ψ, ϕ in a Hilbert space **H**, the transition probability $|\langle\psi, \phi\rangle|^2$ does not change if we multiply ψ or ϕ by a phase.

So, pure states in quantum mechanics are really given, not by unit vectors, but by equivalence classes of unit vectors where

$$\psi' \sim \psi \text{ iff } \psi' = c\psi \text{ for some } c \in \mathbb{C} \text{ with } |c| = 1$$

The set of these equivalence classes is the **projective space** **PH**.

Wigner's Theorem. Given a Hilbert space **H**, any map from **PH** to itself that preserves transition probabilities comes from either

- a unitary operator $U\colon \mathbf{H} \to \mathbf{H}$

$$U(\psi+\phi) = U\psi+U\phi \qquad U(c\psi) = c\,U\psi \qquad \langle U\phi, U\psi\rangle = \langle \phi,\psi\rangle$$

or

- an antiunitary operator $J\colon \mathbf{H} \to \mathbf{H}$

$$J(\psi+\phi) = J\psi+J\phi \qquad J(c\psi) = \overline{c}\,J\psi \qquad \langle J\phi, J\psi\rangle = \overline{\langle \phi,\psi\rangle}$$

SYMMETRIES THAT SQUARE TO ONE

Some important symmetries that square to the identity:

- P: parity
- C: charge conjugation
- T: time reversal

Systems may or may not have any of these symmetries. They may also be symmetric only under combinations like CP, PT, CT or CPT.

SYMMETRIES THAT SQUARE TO ONE

Suppose $f\colon \mathbf{PH} \to \mathbf{PH}$ preserves transition probabilities and $f^2 = 1$. By Wigner's theorem there are two options:

1. f comes from a unitary U with $U^2 = c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

2. f comes from an antiunitary J with $J^2 = c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

SYMMETRIES THAT SQUARE TO ONE

Suppose $f\colon \mathbf{PH} \to \mathbf{PH}$ preserves transition probabilities and $f^2 = 1$. By Wigner's theorem there are two options:

1. f comes from a unitary U with $U^2 = c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

   Then $V = c^{-1/2}U$ is a unitary with $V^2 = 1$ that also gives f.

2. f comes from an antiunitary J with $J^2 = c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

SYMMETRIES THAT SQUARE TO ONE

Suppose $f\colon \mathbf{PH} \to \mathbf{PH}$ preserves transition probabilities and $f^2 = 1$. By Wigner's theorem there are two options:

1. f comes from a unitary U with $U^2 = c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

   Then $V = c^{-1/2} U$ is a unitary with $V^2 = 1$ that also gives f.

2. f comes from an antiunitary J with $J^2 = c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

   Then multiplying J by a phase does not change J^2.

SYMMETRIES THAT SQUARE TO ONE

Suppose $f \colon \mathbf{PH} \to \mathbf{PH}$ preserves transition probabilities and $f^2 = 1$. By Wigner's theorem there are two options:

1. f comes from a unitary U with $U^2 = c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

   Then $V = c^{-1/2} U$ is a unitary with $V^2 = 1$ that also gives f.

2. f comes from an antiunitary J with $J^2 = c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

   Then multiplying J by a phase does not change J^2.
   Since $Jc = \overline{c}J$ yet $Jc = JJ^2 = J^2 J = cJ$, we have $c = \pm 1$.

SYMMETRIES THAT SQUARE TO ONE

So:

If a symmetry $f \colon \mathbf{PH} \to \mathbf{PH}$ that squares to one is implemented by a unitary U, we can always find such a U with $U^2 = 1$.

But if f is implemented by an antiunitary J, precisely one of two options holds: $J^2 = 1$ or $J^2 = -1$.

If an antiunitary $J\colon \mathbf{H} \to \mathbf{H}$ has $J^2 = 1$ then it acts like complex conjugation!

We can define a *real* Hilbert space

$$\mathbf{H}_{\mathbb{R}} = \{\psi \in \mathbf{H} : \; J\psi = \psi\}$$

and $\mathbf{H}$ is the complexification of this:

$$\mathbf{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathbf{H}_{\mathbb{R}}$$

If an antiunitary $J\colon \mathbf{H} \to \mathbf{H}$ has $J^2 = -1$ then the operators i, $j = J$, and $k = ij$ obey the quaternion relations:

$$i^2 = j^2 = k^2 = ijk = -1$$

We can make $\mathbf{H}$ into a *quaternionic* Hilbert space $\mathbf{H}_{\mathbb{H}}$, and $\mathbf{H}$ is the underlying complex Hilbert space of this:

$$\mathbf{H} = \mathbb{C} \otimes_{\mathbb{C}} \mathbf{H}_{\mathbb{H}}$$

So, $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ all show up in quantum physics!
What makes them special?

Let's define an **algebra** to be a finite-dimensional real vector space A with an associative product that distributes over linear combinations, and a unit $1 \in A$.

A **division algebra** is an algebra where any nonzero element has a multiplicative inverse.

Frobenius' Theorem. There are three division algebras:

- the real numbers, $\mathbb{R}$
- the complex numbers, $\mathbb{C}$, with $i^2 = -1$
- the quaternions, $\mathbb{H}$, with $i^2 = j^2 = k^2 = ijk = -1$

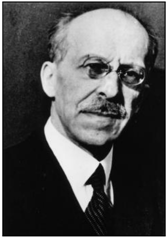
The role of the division algebras in quantum physics becomes even clearer if we focus on systems with symmetry.

A **unitary representation** of a group G on a Hilbert space $\mathbf{H}$ consists of unitaries $\rho(g) \colon \mathbf{H} \to \mathbf{H}$ with

$$\rho(gh) = \rho(g)\rho(h) \qquad \text{and} \qquad \rho(1) = 1$$

We say ρ is **irreducible** if the only closed subspaces $\mathbf{V} \subseteq \mathbf{H}$ with $\rho(g) \colon \mathbf{V} \to \mathbf{V}$ for all g are $\mathbf{V} = \{0\}$ and $\mathbf{V} = \mathbf{H}$.

Schur's Lemma. Suppose ρ is an irreducible unitary representation of a group G on a Hilbert space $\mathbf{H}$. Then the only unitary operators $U\colon \mathbf{H} \to \mathbf{H}$ that commute with all the $\rho(g)$ are phases: $U = c1$ for some $|c| = 1$.

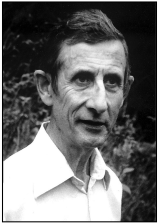


The Threefold Way (Dyson). Suppose ρ is an irreducible unitary representation of a group G on a Hilbert space **H**. Then exactly one of these holds:

-1. There is an antiunitary with $J^2 = -1$ commuting with all the $\rho(g)$. Then ρ is the underlying complex representation of a representation on a quaternionic Hilbert space, and we call ρ **quaternionic**.

0. There is no antiunitary commuting with all the $\rho(g)$.Then we call ρ **complex**.

1. There is an antiunitary with $J^2 = 1$ commuting with all the $\rho(g)$. Then ρ is the complexification of a representation on a real Hilbert space, and we call ρ **real**.



For example:

In the spin-j representation of $\mathrm{SU}(2)$, all the transformations coming from $\mathrm{SU}(2)$ commute with some antiunitary J.

This has $J^2 = 1$ when j is an integer and $J^2 = -1$ when j is a half-integer.

Indeed, the spin-1 representation of $\mathrm{SU}(2)$ on $\mathbb{C}^3$ is the complexification of a *real* representation on $\mathbb{R}^3$.

On the other hand, the spin-$1/2$ representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$ is the underlying complex representation of a *quaternionic* representation on $\mathbb{H}$.

$\mathrm{SU}(2)$ acts on $\mathbb{H}$ as right multiplication by quaternions q with $|q| = 1$.

Any unitary representation ρ of a compact Lie group G is a direct sum

$$\rho = \rho(-1) \oplus \rho(0) \oplus \rho(1)$$

where:

- $\rho(-1)$ is a sum of irreducibles that are quaternionic.
- $\rho(0)$ is a sum of irreducibles that are complex.
- $\rho(1)$ is a sum of irreducibles that are real.

Moreover the set

$$\mathrm{III} = \{-1, 0, 1\} \subseteq \mathbb{R}$$

is closed under multiplication, and given two unitary representations ρ, ρ' we have

$$(\rho \otimes \rho')(j) = \bigoplus_{i, i' \in \mathrm{III} \text{ such that } ii' = j} \rho(i) \otimes \rho'(i')$$

Now, on to the tenfold way!

The tenfold way describes the options for charge conjugation and time reversal, which in condensed matter physics we assume are commuting antiunitary operators:

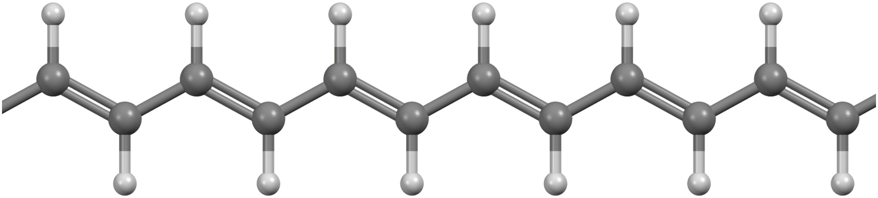
- **time-reversal symmetry**
  with $T^2 = 1$, with $T^2 = -1$, or no T symmetry
- **charge conjugation symmetry**
  with $C^2 = 1$, with $C^2 = -1$, or no C symmetry.

or

- only a combination of both, called S. Since S is unitary we may assume that $S^2 = 1$.

This gives $3 \times 3 + 1 = 10$ options.

For example, the Su–Schrieffer–Heeger model of superconductivity in polyacetylene doesn't have C or T symmetry separately. But it has the combined symmetry: a unitary S with $S^2 = 1$.



More fundamentally, the tenfold way arises from super Hilbert spaces.

A **super Hilbert space** is simply a Hilbert space $\mathbf{H}$ that is written as a direct sum of two parts, $\mathbf{H}_0 \oplus \mathbf{H}_1$.

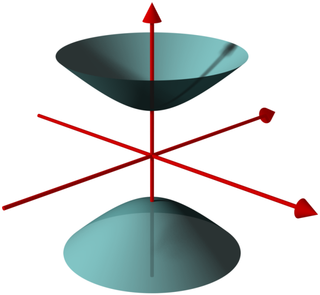
We call states $\psi \in \mathbf{H}_0$ **even** and states $\psi \in \mathbf{H}_1$ **odd**.

We can use super Hilbert spaces in various ways.

We can let $\mathbf{H}_0$ be the bosonic and $\mathbf{H}_1$ the fermionic states.

But condensed matter physics does not mainly apply super Hilbert spaces in this way! Instead....

We can let $\mathbf{H}_0$ be a Hilbert space for particles and $\mathbf{H}_1$ a Hilbert space for antiparticles, or holes.



We can have antiunitaries that are **even**:

$$T \colon \mathbf{H} \to \mathbf{H} \text{ with } T \colon \mathbf{H}_0 \to \mathbf{H}_0, \;\; T \colon \mathbf{H}_1 \to \mathbf{H}_1$$

and antiunitaries that are **odd**:

$$C \colon \mathbf{H} \to \mathbf{H} \text{ with } C \colon \mathbf{H}_0 \to \mathbf{H}_1, \;\; C \colon \mathbf{H}_1 \to \mathbf{H}_0$$

A group G is **$\mathbb{Z}/2$-graded** if it's written as the union of disjoint subsets G_0, G_1 such that

$$\text{if } g \in G_i \text{ and } h \in G_j \text{ then } gh \in G_{i+j} \text{ (with addition mod 2)}$$

A **unitary representation** ρ of a $\mathbb{Z}/2$-graded group G on a super Hilbert space $\mathbf{H}$ is an ordinary unitary representation of G on $\mathbf{H}$ such that

$$\text{if } g \in G_i \text{ and } \psi \in \mathbf{H}_j \text{ then } \rho(g)\psi \in \mathbf{H}_{i+j} \text{ (with addition mod 2)}$$

ρ is **irreducible** if the only closed subspaces $\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_1$, $\mathbf{V}_i \subseteq \mathbf{H}_i$ with $\rho(g)\colon \mathbf{V} \to \mathbf{V}$ for all g are $\mathbf{V} = \{0\}$ and $\mathbf{V} = \mathbf{H}$.

The Tenfold Way. The irreducible unitary representations ρ of a $\mathbb{Z}/2$-graded group G on a super Hilbert space **H** come in 10 types, based on their **commutant**: the set of real-linear operators that commute with $\rho(g)$ for all G.

In 9 of these types the commutant contains:

- an even antiunitary T with either $T^2 = 1$, $T^2 = -1$, or no such T

and

- an odd antiunitary C with either $C^2 = 1$,$C^2 = -1$, or no such C.

In the 10th type the commutant contains:

- no such T or C, but an odd unitary S; we may assume $S^2 = 1$.

Note: phases always give *even* unitaries in the commutant.

The types listed above form a ten-element set. Call this set X.

If a unitary representation ρ of a $\mathbb{Z}/2$-graded group G on a super Hilbert space is a direct sum of irreducibles, then

$$\rho = \bigoplus_{i \in \mathrm{X}} \rho(i)$$

where $\rho(i)$ is a sum of irreducibles of the ith type.

Moreover there is an addition $+$ on the set X such that given two unitary representations ρ, ρ' we have

$$(\rho \otimes \rho')(j) = \bigoplus_{i, i' \in \mathrm{X} \text{ such that } i + i' = j} \rho(i) \otimes \rho'(i')$$

This makes X into a commutative monoid (not a group).

The commutative monoid X is the disjoint union of $\mathbb{Z}/8$ and $\mathbb{Z}/2$, with addition defined by

$$\begin{aligned} i+j &= i+j \bmod 8 \quad \text{if} \quad i,j \in \mathbb{Z}/8 \\ i+j &= i+j \bmod 2 \quad \text{if} \quad i,j \in \mathbb{Z}/2 \\ i+j &= i+j \bmod 2 \quad \text{if} \quad i \in \mathbb{Z}/8, j \in \mathbb{Z}/2 \end{aligned}$$

So, we can write

$$\mathrm{X} = \{0,1,2,3,4,5,6,7,\mathbf{0},\mathbf{1}\}$$

and for example

$$2+3=5, \quad \mathbf{1}+\mathbf{1}=\mathbf{0}, \quad 6+\mathbf{1}=\mathbf{1}$$

$\mathrm{Cliff}_{-1} = \mathbb{R} \oplus \mathbb{R}$ $T^2 = 1$ $C^2 = 1$	$\mathrm{Cliff}_0 = \mathbb{R}$ $T^2 = 1$	$\mathrm{Cliff}_1 = \mathbb{C}$ $T^2 = 1$ $C^2 = -1$
$\mathrm{Cliff}_{-2} = \mathrm{M}_2(\mathbb{R})$ $C^2 = 1$	$\mathrm{Cliff}_0 = \mathbb{C}$ $S^2 = 1$ $\mathrm{Cliff}_1 = \mathbb{C} \oplus \mathbb{C}$	$\mathrm{Cliff}_2 = \mathbb{H}$ $C^2 = -1$
$\mathrm{Cliff}_{-3} = \mathrm{M}_2(\mathbb{C})$ $T^2 = -1$ $C^2 = 1$	$\mathrm{Cliff}_4 \simeq \mathbb{H}$ $T^2 = -1$	$\mathrm{Cliff}_3 = \mathbb{H} \oplus \mathbb{H}$ $T^2 = -1$ $C^2 = -1$

Just as III was secretly the set of *division algebras*, X is secretly the set of *super division algebras!*

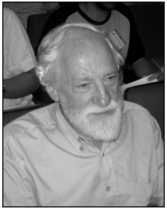
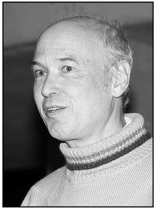
A **superalgebra** is an algebra $A = A_0 \oplus A_1$ such that

if $a \in A_i$ and $b \in A_j$ then $ab \in A_{i+j}$ (with addition mod 2)

We call $a \in A_0$ **even** and $a \in A_1$ **odd**.

A **super division algebra** is a superalgebra where any nonzero element that is either even or odd has a multiplicative inverse.

Example. We can make $\mathbb{C}$ into a super division algebra in two ways. In one, both real and imaginary numbers are even. In the other, real numbers are even and imaginary numbers are odd.

Theorem (Wall, Deligne). There are 10 super division algebras.

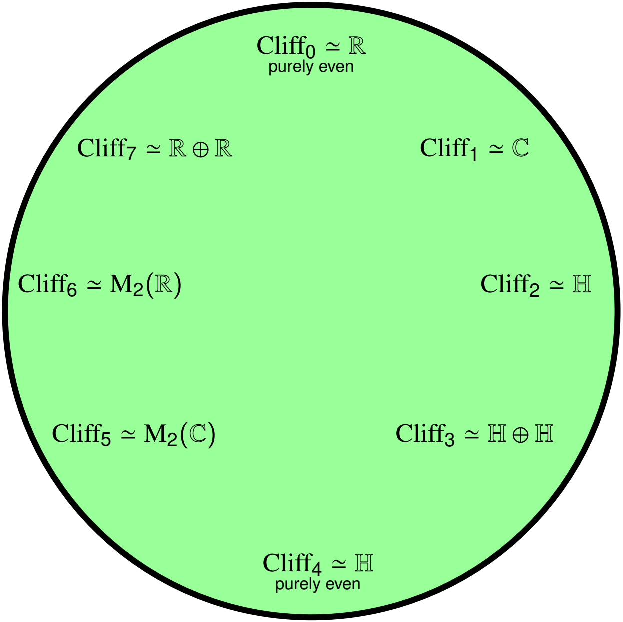
- $\mathrm{Cliff}_0 = \mathbb{R}$ is a super division algebra where every element is even.
- Cliff_1, the free superalgebra on an odd square root of -1. As an algebra $\mathrm{Cliff}_1 \cong \mathbb{C}$.
- Cliff_2, the free superalgebra on 2 anticommuting odd square roots of -1. As an algebra $\mathrm{Cliff}_2 \cong \mathbb{H}$.
- Cliff_3, the free superalgebra on 3 anticommuting odd square roots of -1. As an algebra $\mathrm{Cliff}_3 \cong \mathbb{H} \oplus \mathbb{H}$.

- Cliff_{-1}, the free superalgebra on an odd square root of 1. As an algebra $\mathrm{Cliff}_{-1} \cong \mathbb{R} \oplus \mathbb{R}$.
- Cliff_{-2}, the free superalgebra on 2 anticommuting odd square roots of 1. As an algebra $\mathrm{Cliff}_{-2} \cong \mathrm{M}_2(\mathbb{R})$.
- Cliff_{-3}, the free superalgebra on 3 anticommuting odd square roots of 1. As an algebra $\mathrm{Cliff}_{-3} \cong \mathrm{M}_2(\mathbb{C})$.

Neither Cliff_4 nor Cliff_{-4} is a super division algebra. But both are 'Morita equivalent' to $\mathbb{H}$, a super division algebra where every element is even.

Two superalgebras A and B are **Morita equivalent**, or $A \simeq B$, if they have equivalent categories of representations on super vector spaces. In general

$$\mathrm{Cliff}_{n+8} \simeq \mathrm{Cliff}_n$$

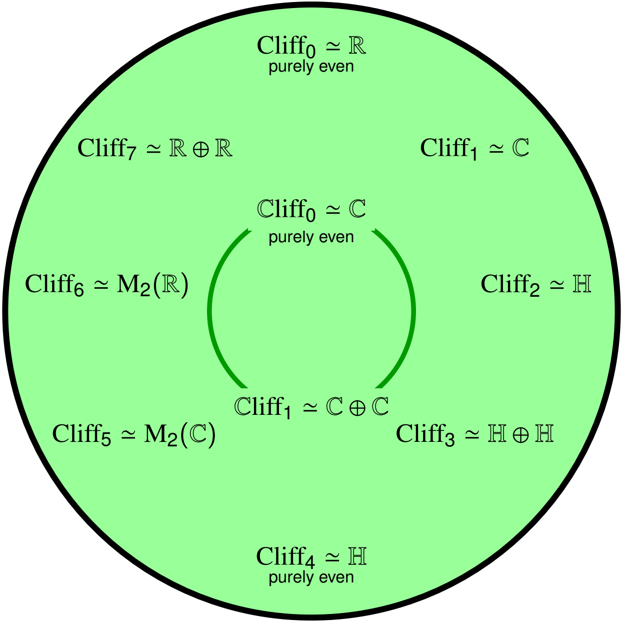

$\mathrm{Cliff}_0 \simeq \mathbb{R}$
purely even
$\mathrm{Cliff}_1 \simeq \mathbb{C}$
$\mathrm{Cliff}_2 \simeq \mathbb{H}$
$\mathrm{Cliff}_3 \simeq \mathbb{H} \oplus \mathbb{H}$
$\mathrm{Cliff}_4 \simeq \mathbb{H}$
purely even
$\mathrm{Cliff}_5 \simeq M_2(\mathbb{C})$
$\mathrm{Cliff}_6 \simeq M_2(\mathbb{R})$
$\mathrm{Cliff}_7 \simeq \mathbb{R} \oplus \mathbb{R}$

The other two super division algebras are *complex* Clifford algebras:

- $\mathbb{C}\mathrm{liff}_0 = \mathbb{C}$ is a complex super division algebra where every element is even.
- $\mathbb{C}\mathrm{liff}_1$, the free complex superalgebra on an odd square root of -1. As an algebra $\mathbb{C}\mathrm{liff} \cong \mathbb{C} \oplus \mathbb{C}$.

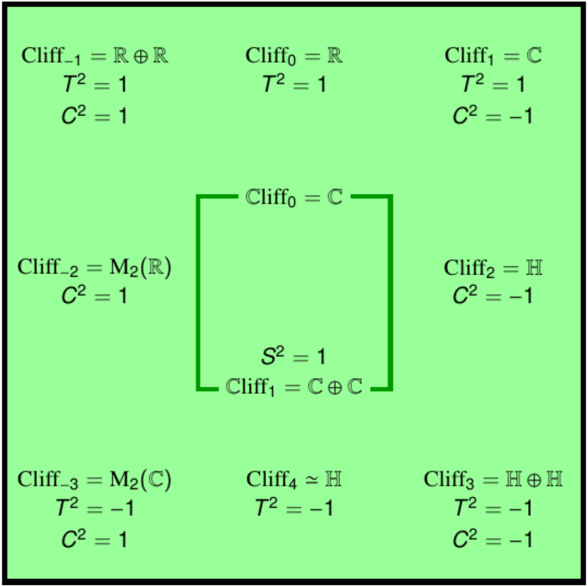
In general

$$\mathbb{C}\mathrm{liff}_{n+2} \simeq \mathbb{C}\mathrm{liff}_n$$


$\mathrm{Cliff}_0 \simeq \mathbb{R}$
purely even
$\mathrm{Cliff}_7 \simeq \mathbb{R} \oplus \mathbb{R}$
$\mathrm{Cliff}_1 \simeq \mathbb{C}$
$\mathbb{C}\mathrm{liff}_0 \simeq \mathbb{C}$
purely even
$\mathrm{Cliff}_6 \simeq M_2(\mathbb{R})$
$\mathrm{Cliff}_2 \simeq \mathbb{H}$
$\mathbb{C}\mathrm{liff}_1 \simeq \mathbb{C} \oplus \mathbb{C}$
$\mathrm{Cliff}_5 \simeq M_2(\mathbb{C})$
$\mathrm{Cliff}_3 \simeq \mathbb{H} \oplus \mathbb{H}$
$\mathrm{Cliff}_4 \simeq \mathbb{H}$
purely even

There is a one-to-one correspondence between:

- The 10 ways unitary and/or antiunitary operators commute with an irreducible unitary representation of a $\mathbb{Z}/2$-graded group on a super Hilbert space.
- The 10 Morita equivalence classes of real and complex Clifford algebras, viewed as super algebras.
- The 10 super division algebras.



$\mathrm{Cliff}_{-1} = \mathbb{R} \oplus \mathbb{R}$
$T^2 = 1$
$C^2 = 1$
$\mathrm{Cliff}_0 = \mathbb{R}$
$T^2 = 1$
$\mathrm{Cliff}_1 = \mathbb{C}$
$T^2 = 1$
$C^2 = -1$
$\mathrm{Cliff}_0 = \mathbb{C}$
$\mathrm{Cliff}_{-2} = \mathrm{M}_2(\mathbb{R})$
$C^2 = 1$
$\mathrm{Cliff}_2 = \mathbb{H}$
$C^2 = -1$
$S^2 = 1$
$\mathrm{Cliff}_1 = \mathbb{C} \oplus \mathbb{C}$
$\mathrm{Cliff}_{-3} = \mathrm{M}_2(\mathbb{C})$
$T^2 = -1$
$C^2 = 1$
$\mathrm{Cliff}_4 \simeq \mathbb{H}$
$T^2 = -1$
$\mathrm{Cliff}_3 = \mathbb{H} \oplus \mathbb{H}$
$T^2 = -1$
$C^2 = -1$