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Lagrangian and symplectic techniques in discrete mechanics

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Dedications

ABSTRACT OF THE DISSERTATION

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By a "discrete mechanical system," we mean a system in which time evolution proceeds in integer steps and the configuration space is a finite set. The most widely studied examples are cellular automata on finite lattices. We develop algebraic analogs of many of the basic concepts from differential geometry to overcome obstacles created by working over algebraic structures. We extend the methods of Lagrangian mechanics to treat discrete mechanical systems. In particular, we derive an algebraic analog of the Euler-Lagrange equation for discrete mechanics, starting from a variational principle. As an example of how this analog works, we prove a version of Noether's theorem applicable to this context. We relate our framework to Hamiltonian mechanics, or more precisely, symplectic geometry. We define a Poisson bracket on our analog of phase space, which allows us to develop some of the results on the Poisson algebra of functions on configuration space. In the process, we apply the techniques constructed to the discretizations of some well-known systems, such as particles in potentials, including the harmonic and anharmonic oscillators, the rigid rotating body, and the WZW model. We give characterizations of the bounded solutions of the discrete harmonic oscillator on the integers and demonstrate a difference between the discrete anharmonic oscillator and its continuous analog. We give a definition of complete integrability for a discrete mechanical system on a smooth real affine algebraic variety and give a criterion for the complete integrability of such systems. We show that as the time steps of a discrete system decrease to zero, a solution of the discrete system converges uniformly to a solution of the corresponding continuous system. We apply this result to the discrete analog of a particle in a polynomial potential and the rigid rotating body.

Contents

Chapter 1

Introduction

By a "discrete mechanical system," we mean a system in which time evolution proceeds in integer steps and the configuration space is a finite set. We distinguish these systems from "discrete dynamical systems", in which the configuration space is a manifold. The most widely known example of discrete mechanical systems are cellular automata on finite lattices. Recently, as physicists and computer scientists have become more familiar with using digital computers to approximately simulate physical systems, there has been an increased interest in discrete mechanics. But as classical field theories may be treated as a special case on classical mechanical systems, it is useful to treat cellular automata as a special case of discrete mechanical systems.

It seems worthwhile to attempt to extend standard techniques in classical mechanics, such as the Lagrangian and Hamiltonian formalisms, to discrete mechanics. To do so, we first develop algebraic analogs of many of the basic concepts from differential geometry, such as substituting an algebraic definition of differential forms for the classical differential calculus, to overcome some of the analytical obstacles created by working over algebraic structures. We begin by deriving an algebraic analog of the Euler-Lagrange equation for discrete mechanics, starting from a variational principle. As an example of how this analog works, we prove a version of Noether's theorem applicable to this context.

We relate our framework to Hamiltonian mechanics, or more precisely, symplectic geometry. With the more algebraic definitions used, we define a Poisson bracket on our analog of phase space, which allows us to develop some of the results on the Poisson algebra of functions on configuration space. In the process, we apply the techniques constructed to the discretizations of some well-known systems, such as particles in potentials, including the harmonic and anharmonic oscillators, the rigid rotating body, and the WZW model. We give characterizations of the bounded solutions of

the rigid rotating body, and the WZW model. We give characterizations of the bounded solutions of the discrete harmonic oscillator on the integers and demonstrate a difference between the discrete anharmonic oscillator and its continuous analog. We give a definition of complete integrability for a discrete mechanical system on a smooth real affine algebraic variety and give a criterion for the complete integrability of such systems. We show that as the time steps of a discrete system decrease to zero, a solution of the discrete system converges uniformly to a solution of the corresponding continuous system. We apply this result to the discrete analog of a particle in a polynomial potential and the rigid rotating body.

For somewhat related work on extending notions of classical mechanics to more general algebraic settings see, for example [2, 12, 15]. For related work in discrete dynamical systems see [4, 17] and the references therein. For applications of symplectic mechanics to cellular automata see [5]. For related work on the integrability of discrete dynamical systems, see [10, 17].

Chapter 2

Generalization of Phase Space

Since the configuration space for our algebraic models may be no more than a ring or group, which may have no specific topological or analytic properties, there is not necessarily a concept of tangent vectors, tangent bundles, cotangent bundles, etc. A new formalism must be developed with structures which perform the same tasks as the above, but are also able to be utilized under very general conditions. In this chapter, we explain how the basic concepts from differential geometry can be replaced by more general algebraic structures and show how some standard tools can be used in this new setting. First, instead of working directly with configuration space, we work with the algebraic functions on the configuration space, which form a commutative algebra A over configuration space. For example, if the configuration space is an n-dimensional vector space over a field k, we use the algebra of polynomial functions in n variables over k, $k[x_1, \ldots, x_n]$.

2.1 Algebraic Analogs

The analog of a vector field on configuration space is a derivation of A, that is, a k-linear map $v\colon A \to A$ such that $v(ab) = av(b) + v(a)b$ for all a, $b \in A$.

The space of all derivations of A is itself a Lie algebra over A with the bracket structure given by the commutator of derivations and is denoted $\mathrm{Der}(A)$.

In what follows, we will need an algebraic analog of the differential calculus. For this, we use the concept of Kähler differentials [3, 9], or more generally, algebraic differential forms.

Definition 2.1. *A **graded algebra** over k is an algebra Ω, that is also a graded module:*

$$\Omega = \bigoplus_{p=0}^{\infty} \Omega^p$$

in such a way that if $\omega \in \Omega^p$ and $\mu \in \Omega^q$, then the product $\omega\mu \in \Omega^{p+q}$.

Definition 2.2. *An element $\omega \in \Omega^p$ is said to be of **degree** p.*

Definition 2.3. *A map $f\colon \Omega \to \Omega'$ is said to be of **degree** q if for any $\omega \in \Omega^p$, $f(\omega) \in {\Omega'}^{p+q}$.*

Definition 2.4. *A map $f\colon \Omega \to \Omega'$ is called a **graded algebra morphism** if it is an algebra homomorphism of degree 0, i.e. it preserves grading.*

Definition 2.5. *A **graded derivation of degree** q of a graded algebra Ω is a map $\delta\colon \Omega \to \Omega$ such that if $\omega \in \Omega^p$:*

$$\delta(\omega\mu) = \delta(\omega)\mu + (-1)^{pq}\omega\delta\mu.$$

Definition 2.6. *A **differential graded algebra** is a graded algebra Ω equipped with a graded derivation d of degree 1 such that $d^2 = 0$. This map d is called the **differential** of Ω.*

Definition 2.7. *Given two differential graded algebras Ω and Ω' with differentials d and d', a **differential graded algebra morphism** $f\colon \Omega \to \Omega'$ is an algebra homomorphism that preserves grading and intertwines the differentials, i.e. $d'f = fd$.*

Definition 2.8. *A graded algebra Ω is called **graded commutative** if for $\omega \in \Omega^p$ and $\mu \in \Omega^q$, then $\omega\mu = (-1)^{pq}\mu\omega$.*

Definition 2.9. *If B is a commutative k-algebra, the **algebraic differential forms** $\Omega(B)$ are the graded commutative differential graded algebra with $\Omega^0(B) = B$ defined by the following universal mapping property: given any graded commutative differential graded algebra Ω and given any algebra homomorphism $f\colon B \to \Omega^0$, there exists a unique differential graded algebra homomorphism $f_*\colon \Omega(B) \to \Omega$ such that the following diagram commutes*

$$\begin{array}{ccc} \Omega(B) & \xrightarrow{f_*} & \Omega \\ \uparrow & & \uparrow \\ B & \xrightarrow{f} & \Omega^0 \end{array}$$

We write the product in $\Omega(B)$ as a wedge product and the differential as d. We refer to the elements of $\Omega^p(B)$ as p-forms.

If B is the algebra of smooth functions on a manifold M, $\Omega(B)$ is isomorphic to the usual space of differential forms. Concretely, $\Omega(B)$ is the algebra generated by B and the elements da, where $a \in B$, with the relations:

$$\begin{aligned} d(\lambda a) &= \lambda\, da, & d(a+b) &= da + db, \\ d(ab) &= da \wedge b + a \wedge db, & a \wedge db &= db \wedge a, \\ da \wedge db &= -db \wedge da, & da \wedge da &= 0, \end{aligned}$$

for all a, $b \in B$, $\lambda \in k$, with the last necessary only if 2 is not a unit in k.

Definition 2.10. *Given a derivation v of B, we define a degree 1 graded derivation $\imath_v$ of $\Omega(B)$, called the* **interior product** *with v as follows: If $\omega \in \Omega^p(B)$, $\imath_v\omega \in \Omega^{p-1}(B)$ is given by*

$$(\imath_v\omega)(v_1, \ldots, v_{p-1}) = \omega(v, v_1, \ldots, v_{p-1}).$$

Definition 2.11. *If v is a derivation, the* **Lie derivative** *in the direction of v, L_v is the degree 0 derivation uniquely determined by the following properties: given $\omega \in \Omega^p(B)$, $\mu \in \Omega^q(B)$, $f \in B$, $\alpha \in k$,*

$$\begin{aligned} L_v(\omega + \mu) &= L_v\omega + L_v\mu \\ L_v(\omega \wedge \mu) &= (L_v\omega) \wedge \mu + \omega \wedge (L_v\mu) \\ L_v(\alpha\omega) &= \alpha L_v\omega \\ L_v f &= vf \\ L_v(d\omega) &= d(L_v\omega) \end{aligned}$$

The following result nicely ties together the differential, interior product, and Lie derivative on $\Omega(B)$.

Proposition 2.1. (Weil's Formula) *Let v be a derivation on B. Then*

$$L_v = (i_v d + d i_v)$$

Proof - Since $\Omega(B)$ is generated by elements of the form $a \wedge db$, we need only show the result for these. First

$$\begin{aligned} L_v(a \wedge db) &= L_v a \wedge db + a \wedge L_v(db) \\ &= va \wedge db + a \wedge d(L_v b) \\ &= va \wedge db + a \wedge dvb. \end{aligned}$$

And

$$\begin{aligned}(i_v d + d i_v)(a \wedge db) &= i_v d(a \wedge db) + d i_v(a \wedge db) \\ &= i_v(da \wedge db) + d(a \wedge i_v db) \\ &= i_v(da) \wedge db - da \wedge i_v(db) + da \wedge vb + a \wedge dvb \\ &= va \wedge db + a \wedge dvb\end{aligned}$$

Hence, $L_v = (i_v d + d i_v)$.

□

As $\mathrm{Der}(B)$ is our analog of vector fields on configuration space and $\Omega^1(B)$ is our analog of one-forms on configuration space, we want the notion of a pairing between vector fields and one-forms.

Proposition 2.2. *For any commutative k-algebra B, there exists a unique B-bilinear pairing* $\langle \cdot, \cdot \rangle \colon \Omega^1(B) \otimes \mathrm{Der}(B) \to B$ *such that* $\langle da, v \rangle = v(a)$ *for all* $a \in B$, $v \in \mathrm{Der}(B)$.

Proof - Since $\Omega^1(B)$ is generated by elements of the form da, where $a \in B$, the pairing is unique. Now

$$\begin{aligned}\langle da, \alpha v + w \rangle &= (\alpha v + w)(a) \\ &= \alpha v(a) + w(a) \\ &= \alpha \langle da, v \rangle + \langle da, w \rangle,\end{aligned}$$

and

$$\begin{aligned}\langle \alpha \wedge da + db, v \rangle &= \langle d(\alpha a + b), v \rangle \\ &= v(\alpha a + b) \\ &= \alpha v(a) + v(b) \\ &= \alpha \langle da, v \rangle + \langle db, v \rangle.\end{aligned}$$

Therefore, the pairing is bilinear.

□

2.2 Symplectic Structures

Recall that a symplectic structure ω on a space X is a closed, non-degenerate 2-form on X, i.e. $\omega \in \Omega^2(X)$, with $d\omega = 0$, such that if $\omega(a, \cdot) \equiv 0$, then $a = 0$. For example, if X is the phase space for a mechanical system with coordinates (p_i, q_i) representing momentum and position respectively, the canonical symplectic structure on X is given by $\sum_i dp_i \wedge dq_i$.

Given a symplectic structure ω on the phase space X, one constructs a Poisson bracket as follows: for every vector field v on configuration space, we obtain an invertible map α from the vector fields on X to the 1-forms on X given by

$$\alpha(v)(w) = \omega(w, v).$$

Then given a function F on X, the corresponding Hamiltonian vector field v_F is given by

$$v_F = \alpha^{-1}(dF).$$

Now, given two functions F and G on X, one defines

$$\{F, G\} = v_F G.$$

We note that in some cases the symplectic structure used in this process need only be non-degenerate in the sense that the map α is invertible.

To define the algebraic analog of the above, let A be a commutative k-algebra and let ω be an element of $\Omega^2(A)$. We define a map $\alpha\colon \mathrm{Der}(A) \to \Omega^1(A)$ by $v \mapsto \alpha(v)$, where $\alpha(v)(w) = \omega(w, v)$. We say that a closed 2-form ω

is a symplectic structure on A provided the map $\alpha\colon \mathrm{Der}(A) \to \Omega^1(A)$ defined above is an isomorphism of A-modules. If this is the case, given an element F of A, we can define the derivation v_F corresponding to the Hamiltonian vector field with respect to F by

$$v_F = \alpha^{-1}(dF).$$

This allows us to define a Poisson bracket on $A \times A$ as the following: given two elements F and G of A, define

$$\{F, G\} = v_F G = \langle v_F, dG \rangle.$$

This implies

$$\{F, G\} = \omega(v_F, v_G).$$

Proposition 2.3. *If ω is a non-degenerate 2-form on A, the above bracket makes A into a Poisson algebra.*

Lemma 2.1. *If ω is a symplectic structure on A and $f \in A$, then $L_{v_F}\omega = 0$.*

Proof - By Weil's formula,

$$L_{v_F}\omega = (\iota_{v_F} d + d\iota_{v_F})\omega = \iota_{v_F} d\omega + d\iota_{v_F}\omega = 0 - ddF = 0$$

$\square$

Proof of Proposition - The linearity of the bracket is clear since ω and α^{-1} are linear. Also, $\{F, G\} = -\{G, F\}$ since ω is skew-symmetric. To see that the bracket is a bi-derivation, note that $\{F, \cdot\} = v_F$ is a derivation. For the Jacobi identity, by the above lemma,

$$0 = (L_{v_f})\omega(v_g, v_h)$$

$$
\begin{aligned}
&= v_f\omega(v_g, v_h) - \omega([v_f, v_g], v_h) + \omega([v_f, v_h], v_g) \\
&= \{f,\{g,h\}\} + \iota_{v_h}[v_f, v_g] - \iota_{v_g}[v_f, v_h] \\
&= \{f,\{g,h\}\} - [v_f, v_g]h + [v_f, v_h]g \\
&= \{f,\{g,h\}\} - v_f(v_g h) + v_g(v_f h) + v_f(v_h g) - v_h(v_f g) \\
&= \{f,\{g,h\}\} - \{f,\{g,h\}\} + \{g,\{f,h\}\} + \{f,\{h,g\}\} - \{h,\{f,g\}\} \\
&= \{g,\{f,h\}\} + \{f,\{h,g\}\} + \{h,\{g,f\}\}
\end{aligned}
$$

□

Proposition 2.4. *Let f, g be elements of A. Then*

$$v_{\{f,g\}} = [v_f, v_g].$$

In other words, the map $f \mapsto v_f$ is a Lie algebra homomorphism.

Proof - From the Jacobi identity,

$$
\begin{aligned}
0 &= \{h,\{f,g\}\} + \{f,\{g,h\}\} + \{g,\{h,f\}\} \\
&= -\{\{f,g\},h\} + \{f,\{g,h\}\} - \{g,\{f,h\}\} \\
&= -v_{\{f,g\}}h + v_f(v_g h) - v_g(v_f h) \\
&= -v_{\{f,g\}}h + [v_f, v_g]h
\end{aligned}
$$

Hence, $v_{\{f,g\}} = [v_f, v_g]h.$

□

Example 2.1. *The canonical symplectic structure on the phase space for a mechanical system with coordinates (p_i, q_i) is given by $\sum_i dp_i \wedge dq_i$, as noted above. Using the above process, we find*

$$\{p_i, p_j\} = \alpha^{-1}(dp_i)(p_j) = \partial_{q_i}(p_j) = 0$$

$$\{q_i, q_j\} = \alpha^{-1}(dq_i)(q_j) = -\partial_{p_i}(q_j) = 0$$

$$\{p_i, q_j\} = \alpha^{-1}(dp_i)(q_j) = -\partial_{q_i}(q_j) = -\delta_{ij}$$

Chapter 3

Discrete Systems

Now that we have defined algebraic analogs for the major structures and tools, we are ready to develop a discrete Lagrangian formalism. In this chapter, we construct analogs of some major tools from classical mechanics. We then apply these tools to the discrete analogs of a particle in a polynomial potential and automata corresponding to discrete wave equations. In addition, we characterize the bounded solutions for the harmonic oscillator on the integers and point out differences between a discrete anharmonic oscillator on the integers and its continuous counterpart.

3.1 Euler-Lagrange Equations

Recall from classical mechanics that the Lagrangian Formalism is based on the fact that a system has a Lagrangian, which defines an action functional. This action functional, when extremized, yields the equations of motion for the system. The process of extremization gives rise to the Euler-Lagrange equations, which can be derived as follows: given the Lagrangian $\mathcal{L}$ for a system and a path $q(t)$ in configuration space, define the action functional S

by

$$S = \int_0^T \mathcal{L}(q, \dot{q}) dt.$$

Then consider the variation of S with fixed endpoints.

$$\begin{aligned}
\delta S &= \delta \int_0^T \mathcal{L}(q, \dot{q}) dt \\
&= \int_0^T \delta \mathcal{L}(q, \dot{q}) dt \\
&= \int_0^T \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q} \delta q dt + \int_0^T \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}} \delta \dot{q} dt \\
&= \int_0^T \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}} \right) \delta q dt.
\end{aligned}$$

The only way that this variation can vanish is if

$$\frac{\partial \mathcal{L}(q, \dot{q})}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}} = 0.$$

We call the above equation the **Euler-Lagrange equation**. The solution to this equation describes the motion of the system defined by $\mathcal{L}$.

For the discretization of this process, we need to develop a variational principle applicable in a more general setting and explain how to obtain equations of motion from this variational principle. In the discrete Lagrangian formalism, the system takes values in an arbitrary commutative ring k instead of the real numbers, and, as described before, we work with the algebraic functions on this ring, which form a commutative algebra A over k. We will consider A as the configuration space for the system. Now, the algebra $H = A^{\otimes(T+1)} = A \otimes \cdots \otimes A$ is to be thought of as the functions on the space of "histories", where time takes values in the discrete set $\{0, \ldots, T\}$. In what follows, it will be convenient to sometimes use the notation $H = A_0 \otimes A_1 \otimes \cdots \otimes A_T$, where the algebras A_i are simply copies of A, with A_i thought of as the functions on configuration space at time i.

The Lagrangian for the system $\mathcal{L}$ is now a fixed element of $A \otimes A$. In the algebra H, let

$$S = \sum_{i=0}^{T-1} \mathcal{L}_i,$$

where $\mathcal{L}_i = 1 \otimes \cdots \otimes \mathcal{L} \otimes \cdots \otimes 1$ with $\mathcal{L}$ occupying the ith and $(i+1)$st slots. The element S corresponds to the action functional in classical mechanics. We now proceed to determine the variation of the action, and describe in what sense it must vanish to obtain the discrete analog of the Euler-Lagrange equations.

Given the algebra A, we form the algebraic differential forms on A, $\Omega(A)$. It can be shown for any algebra A that

$$\Omega^p(A \otimes A) = \bigoplus_{q=0}^{p} \Omega^q(A) \otimes \Omega^{p-q}(A)$$

and so, by induction,

$$\Omega^1(H) = \Omega^1(A_1 \otimes \cdots \otimes A_n) = \bigoplus_{i=0}^{n} A_0 \otimes A_1 \otimes \cdots \Omega^1(A_i) \otimes \cdots \otimes A_n.$$

Let $p_i \colon \Omega^1(H) \to \Omega^1(H)$ denote projection onto the ith summand. We define $d_i = p_i d$ and note $d = \sum_{i=0}^{n} d_i$. We may now define a new operator $\delta = \sum_{i=1}^{n-1} d_i$, which we think of as a variation of an element of H keeping the first and last summands fixed. We will use this new operator to extremize the discrete action functional.

Proposition 3.1. $\delta S = \sum_{i=1}^{T-1} d_i \mathcal{L}_i + d_i \mathcal{L}_{i-1}$.

Proof - For $j \neq i, i+1$, $d_j \mathcal{L}_i = 0$, hence

$$\delta S = \delta \sum_{i=0}^{T-1} \mathcal{L}_i = \sum_{i=0}^{T-1} \delta \mathcal{L}_i = \sum_{i=1}^{T-1} d_i \mathcal{L}_i + d_i \mathcal{L}_{i-1}.$$

$\square$

The vanishing of the expression $d_i\mathcal{L}_i + d_i\mathcal{L}_{i-1}$ is supposed to correspond to the Euler-Lagrange equation in this framework. But we must describe precisely in what sense it vanishes. It certainly does not vanish as a 1-form on the whole space of histories; rather, it should only vanish on trajectories satisfying the equations of motion. In the discrete-time context, the equations of motion express the configuration at a particular time as a function of the configurations at the two preceding times. We formalize this in terms of a homomorphism $\varphi\colon A_2 \to A_0 \otimes A_1$. Such a homomorphism determines a homomorphism $\Phi\colon A_1 \otimes A_2 \to A_0 \otimes A_1$ by

$$\begin{aligned} a \otimes 1 &\mapsto 1 \otimes a \\ 1 \otimes a &\mapsto \varphi(a). \end{aligned}$$

which plays the role of a time evolution map. (If it is puzzling that the indices *decrease* by 1 in this map, simply recall that a map from a space X to a space Y allows one to pull back functions on Y to functions on X.) We say that φ, or alternatively Φ, **satisfies the Euler-Lagrange equation** provided

$$\Phi_* d_1\mathcal{L}_1 + d_1\mathcal{L}_0 = 0$$

where $\Phi_*\colon \Omega^1(A_1 \otimes A_2) \to \Omega^1(A_0 \otimes A_1)$ is the map induced by Φ, and we regard $A_0 \otimes A_1$ and $A_1 \otimes A_2$ as subalgebras of H in order to define the differential d_1.

3.2 Noether's Theorem

Classically, Noether's theorem states that to any infinitesimal symmetry of a lagrangian system, there corresponds a conserved quantity, and gives an expression for this quantity:

Let M be a smooth manifold, $\mathcal{L}: TM \to \mathbb{R}$ a smooth function on its tangent bundle TM. Let $h^s: M \to M$, $s \in \mathbb{R}$, be a one-parameter group of diffeomorphisms which preserves $\mathcal{L}$. Then the lagrangian system of equations corresponding to $\mathcal{L}$ has a first integral $I: TM \to \mathbb{R}$.

In local coordinates q on M the integral I is written in the form

$$I(q, \dot{q}) = \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{dh^s(q)}{ds}\Big|_{s=0}.$$

We consider an analog of Noether's theorem in the discrete setting. Given a derivation v on A, we construct a derivation D on $A \otimes A$ by $D = 1 \otimes v + v \otimes 1$; that is,

$$D(a \otimes b) = a \otimes v(b) + v(a) \otimes b$$

A derivation v is called an **infinitesimal symmetry** of the Lagrangian $\mathcal{L} \in A \otimes A$ provided $(1 \otimes v + v \otimes 1)\mathcal{L} = 0$.

Theorem 3.1. *Suppose Φ is a time evolution map satisfying the Euler-Lagrange equation and v an infinitesimal symmetry of $\mathcal{L}$. Then the quantity $F_v \in A \otimes A$ defined by $F_v = -(v \otimes 1)\mathcal{L}$ is preserved by Φ, i.e., $\Phi(F_v) = F_v$.*

Proof - Here we use the pairing between 1-forms and derivations. Also, any isomorphism $\Phi: B \to C$ of such algebras induces a map $\Phi_*: \mathrm{Der}(C) \to \mathrm{Der}(B)$ given by $\Phi_* v = \Phi \circ v \circ \Phi^{-1}$. We thus have

$$\Phi F_v \;=\; -\Phi((v \otimes 1)\mathcal{L}) = -\Phi\langle d\mathcal{L}, v \otimes 1\rangle = -\Phi\langle d_1\mathcal{L}, v \otimes 1\rangle$$

$$
\begin{aligned}
&= -\langle \Phi_* d_1\mathcal{L}, \Phi_*(v\otimes 1)\rangle = \langle d_2\mathcal{L}, \Phi_*(v\otimes 1)\rangle \\
&= \langle d_2\mathcal{L}, 1\otimes v\rangle = \langle d\mathcal{L}, 1\otimes v\rangle = (1\otimes v)\mathcal{L} = -(v\otimes 1)\mathcal{L} = F_v
\end{aligned}
$$

□

Theorem 3.2. *The infinitesimal symmetries of a Lagrangian* $\mathcal{L}$ *form a Lie subalgebra of* $\mathrm{Der}(A\otimes A)$.

Proof - Clearly, linear combinations of infinitesimal symmetries are infinitesimal symmetries. If v and w are infinitesimal symmetries,

$$
\begin{aligned}
([v,w]\otimes 1 + 1\otimes [v,w])\mathcal{L} &= (v\otimes 1)[(w\otimes 1)\mathcal{L}] - (w\otimes 1)[(v\otimes 1)\mathcal{L}] \\
&\quad +(1\otimes v)[(1\otimes w)\mathcal{L}] - (1\otimes w)[(1\otimes v)\mathcal{L}] \\
&= -(v\otimes 1)[(1\otimes w)\mathcal{L}] + (w\otimes 1)[(1\otimes v)\mathcal{L}] \\
&\quad -(1\otimes v)[(w\otimes 1)\mathcal{L}] + (1\otimes w)[(v\otimes 1)\mathcal{L}] \\
&= -(v\otimes w)\mathcal{L} + (w\otimes v)\mathcal{L} - (w\otimes v)\mathcal{L} + (v\otimes w)\mathcal{L} \\
&= 0.
\end{aligned}
$$

So $[v,w]$ is an infinitesimal symmetry of $\mathcal{L}$. □

3.3 Discrete Symplectic Geometry

In continuous time classical mechanics the phase space is defined to be the cotangent bundle over the configuration space. In this discrete formulation, the Cartesian product of the configuration space with itself in fact plays the role of a phase space. Hence, the algebra $A\otimes A$ is the analog of the functions on phase space. To clarify this analogy, we would like to define a symplectic structure on $A\otimes A$ that is preserved by time evolution.

In this context, it is more convenient to think of the time evolution map Φ as a map from a fixed copy of $A \otimes A$ to itself. This requires a change of indexing from Section 3.1; we say that $\Phi: A_1 \otimes A_2 \to A_1 \otimes A_2$ satisfies the Euler-Lagrange equation if

$$\Phi_* d_1 \mathcal{L} + d_2 \mathcal{L} = 0,$$

where $\mathcal{L} = \mathcal{L}_1 \in A_1 \otimes A_2$. Note that $d\mathcal{L} = d_1\mathcal{L} + d_2\mathcal{L}$. We define the closed 2-form $\omega \in \Omega^2(A \otimes A)$ by $\omega = -dd_1\mathcal{L} = dd_2\mathcal{L}$. This plays the role of a symplectic structure on phase space. In fact, it is always preserved by time evolution:

Theorem 3.3. *ω is preserved by the time evolution map Φ, that is, $\Phi_*\omega = \omega$, provided Φ satisfies the Euler-Lagrange equations.*

Proof -

$$\Phi_*\omega = -\Phi_* dd_1\mathcal{L} = -d\Phi_* d_1\mathcal{L} = dd_2\mathcal{L} = \omega.$$

□

Note that, as defined, the symplectic structure ω defined above may not be non-degenerate. In what follows, we assume that ω as defined above is non-degenerate, i.e. the map it induces between $\mathrm{Der}(A \otimes A)$ and $\Omega^1(A \otimes A)$ is actually a k-module isomorphism.

We now develop some tools to prove conserved quantities Poisson-commute. The following lemma gives a simple formula for the derivation generated by a conserved quantity via the discrete Noether's theorem.

Theorem 3.4. *Suppose $\omega = dd_2\mathcal{L}$ is a non-degenerate symplectic structure and w is an infinitesimal symmetry of $\mathcal{L}$. Let F_w be the conserved quantity* *obtained from Noether's theorem. Then*

$$v_{F_w} = \alpha^{-1}(d[(w \otimes 1)\mathcal{L}]) = 1 \otimes w + w \otimes 1.$$

Proof - Consider the decomposition

$$d[(w \otimes 1)\mathcal{L}] = d_1[(w \otimes 1)\mathcal{L}] + d_2[(w \otimes 1)\mathcal{L}].$$

First, we show $\alpha^{-1}(d_1[(w \otimes 1)\mathcal{L}]) = 1 \otimes w$. Note that

$$\alpha(1 \otimes w)(1 \otimes u) = \omega(1 \otimes w, 1 \otimes u) = 0.$$

Also

$$(d_1[(w \otimes 1)\mathcal{L}])(1 \otimes u) = 0,$$

so $\alpha(1 \otimes w)$ and $d_1[(w \otimes 1)\mathcal{L}]$ agree on elements of the form $1 \otimes u$. Similarly,

$$\begin{aligned}
\alpha(1 \otimes w)(u \otimes 1) &= \omega(1 \otimes w, u \otimes 1) \\
&= d(d_2\mathcal{L})(1 \otimes w, u \otimes 1) \\
&= (1 \otimes w)(d_2\mathcal{L})(u \otimes 1) - (u \otimes 1)(d_2\mathcal{L})(1 \otimes w) \\
&\quad -(d_2\mathcal{L})([1 \otimes w, u \otimes 1]) \\
&= -(u \otimes 1)[(1 \otimes w)\mathcal{L}].
\end{aligned}$$

Since w is an infinitesimal symmetry of $\mathcal{L}$, we have $\alpha(1 \otimes w)(1 \otimes u) = (uw \otimes 1)\mathcal{L}$. Also,

$$d_1[(w \otimes 1)\mathcal{L}](u \otimes 1) = (u \otimes 1)[(w \otimes 1)\mathcal{L}] = (uw \otimes 1)\mathcal{L},$$

so $\alpha(1 \otimes w)$ and $d_1[(w \otimes 1)\mathcal{L}]$ also agree on elements of the form $u \otimes 1$. Since any derivation on $A \otimes A$ can be written as a combination of elements of the form $1 \otimes u$ and $u \otimes 1$, we have $d_1[(w \otimes 1)\mathcal{L}] = \alpha(1 \otimes w)$. Since α is an isomorphism, we have $\alpha^{-1}(d_1[(w\otimes 1)\mathcal{L}]) = 1\otimes w$. Now, we show $\alpha^{-1}(d_2[(w\otimes 1)\mathcal{L}]) = w\otimes 1$. As above,

$$\alpha(w \otimes 1)(u \otimes 1) = \omega(w \otimes 1, u \otimes 1) = 0$$

and

$$d_2[(w \otimes 1)\mathcal{L}](u \otimes 1) = 0,$$

so $\alpha(w \otimes 1)$ and $d_2[(w \otimes 1)\mathcal{L}]$ agree on elements of the form $u \otimes 1$. Similarly

$$\begin{aligned}
\alpha(w \otimes 1)(1 \otimes u) &= \omega(w \otimes 1, 1 \otimes u) \\
&= d(d_2\mathcal{L})\omega(w \otimes 1, 1 \otimes u) \\
&= (w \otimes 1)(d_2\mathcal{L})(1 \otimes u) - (1 \otimes u)(d_2\mathcal{L})(w \otimes 1) \\
&\quad -(d_2\mathcal{L})([w \otimes 1, 1 \otimes u]) \\
&= (w \otimes 1)[(1 \otimes u)\mathcal{L}] \\
&= (w \otimes u)\mathcal{L},
\end{aligned}$$

while

$$(d_2[(w \otimes 1)\mathcal{L}])(1 \otimes u) = (w \otimes u)\mathcal{L},$$

hence $\alpha(w \otimes 1)$ and $d_2[(w \otimes 1)\mathcal{L}]$ agree on elements of the form $1 \otimes u$. So, as above, $\alpha^{-1}(d_2[(w \otimes 1)\mathcal{L}]) = w \otimes 1$. Thus, we have

$$\alpha^{-1}(d[(w \otimes 1)\mathcal{L}]) = 1 \otimes w + w \otimes 1.$$

□

We can restate the above result as follows: If w is an infinitesimal symmetry of $\mathcal{L}$ and F_w is the conserved quantity obtained from w through Noether's theorem, then

$$dF_w = \alpha(1 \otimes w + w \otimes 1).$$

We now can prove the following proposition.

Theorem 3.5. *Suppose ω is a non-degenerate symplectic structure and v an infinitesimal symmetry of $\mathcal{L}$. The map $v \mapsto F_v$ is a Lie algebra homomorphism from the Lie algebra of infinitesimal symmetries of $\mathcal{L}$ to the Poisson algebra $A \otimes A$.*

Proof -

$$\begin{aligned}
\{F_v, F_w\} &= \{(v \otimes 1)\mathcal{L}, (w \otimes 1)\mathcal{L}\} \\
&= \alpha^{-1}(d[(v \otimes 1)\mathcal{L}])(w \otimes 1)\mathcal{L} \\
&= (1 \otimes v + v \otimes 1)[(w \otimes 1)\mathcal{L}] \\
&= (w \otimes v)\mathcal{L} + (vw \otimes 1)\mathcal{L} \\
&= -(wv \otimes 1)\mathcal{L} + (vw \otimes 1)\mathcal{L} \\
&= [(vw - wv) \otimes 1]\mathcal{L} \\
&= ([v, w] \otimes 1)\mathcal{L}
\end{aligned}$$

□

We conclude with a curious relationship which we do not understand, but may be important.

Proposition 3.2. *Suppose ω is a non-degenerate symplectic structure and $w \in \mathrm{Der}(A)$. Then*

$$\alpha^{-1}(d_1[(1 \otimes w)\mathcal{L}]) = 1 \otimes w.$$

Proof - First, note $(d_1[(1 \otimes w)\mathcal{L}])(1 \otimes u) = 0$ and $(d_1[(1 \otimes w)\mathcal{L}])(u \otimes 1) = (u \otimes w)\mathcal{L}$. Also, $\alpha(1 \otimes w)(1 \otimes u) = \omega(1 \otimes u, 1 \otimes w) = 0$ and $\alpha(1 \otimes w)(u \otimes 1) = \omega(u \otimes 1, w \otimes 1) = (u \otimes w)\mathcal{L}$.

Hence, $d_1[(1 \otimes w)\mathcal{L}]$ and $\alpha(1 \otimes w)$ agree on things of the form $(1 \otimes u)$ and $(u \otimes 1)$ and hence

$$d_1[(1 \otimes w)\mathcal{L}] = \alpha(1 \otimes w).$$

Since α is an isomorphism,

$$\alpha^{-1}(d_1[(1 \otimes w)\mathcal{L}]) = 1 \otimes w.$$

□

3.4 Examples

3.4.1 Particle in a 1-Dimensional Potential

Suppose 2 is a unit in a commutative ring k and the algebra $A = k[x]$, so $A \otimes A \cong k[q_1, q_2]$ and $H \cong k[q_0, \ldots, q_T]$, the polynomials in $T + 1$ variables over k. We then consider the Lagrangian $\mathcal{L}_i$ for a particle in a polynomial potential V as a function of consecutive positions q_i and q_{i+1} of the particle,

$$\mathcal{L}_i = \mathcal{L}(q_i, q_{i+1}) = \frac{1}{2}m\dot{q}_i^2 - V(q_i),$$

where we define $\dot{q}_i = q_{i+1} - q_i$ and where m is a ring element representing the mass of the particle.

Applying the Euler-Lagrange equation yields the equation of motion

$$m(\dot{q}_i - \dot{q}_{i-1}) = -V'(q_i).$$

This is the discrete analog of Newton's equation of motion for a particle moving in a potential, $m\ddot{q} = -V'(q)$.

Consider the case when $V = 0$, that is, the free particle. The Lagrangian $\mathcal{L} = \frac{1}{2}m\dot{q}_1^2 = \frac{1}{2}m(q_2-q_1)^2$ is translation-invariant, that is, it has the derivation

$v = \partial_x$ as an infinitesimal symmetry:

$$(v \otimes 1 + 1 \otimes v) = \partial_{q_1} + \partial_{q_2}$$

and

$$(\partial_{q_1} + \partial_{q_2})\mathcal{L} = 0.$$

The corresponding conserved quantity is momentum:

$$F_v = -(v \otimes 1)\mathcal{L} = -\partial_{q_1}\frac{1}{2}m(q_2 - q_1)^2 = m\dot{q}_1.$$

To compute the symplectic structure ω, consider $\mathcal{L} \in A_1 \otimes A_2 \cong k[q_1, q_2]$. Then

$$\begin{aligned} d_1\mathcal{L} &= \partial_{q_1}\mathcal{L}(q_1, q_2)\ dq_1 \\ d_2\mathcal{L} &= \partial_{q_2}\mathcal{L}(q_1, q_2)\ dq_2. \end{aligned}$$

Hence

$$\omega = \partial_{q_1}\partial_{q_2}\mathcal{L}(q_1, q_2)\ dq_1 \wedge dq_2.$$

Recalling that the discrete Lagrangian of a particle moving in a potential is given by $\frac{1}{2}m(q_2 - q_1)^2 - V(q_1)$ and assuming 2 is a unit in k, we see that:

$$\omega = \partial_{q_1}\partial_{q_2}\left[\frac{1}{2}m(q_2 - q_1)^2 - V(q_1)\right]\ dq_1 \wedge dq_2 = m\ dq_2 \wedge dq_1.$$

If we define the "momentum" p_1 to be $m\dot{q}_1$, we have the familiar formula $\omega = dp_1 \wedge dq_1$.

Using the time evolution defined in the previous chapter and assuming m is a unit, we have

$$\Phi(q_1) = q_2, \qquad \Phi(q_2) = 2q_2 - q_1 - m^{-1}V'(q_2),$$

hence the symplectic structure ω is preserved:

$$\begin{aligned}
\Phi_*\omega &= \Phi_*(m\ dq_2 \wedge dq_1) \\
&= m(2dq_2 - dq_1 - m^{-1}V''(q_2)\ dq_2) \wedge dq_2 \\
&= \omega.
\end{aligned}$$

Example. 3.1 Suppose we take our base ring to be the integers and let $V(q_i) = kq_i^2$, $k \geq 0$, the discrete analog of the harmonic oscillator. We consider the most general quadratic Lagrangian in q_i, and $\dot{q}_i$,

$$\mathcal{L}(q_i, \dot{q}_i) = a\dot{q}_i^2 + bq_i\dot{q}_i + cq_i^2$$

where a, b, and c are integers, $a > 0$. Using $\dot{q}_i = q_{i+1} - q_i$, we can rewrite this Lagrangian as

$$\begin{aligned}
\mathcal{L}(q_i, q_{i+1}) &= a(q_{i+1} - q_i)^2 + bq_i(q_{i+1} - q_i) + cq_i^2 \\
&= aq_{i+1}^2 - 2aq_iq_{i+1} + aq_i^2 + bq_iq_{i+1} - bq_i^2 + cq_i^2 \\
&= aq_{i+1}^2 + (b - 2a)q_iq_{i+1} + (a - b + c)q_i^2.
\end{aligned}$$

Applying the discrete Euler-Lagrange equation $\dfrac{\partial\mathcal{L}(q_{i-1}, q_i)}{\partial q_i} + \dfrac{\partial\mathcal{L}(q_i, q_{i+1})}{\partial q_i} = 0$ yields

$$\begin{aligned}
0 &= 2aq_i + (b - 2a)q_{i-1} + (b - 2a)q_{i+1} + 2(a - b + c)q_i \\
&= (b - 2a)q_{i+1} + (4a - 2b + 2c)q_i + (b - 2a)q_{i-1} \\
&= (b - 2a)\left[q_{i+1} - \left(2 + \frac{2c}{2a - b}\right)q_i + q_{i-1}\right]
\end{aligned}$$

So we see

$$q_{i+1} = \left(2 + \frac{2c}{2a - b}q_i\right) - q_{i-1}.$$

This time evolution equation will have integral coefficients and hence solutions will stay in the integers provided $\dfrac{2c}{2a-b}$ is an integer. Solving the time evolution equation as a recursion relation, we see

$$q_n = \alpha(R + \sqrt{R^2-1})^n + \beta(R - \sqrt{R^2-1})^n,$$

where

$$\alpha = -\frac{(R-\sqrt{R^2-1})q_0 - q_1}{2\sqrt{R^2-1}}, \qquad \beta = \frac{(R+\sqrt{R^2-1})q_0 - q_1}{2\sqrt{R^2-1}},$$

and $R = 1 + \dfrac{c}{2a-b}$ is a half-integer, and provided that $\frac{c}{2a-b} \neq 0, -2$ so the roots of the characteristic polynomial for the recursion relation are distinct. Note that

$$(R + \sqrt{R^2-1})(R - \sqrt{R^2-1}) = 1,$$

so $|R + \sqrt{R^2-1}| = \frac{1}{|R-\sqrt{R^2-1}|}$.

We now determine if any bounded solutions in the integers exist. Consider the following cases:

Case 1) – If α or β is zero, the only integral evolutions will occur when $\frac{c}{2a-b} = 0$ or -2. But the roots of the characteristic polynomial for this recursion relation would not be distinct in these cases, so a different recursion relation, described below, must be used.

Case 2) – If $\alpha, \beta \neq 0$ and $\frac{c}{2a-b} \geq \frac{1}{2}$, then $R + \sqrt{R^2-1} > 1$ and hence any solution will be unbounded.

Case 3) – If $\alpha, \beta \neq 0$ and $\frac{c}{2a-b} \leq -\frac{5}{2}$, then $R - \sqrt{R^2-1} < -1$ and hence any solution will be unbounded.

Case 4) – If $\alpha, \beta \neq 0$ and $\frac{c}{2a-b} = -\frac{1}{2}$, then $R + \sqrt{R^2-1} = \frac{1}{2} + \imath\frac{\sqrt{3}}{2}$, which has order 6. Hence, all solutions will be of order 6.

Case 5) – If α, $\beta \neq 0$ and $\frac{c}{2a-b} = -1$, then $R + \sqrt{R^2-1} = \imath$, which has order 4. Hence, all solutions will be of order 4.
Case 6) – If α, $\beta \neq 0$ and $\frac{c}{2a-b} = -\frac{3}{2}$, then $R + \sqrt{R^2-1} = -\frac{1}{2} + \imath\frac{\sqrt{3}}{2}$, which has order 3. Hence, all solutions will be of order 3.

The remaining cases are when $\frac{c}{2a-b} = 0$ or -2. For the case $\frac{c}{2a-b} = 0$, the recursion relation is

$$q_n = (q_1 - q_0)n + q_0$$

which will be unbounded unless $q_1 = q_0$, so the only bounded solutions are constant. For the case $\frac{c}{2a-b} = -2$, the recursion relation is

$$q_n = (-1)^{n+1}(q_1 + q_0)n + (-1)^n q_0$$

which will be unbounded unless $q_1 = -q_0$, and in this case the solutions will have order 2. Thus, we have proved:

Proposition 3.3. *The discrete harmonic oscillator in $\mathbb{Z}$ has bounded solutions of orders 1, 2, 3, 4, or 6 and only these orders.*

Note that each corresponds to the occurrence of a root of unity in the solution of the recursion relation.

Example. 3.2 If we again take our base ring to be the integers and let $V(q) = cq^2 + dq^4$, $d \geq 0$, we have the discrete analog of the anharmonic oscillator. To allow a variety of kinetic terms, we shall consider any Lagrangian of the form:

$$\mathcal{L}(q_i, \dot{q}_i) = a\dot{q}_i^2 + bq_i\dot{q}_i + cq_i^2 + dq_i^4$$

where a, b, c, and d are integers and a, $d > 0$. Using $\dot{q}_i = q_{i+1} - q_i$, we can rewrite this Lagrangian as

$$\mathcal{L}(q_i, q_{i+1}) = a(q_{i+1} - q_i)^2 + bq_i(q_{i+1} - q_i) - cq_i^2 - dq_i^4$$

$$= aq_{i+1}^2 - 2aq_iq_{i+1} + aq_i^2 + bq_iq_{i+1} - cq_i^2 - dq_i^4$$
$$= aq_{i+1}^2 + (b-2a)q_iq_{i+1} + (a-b-c)q_i^2 - dq_i^4.$$

Applying the discrete Euler-Lagrange equations for the integers yields

$$\begin{aligned} 0 &= 2aq_i + (b-2a)q_{i-1} + (b-2a)q_{i+1} + 2(a-b-c)q_i - 4dq_i^3 \\ &= (b-2a)q_{i+1} + (4a-2b-2c)q_i + (b-2a)q_{i-1} - 4dq_i^3 \\ &= (b-2a)\left[q_{i+1} - \left(2 + \frac{2c}{2a-b}\right)q_i + q_{i-1} + \frac{4dq_i^3}{b-2a}\right]. \end{aligned}$$

So we see

$$q_{i+1} = \left(2 + \frac{2c}{2a-b}\right)q_i - q_{i-1} + \frac{4dq_i^3}{2a-b}.$$

This evolution will stay in the integers provided $\frac{2c}{2a-b}$ and $\frac{4d}{2a-b}$ are both integers. The discrete anharmonic oscillator on $\mathbb{Z}$ differs from the discrete harmonic oscillator on $\mathbb{Z}$ in that there are no choice for a, b, c, and d which always gives bounded solutions.

Proposition 3.4. *No Lagrangian of the form*

$$\mathcal{L}(q_i, \dot{q}_i) = a\dot{q}_i^2 + bq_i\dot{q}_i + cq_i^2 + dq_i^4$$

on the integers yields bounded solutions for all choices of initial data.

Proof - We will prove this by cases: Let $m = \frac{4d}{2a-b}$ and $n = \frac{2c}{2a-b}$. Suppose $q_i > q_{i-1}$. Then $q_{i+1} > mq_i^3 + (1+n)q_i$.

Case 1) – Suppose $m > 0$, $n > -1$. Then for any $q_i > 0$, $q_{i+1} > q_i$. So the choice of $q_0 = 0$, $q_1 = 1$, will give an unbounded solution.

Case 2) – Suppose $m > 0$, $n \leq -1$ or $m < 0$, $n > -1$. Then for $q_i > \sqrt{-\frac{n}{m}}$, $q_{i+1} > q_i$. So the choice of $q_0 = 0$, $q_1 > \sqrt{-\frac{n}{m}}$ gives an unbounded solution.

Case 3) – Suppose $m < 0$, $n \leq -1$. Then for any $q_i < 0$, $q_{i+1} > q_i$. So the choice of $q_0 = -2$, $q_1 = -1$, will give an unbounded solution.

Therefore, there is no choice for the coefficients for the above Lagrangian for which all solutions are bounded. $\square$

We note that for certain choices of coefficients, there are bounded solutions. For example, for $\frac{4d}{2a-b} = 1$ and $\frac{2c}{2a-b} = -3$, the initial data $q_0 = 0$, $q_1 = 1$, gives the solution

$$\{0,\ 1,\ 1,\ 0,\ -1,\ -1,\ 0,\ \ldots\}.$$

and we conjecture that the only bounded solutions are those of the following patterns:

$$\{a, \ldots\}$$

$$\{a, -a, \ldots\}$$

$$\{a, -a, 0, \ldots\}$$

$$\{a, 0, -a, 0, \ldots\}$$

$$\{a, a, 0, -a, -a, 0, \ldots\}$$

We note that these solutions are of order 1, 2, 3, 4, or 6, exactly as the bounded solutions to the discrete harmonic oscillator were.

3.4.2 Particle in a Spherically Symmetric Potential

In this example, we work with a n-dimensional configuration space, so that $A = k[x_1, \ldots, x_n]$, and let $\mathcal{L}$ be the analog of the Lagrangian for a particle in a spherically symmetric potential,

$$\mathcal{L}(x_1, \ldots, x_n, y_1, \ldots y_n) = \frac{1}{2} m \sum_{i=1}^{n} (y_i - x_i)^2 - V(x_1^2 + \cdots + x_n^2).$$

where x_i and y_i correspond to the coordinates of the particles at two consecutive times and m is an invertible ring element corresponding to the mass of the particle. If we denote the coordinates at the next time by z_i, the Euler-Lagrange equations are given by

$$\frac{\partial \mathcal{L}(x_1,\ldots,x_n,y_1,\ldots,y_n)}{\partial y_i}+\frac{\partial \mathcal{L}(y_1,\ldots,y_n,z_1,\ldots,z_n)}{\partial y_i}=0$$

Applying these, we see

$$m(z_i-2y_i+x_i)+2y_iV'(y_1^2+\cdots+y_n^2)=0$$

or

$$z_i=2y_i-x_i-2m^{-1}y_iV'(y_1^2+\cdots+y_n^2).$$

If $k=\mathbb{Z}$ this Lagrangian describes the motion of a particle in the lattice $\mathbb{Z}^n$ in discrete time steps. The Lagrangian $\mathcal{L}$ has as infinitesimal symmetries the derivations corresponding to infinitesimal rotations $x_i\partial_{x_j}-x_j\partial_{x_i}$:

$$\begin{aligned}
[x_i\partial_{x_j}-x_j\partial_{x_i}]\mathcal{L} &= [x_i\partial_{x_j}-\frac{1}{2}mx_j\partial_{x_i}]\sum_{i=1}^{n}(y_i-x_i)^2-V(x_1^2+\cdots+x_n^2)\\
&= x_i[-m(y_j-x_j)-2x_jV'(x_1^2+\cdots+x_n^2)]\\
&\quad -x_j[-m(y_i-x_i)-2x_iV'(x_1^2+\cdots+x_n^2)]\\
&= m(x_jy_i-x_iy_j)
\end{aligned}$$

and

$$\begin{aligned}
[y_i\partial_{y_j}-y_j\partial_{y_i}]\mathcal{L} &= \frac{1}{2}m[y_i\partial_{y_j}-y_j\partial_{y_i}]\sum_{i=1}^{n}(y_i-x_i)^2-V(x_1^2+\cdots+x_n^2)\\
&= y_i[m(y_j-x_j)]-y_j[m(y_i-x_i)]\\
&= m(x_iy_j-x_jy_i)
\end{aligned}$$

so

$$[x_i\partial_{x_j}-x_j\partial_{x_i}]\mathcal{L}+[y_i\partial_{y_j}-y_j\partial_{y_i}]\mathcal{L}=0.$$

The conserved quantities corresponding to these infinitesimal symmetries are given by

$$\begin{aligned}
F_{ij} &= -(x_i\partial_{x_j} - x_j\partial_{x_i})\mathcal{L} \\
&= -(x_i\partial_{x_j} - x_j\partial_{x_i})\frac{1}{2}m\sum_{i=1}^{n}(y_i - x_i)^2 - V(x_1^2 + \cdots + x_n^2) \\
&= -x_i[m(y_j - x_j) - 2x_jV'(x_1^2 + \cdots + x_n^2)] \\
&\qquad +x_j[m(y_i - x_i) - 2x_iV'(x_1^2 + \cdots + x_n^2)] \\
&= -mx_j(y_i - x_i) + mx_i(y_j - x_j) \\
&= m(x_iy_j - x_jy_i).
\end{aligned}$$

These quantities correspond to the components of the angular momentum. Hence, angular momentum is conserved, even though the lattice does not have $SO(n)$ symmetry.

The symplectic structure corresponding to this Lagrangian is given by

$$\begin{aligned}
\omega &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_i}\mathcal{L}dx_i \wedge dy_i \\
&= \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}\frac{1}{2}m\sum_{i=1}^{n}(y_i - x_i)^2 - V(x_1^2 + \cdots + x_n^2)dx_i \wedge dy_j \\
&= \frac{\partial}{\partial x_i}\delta_{ij}\sum_{i=1}^{n}m(y_i - x_i)dx_i \wedge dy_j \\
&= \sum_{i=1}^{n} -m dx_i \wedge dy_i,
\end{aligned}$$

which, as before, corresponds to the standard symplectic structure on $\mathbb{R}^n$, $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$.

3.4.3 Automata Corresponding to Discrete Wave Equations

Now we consider a cellular automaton based on a $1+1$ dimensional discrete wave equation. Here the configuration space is the space k^N which corresponds to the values of a field φ at each of the N sites of the automaton. The Lagrangian for this system in the continuous case is given by

$$\frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2 - V(\varphi),$$

where V is a polynomial. Discretizing this Lagrangian gives

$\mathcal{L}(\varphi(x,t),\varphi(x,t+1)) =$

$$\sum_x (\varphi(x,t+1)-\varphi(x,t))^2 - (\varphi(x+1,t)-\varphi(x,t))^2 - V(\varphi(x,t)).$$

The Euler-Lagrange equations thus take the form

$$\frac{\partial\mathcal{L}(\varphi(x,t-1),\varphi(x,t))}{\partial\varphi(x,t)} + \frac{\partial\mathcal{L}(\varphi(x,t),\varphi(x,t+1))}{\partial\varphi(x,t)} = 0.$$

Now

$$\frac{\partial\mathcal{L}(\varphi(x,t-1),\varphi(x,t))}{\partial\varphi(x,t)} = \varphi(x,t)-\varphi(x,t-1)$$

and

$\dfrac{\partial\mathcal{L}(\varphi(x,t),\varphi(x,t+1))}{\partial\varphi(x,t)} =$

$$\varphi(x+1,t)-\varphi(x,t+1)-\varphi(x,t)+\varphi(x-1,t)-V'(\varphi(x,t)),$$

so the Euler-Lagrange equations yield

$$-\varphi(x,t-1)+\varphi(x+1,t)-\varphi(x,t+1)+\varphi(x-1,t)-V'(\varphi(x,t)) = 0.$$

or

$$\varphi(x,t+1)-\varphi(x+1,t)-\varphi(x-1,t)+\varphi(x,t-1) = -V'(\varphi(x,t)),$$

which is the discrete analog of the nonlinear wave equation $\Box\varphi = -V'(\varphi)$.

The symplectic structure corresponding to this Lagrangian takes the form

$$\begin{aligned}
\omega &= \frac{\partial}{\partial\varphi(x,t)}\frac{\partial}{\partial\varphi(x,t+1)}\mathcal{L}d[\varphi(x,t)] \wedge d[\varphi(x,t+1)] \\
&= \frac{\partial^2}{\partial\varphi(x,t)\partial\varphi(x,t+1)}\sum_x(\varphi(x,t+1)-\varphi(x,t))^2 \\
&= \qquad -(\varphi(x+1,t)-\varphi(x,t))^2 - V(\varphi(x,t))d[\varphi(x,t)] \wedge d[\varphi(x,t+1)] \\
&= \frac{\partial}{\partial\varphi(x,t)}\sum_x \varphi(x,t)d[\varphi(x,t)] \wedge d[\varphi(x,t+1)] \\
&= \sum_x d[\varphi(x,t)] \wedge d[\varphi(x,t+1)].
\end{aligned}$$

Expressing ω in terms of $\varphi(x,t)$ and $\dot\varphi(x,t)$, we see

$$\omega = \sum_x d[\varphi(x,t)] \wedge d[\dot\varphi(x,t)].$$

Chapter 4

The Continuum Limit

We have developed a full Lagrangian formalism for discrete mechanical systems which contains discrete machinery analogous to the important tools from classical mechanics. In this chapter, we show that as the time step decreases to zero, the solutions to a discrete mechanical system will converge to the solutions of the analogous classical mechanical system. In particular, we show that for a particle in a polynomial potential on a submanifold of $\mathbb{R}^n$, the solution to the discrete mechanical system converges to the solution of the classical problem.

4.1 First Order Equations

Let $x: \mathbb{R} \to \mathbb{R}^n$ and consider the first order differential equation

$$x'(t) = f(x(t)).$$

Given a discrete step size ϵ, we define the discrete analog of $x'(t)$, which we denote here as $\dot{x}(t)$ by

$$\dot{x}(t) = \frac{x(t+\epsilon) - x(t)}{\epsilon}.$$

Hence, the discrete analog of this first order equation is just the corresponding difference equation. In the previous chapters, we only considered a step size of 1, so $\dot{x}(t) = x(t+1) - x(t)$. If we allow ϵ to tend to zero, a solution to the discrete equation $\dot{x}(t) = f(x(t))$ will converge to the solution of the continuous equation $x'(t) = f(x(t))$ with the same initial data.

Theorem 4.1. *Let U be a submanifold of $\mathbb{R}^n$ and $f: U \to \mathbb{R}^n$ be $\mathcal{C}^1$. Let $A(t)$ be a solution to the discrete equation $\dot{x}(t) = f(x(t))$ and $C(t)$ be a solution to the continuous equation $x'(t) = f(x(t))$ on some time interval $[a,b]$, each staying inside U. Take $\epsilon = \dfrac{b-a}{n}$. If $A(t)$ and $C(t)$ are given the same initial data, i.e. if $A(a) = C(a)$, then as the number n of time steps increases, $\|A - C\| \to 0$. Hence, the solution to the discrete equation $\dot{x} = f(x(t))$ converges uniformly to the corresponding solution to the continuous equation $x'(t) = f(x(t))$.*

We begin with some lemmas necessary for the proof. First, we state a useful result which will be used frequently.

Lemma 4.1. *Let U be a submanifold of $\mathbb{R}^n$. Suppose $f: U \to \mathbb{R}^n$ is a $\mathcal{C}^1$. In addition, assume $x(t)$ is a solution to either $x'(t) = f(x(t)$ or $\dot{x} = f(x(t)$, and $x(t)$ stays in U for all $a \leq t \leq b$. Then for any compact set $S \subset U$, there exists a compact set $T \subset U$ such that if $x(a) \in S$, then $x(t) \in T$ for all $a \leq t \leq b$. Moreover, the set T is independent of the size of the time step ϵ.*

Next, we bound the difference between two solutions to the discrete equation $\dot{x} = f(x(t))$.

Lemma 4.2. *Suppose $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$. If $A(t)$ and $B(t)$ are solutions to the discrete equation $\dot{x}(t) = f(x(t))$ on $[a,b]$ staying in U, with $\|A(a) - B(a)\| \leq \delta$, then $\|A(a+\epsilon) - B(a+\epsilon)\| \leq \delta(1+K\epsilon)$, where K is a constant.*

Proof - If $A(t)$ and $B(t)$ are solutions to the given discrete equation, then

$$\begin{aligned} A(a+\epsilon) &= A(a) + \epsilon A'(a) \\ B(a+\epsilon) &= B(a) + \epsilon B'(a). \end{aligned}$$

Note, $A(a)$ and $B(a)$ are contained in a compact set V. Since f is $\mathcal{C}^1$ and the solutions stay in U, $\|f\|$ and $\|f'\|$ attain maximum values on V. Thus, in particular, there exists a constant K, depending on V, for which $\|f'\| \leq K$. Moreover, by Lemma 4.1, K is independent of the time step ϵ. Now,

$$\begin{aligned} \|B(a+\epsilon) - A(a+\epsilon)\| &= \Big\|\{B(a) + \epsilon B'(a)\} - \{A(a) + \epsilon A'(a)\}\Big\| \\ &= \Big\|\{B(a) - A(a)\} + \epsilon\{B'(a) - A'(a)\}\Big\| \\ &= \Big\|\delta + \epsilon\{B'(a) - A'(a)\}\Big\| \\ &= \Big\|\delta + \epsilon[f(B(a)) - f(A(a))]\Big\| \\ &= \Big\|\delta + \epsilon[f'(c)(B(a) - A(a))] \\ &\leq \delta + \epsilon\Big\|f'(c)(B(a) - A(a))\Big\| \\ &= \delta + \epsilon\|f'(c)\|\delta \\ &\leq \delta + \epsilon K\delta \\ &= \delta(1+K\epsilon) \end{aligned}$$

□

Now, we bound the difference between a solution to the discrete equation $\dot{x} = f(x(t))$ and a solution to the continuous equation $x'(t) = f(x(t))$ which have the same initial data.

Lemma 4.3. *Suppose $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$. Let $B(t)$ represent the solution to the discrete equation $\dot{x} = f(x(t))$ and $C(t)$ represent the corresponding solution to the continuous equation $x'(t) = f(x(t))$ with the same initial data, i.e. $B(a) = C(a)$. Then $\|C(a+\epsilon) - B(a+\epsilon)\| \leq M\epsilon^2$, where M is a constant.*

Proof - First note, the values of a solution on the compact set $[a, a+\epsilon]$ live in a compact set V and hence f and its derivative attain maximum values on V. Since $x'(t) = f(x(t))$, $\|x'\|$ therefore has a maximum value. Also, $\|x\|$ attains a maximum value on $[a, a+\epsilon]$. Moreover, by Lemma 4.1, these values do not depend on the step size ϵ of $B(t)$. Now, if $B(t)$ is a solution to the discrete equation, then

$$B(a+\epsilon) = x(a) + \epsilon x'(a).$$

On the other hand, if $C(t)$ is a solution to the continuous equation, then

$$C(a+\epsilon) = x(a) + \int_a^{a+\epsilon} x'(t)dt.$$

Hence

$$\begin{aligned}
\|C(a+\epsilon) - B(a+\epsilon)\| &= \left\|\left(x(a) + \int_a^{a+\epsilon} x'(t)dt\right) - (x(a) + \epsilon x'(a))\right\| \\
&= \left\|(x(a) - x(a)) + \int_a^{a+\epsilon} x'(t) - \epsilon x'(a)\right\| \\
&= \left\|\int_a^{a+\epsilon} x'(t)dt - \epsilon x'(a)\right\| \\
&= \left\|\int_a^{a+\epsilon} [x'(t) - x'(a)]dt\right\| \\
&= \left\|\int_a^{a+\epsilon} [f(x(t)) - f(x(a))]dt\right\| \\
&\leq \int_a^{a+\epsilon} \|f(x(t)) - f(x(a))\|dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_a^{a+\epsilon} K\|x(t)-x(a)\|dt \\
&\leq KK' \int_a^{a+\epsilon} \|t-a\|dt \\
&\leq KK' \int_a^{a+\epsilon} \epsilon dt \\
&= KK'\epsilon^2 \\
&= M\epsilon^2
\end{aligned}
$$

□

Using these, we can get a bound on the difference between any solution to the continuous equation $x'(t) = f(x(t))$ and any solution to the discrete equation $\dot{x}(t) = f(x(t))$ after evolving one time step.

Lemma 4.4. *Suppose $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$. Let $A(t)$ be a solution to $\dot{x}(t) = f(x(t))$ and $C(t)$ be a solution of $x'(t) = f(x(t))$. Suppose both solutions start at $t = a$ and stay in U, Then $\|A(a+\epsilon) - C(a+\epsilon)\| \leq \delta(1 + M\epsilon) + M^2\epsilon^2$, where $\delta = \|A(a) - C(a)\|$.*

Proof - First,

$$\|A(t) - C(t)\| \leq \|A(t) - B(t)\| + \|B(t) - C(t)\|$$

where $B(t)$ is the solution to $\dot{x}(t) = f(x(t))$ corresponding to the same initial data as $C(t)$. Then, using the above lemmas,

$$\|A(a+\epsilon) - C(a+\epsilon)\| \leq \delta(1 + K\epsilon) + KK'\epsilon^2$$

Setting $M = \max(K, K')$, we see

$$\|A(a+\epsilon) - C(a+\epsilon)\| \leq \delta(1 + M\epsilon) + M^2\epsilon^2.$$

$\square$

Now, if we evolve over n time steps of size ϵ, we can get a bound on the difference between a solution of $x'(t) = f(x(t))$ and a solution of $\dot{x}(t) = f(x(t))$ on the interval $[a, b]$.

Lemma 4.5. *Suppose $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$. Let $A(t)$ and $C(t)$ be solutions to $\dot{x}(t) = f(x(t))$ and $x'(t) = f(x(t))$ respectively with $\|A(a) - C(a)\| = \delta_0$ and stay in U. Set*

$$\delta_n = \delta_{n-1}(1 + M\epsilon) + M^2\epsilon^2.$$

Then $\|A(a + n\epsilon) - C(a + n\epsilon)\| < \delta_n$, where

$$\delta_n = \delta_0(1 + M\epsilon)^n + M\epsilon[(1 + M\epsilon)^n - 1]$$

Proof - By induction, we have

$$\begin{aligned}
\delta_n &= \delta_{n-1}(1 + M\epsilon) + M^2\epsilon^2 \\
&= [\delta_0(1 + M\epsilon)^{n-1} + M\epsilon[(1 + M\epsilon)^{n-1} - 1]](1 + M\epsilon) + M^2\epsilon^2 \\
&= \delta_0(1 + M\epsilon)^n + M\epsilon[(1 + M\epsilon)^{n-1} - 1]](1 + M\epsilon) + M^2\epsilon^2 \\
&= \delta_0(1 + M\epsilon)^n + M\epsilon[(1 + M\epsilon)^n - (1 + M\epsilon)] + M^2\epsilon^2 \\
&= \delta_0(1 + M\epsilon)^n + M\epsilon[(1 + M\epsilon)^n - 1]
\end{aligned}$$

$\square$

We now prove the theorem.

Proof of Theorem 4.1 - From above,

$$\|A(b) - C(b)\| = \|A(a + n\epsilon) - C(a + n\epsilon)\| \leq \delta_n$$

where

$$\delta_0(1 + M\epsilon)^n + M\epsilon[(1 + M\epsilon)^n - 1],$$

M is constant, and $\delta_0 = \|A(a) - C(a)\|$. From the hypotheses, we have

$$\delta_n = \frac{M(b-a)}{n}\left[\left(1 + \frac{M(b-a)}{n}\right)^n - 1\right].$$

As n tends to infinity, $\left(1 + \frac{M(b-a)}{n}\right)^n$ tends to $e^{M(b-a)}$. Hence δ_n tends to zero and $A(t)$ converges uniformly to $C(t)$.

$\square$

4.2 Second Order Equations

Given a smooth function f and the second order equation

$$x''(t) = f(x(t), x'(t)),$$

we can rewrite it as a system of two first order equations

$$\begin{aligned} y(t) &= x'(t) \\ y'(t) &= f(x(t), x'(t)). \end{aligned}$$

In order to do the same for the discrete case, we need an analogous way to change the second order discrete equation

$$\ddot{x}_t = f(x_t, \dot{x}_t),$$

where

$$\ddot{x}_{n\epsilon} = \frac{x_{(n+1)\epsilon} - 2x_{n\epsilon} + x_{(n-1)\epsilon}}{\epsilon^2}$$

and

$$\dot{x}_{n\epsilon} = \frac{x_{(n+1)\epsilon} - x_{n\epsilon}}{\epsilon},$$

into a system of first order discrete equations.

Consider the system

$$\begin{aligned} y_t &= \dot{x}_t \\ \dot{y}_t &= f(x_t, \dot{x}_t). \end{aligned}$$

Given initial data for the continuous system, we obtain initial data for the discrete system as follows: given $x(0)$ and $x'(0)$, we define $x_0 = x(0)$ and $x_\epsilon = x(0) + \epsilon x'(0)$. Now, when we think of this data as initial data for the discrete system, we have

$$\begin{aligned} y_0 &= \frac{x_\epsilon - x_0}{\epsilon} \\ \dot{y}_0 &= f\left(x_0, \frac{x_\epsilon - x_0}{\epsilon}\right). \end{aligned}$$

Using the definition of x_ϵ, we see

$$\dot{y}_0 = f(x_0, x'(0)).$$

Hence, the discrete first order system has the same initial data as the continuous first order system. Therefore, by Theorem 4.1, as $\epsilon \to 0$, the solution to the discrete equation $\ddot{x}_t = f(x_t, \dot{x}_t)$ converges uniformly to the solution to the continuous equation $x''(t) = f(x(t), x'(t))$.

Applying these results to the discrete system of a particle in a polynomial potential, we have

Theorem 4.2. *Let U be a submanifold of $\mathbb{R}^n$. Suppose $A(t)$ is a solution to the discrete equation $\ddot{x}(t) = -V'(x(t))$ and $C(t)$ is a solution to the continuous equation $x''(t) = -V'(x(t))$, with both on the interval $[a, b]$, and both staying in U. If $A(a) = C(a)$, then, as $\epsilon \to 0$, $A(t)$ converges uniformly to $C(t)$.*

Chapter 5

Discrete Rigid Rotating Body

5.1 Introduction

We now turn our attention to the the case of the the rigid rotating body, which is a system whose configuration space is $SO(N)$. It has been shown [8] that the continuous model of the rigid rotating body on $\mathbb{R}^N$ is completely integrable for all N. Here, instead of starting with the functions on $SO(N,k)$, k a field, we take our algebra to be the functions on $GL(N,k)$, the set of invertible $n \times n$ matrices with entries in k. We remind the reader that this algebra on $GL(N,k)$ can be defined in the following way:

For k a field, consider $k[x_i^j, \det^{-1}]$, $1 \leq i,j \leq N$. We then define

$$A = k[x_i^j, \det^{-1}]/\langle \det \det^{-1} - 1 \rangle$$

($\det^{-1}$ is adjoined since det may not have an inverse in $k[x_i^j]$). We note that A is the algebra of all polynomials in x_i^j and $\det^{-1}$. In what follows we will refer to $GL(N,k)$ and $SO(N,k)$ as $GL(N)$ and $SO(N)$ respectively. In this case, the Euler-Lagrange equations have the form

$$d\mathcal{L}(g,h) + d\mathcal{L}(h,k) = 0$$

where g, h, and k represent the configurations at times i, $i+1$, and $i+2$ respectively. Now, since

$$d = \sum_{i,j=1}^{n} \frac{d}{dh_i^j} dh_i^j$$

and the elements dh_i^j are independent in $\Omega^1(A)$, we know for $f \in A$,

$$df = 0 \iff \frac{d}{dh_i^j} = 0 \qquad \forall\, i,\ j \ = \ 1, \ldots n.$$

Hence, we can rewrite the Euler-Lagrange equations as

$$\frac{d}{dh_i^j}\mathcal{L}(g,h) + \frac{d}{dh_i^j}\mathcal{L}(h,k) = 0,$$

with

$$\frac{d}{dh_i^j}\mathcal{L}(g,h) = \frac{d}{dt}\mathcal{L}(g, h + tE_i^j)|_{t=0},$$

where E_i^j is the elementary matrix with a 1 in the ith row and jth column and 0 elsewhere.

We take the Lagrangian for our system to be

$$\mathcal{L}(g,h) = -\mathrm{tr}(hIg^{-1})$$

where $I \in GL(N)$ represents the moment of inertia for the body. Now, we determine the equations of motion for the system. Noting that

$$\frac{d}{dh_i^j}\mathrm{tr}(kIh^{-1}) = -\mathrm{tr}(kIh^{-1}E_i^j h^{-1}),$$

the Euler-Lagrange equations imply

$$\begin{aligned} \frac{d}{dh_i^j}\mathcal{L}(g,h) + \frac{d}{dh_i^j}\mathcal{L}(h,k) &= -\mathrm{tr}(E_i^j I g^{-1}) + \mathrm{tr}(kIh^{-1}E_i^j h^{-1}) \\ &= \mathrm{tr}(E_i^j(-Ig^{-1} + h^{-1}kIh^{-1})). \end{aligned}$$

But this expression can only vanish for all i, j if

$$k = hIg^{-1}hI^{-1}.$$

Conversely, if we have $k = hIg^{-1}hI^{-1}$, we see

$$\begin{aligned}\frac{d}{dh_i^j}\mathcal{L}(g,h) + \frac{d}{dh_i^j}\mathcal{L}(h,k) &= -\mathrm{tr}(E_i^j I g^{-1}) + \mathrm{tr}(kIh^{-1}E_i^j h^{-1}) \\ &= -\mathrm{tr}(E_i^j I g^{-1}) + \mathrm{tr}(hIg^{-1}hI^{-1}Ih^{-1}E_i^j h^{-1}) \\ &= -\mathrm{tr}(E_i^j I g^{-1}) + \mathrm{tr}(Ig^{-1}E_i^j) \\ &= 0\end{aligned}$$

Since A is finitely generated over k, we know [6] that

$$\mathrm{Der}(A) \cong \Omega^1(A).$$

Since $\Omega^1(A)$ is generated by the elements dx_i^j, we need only consider these elements. Using the above isomorphism, we have a generating set $\partial_{x_i^j}$ for $\mathrm{Der}(A)$. Recall

$$\Omega^1(A \otimes A) \cong \Omega^1(A) \otimes A \oplus A \otimes \Omega^1(A).$$

So we have

$$\mathrm{Der}(A \otimes A) \cong \mathrm{Der}(A) \otimes A \oplus A \otimes \mathrm{Der}(A).$$

This gives us the generators $dg_i^j = d(x_i^j \otimes 1)$ and $dh_k^l = d(1 \otimes x_k^l)$ of $\Omega^1(A \otimes A)$ and the generators $\partial_{g_i^j} = \partial_{(x_i^j \otimes 1)}$ and $\partial_{h_k^l} = \partial_{(1 \otimes x_k^l)}$ of our equivalent of vector fields, $\mathrm{Der}(A \otimes A)$.

We now compute the symplectic structure on $A \otimes A$ defined by the Lagrangian $\mathcal{L}$. Using g_i^j and h_k^l as generators for the functions on $GL(N) \times GL(N)$, we have

$$\begin{aligned}\omega &= dd_2\mathcal{L}(g,h) \\ &= \frac{d}{dg_i^j}\frac{d}{dh_k^l}[-\mathrm{tr}(hIg^{-1})]dg_i^j \wedge dh_k^l\end{aligned}$$

$$
\begin{aligned}
&= -\frac{d^2}{dsdt}\mathrm{tr}((h+te_l^k)I(g+se_j^i)^{-1})dg_i^j \wedge dh_k^l \\
&= \mathrm{tr}(e_l^k I g^{-1} e_j^i g^{-1}) dg_i^j \wedge dh_k^l \\
&= (g^{-1})_l^i I_a^k (g^{-1})_j^a dg_i^j \wedge dh_k^l
\end{aligned}
$$

Instead of taking g_i^j and h_i^j as our generators, we can use the functions $(g^{-1})_i^j$ and h_i^j. We can use $(g^{-1})_i^j$ since it can be written uniquely as a polynomial in the elements g_i^j and $\det^{-1}$. We carry out the above computations using this new generating set and get an equivalent, but simpler-looking symplectic structure:

$$
\begin{aligned}
\omega &= dd_2\mathcal{L}(g,h) \\
&= \frac{d}{(dg^{-1})_i^j}\frac{d}{dh_k^l}[-\mathrm{tr}(hIg^{-1})](dg^{-1})_i^j \ wdh_k^l \\
&= -\frac{d^2}{dsdt}\mathrm{tr}((h+te_k^l)I(g^{-1}+se_i^j))(dg^{-1})_i^j \wedge dh_k^l \\
&= -\mathrm{tr}(e_k^l I e_i^j)(dg^{-1})_i^j \wedge dh_k^l \\
&= -(e_k^l)_a^b I_b^c (e_i^j)_c^a (dg^{-1})_i^j \wedge dh_k^l \\
&= -\delta^{lb}\delta_{ka} I_b^c \delta^{ja}\delta_{ic}(dg^{-1})_i^j \wedge dh_k^l \\
&= -\delta_j^k I_l^i (dg^{-1})_i^j \wedge dh_k^l
\end{aligned}
$$

Recall that the time evolution is given by $k = hIg^{-1}hI^{-1}$. Using this, we can show directly that the symplectic structure is preserved by this time evolution.

$$
\begin{aligned}
\Phi_*\omega &= \Phi_*(-\delta_j^r I_l^i (dg^{-1})_i^j \wedge dh_r^l) \\
&= -\delta_j^r I_l^i (dh^{-1})_i^j \wedge dk_r^l \\
&= -\delta_j^r I_l^i (dh^{-1})_i^j \wedge d(hIg^{-1}hI^{-1})_r^l \\
&= -\delta_j^r I_l^i (dh^{-1})_i^j \wedge d(h_r^m I_m^n (g^{-1})_n^o h_o^p (I^{-1})_p^l)
\end{aligned}
$$

$$
\begin{aligned}
&= -\Big[\delta^r_j I^i_l I^n_m (g^{-1})^o_n h^p_o (I^{-1})^l_p (dh^{-1})^j_i \wedge dh^m_r \\
&+ \delta^r_j I^i_l h^m_r I^n_m h^p_o (I^{-1})^l_p (dh^{-1})^j_i \wedge (dg^{-1})^o_n \\
&+ \delta^r_j I^i_l h^m_r I^n_m (g^{-1})^o_n (I^{-1})^l_p (dh^{-1})^j_i \wedge dh^p_o\Big] \\
&= -\delta^r_j I^i_l h^m_r I^n_m h^p_o (I^{-1})^l_p (-(h^{-1})^a_i dh^b_a (h^{-1})^j_b) \wedge (dg^{-1})^o_n \\
&= -\delta^r_j (I^{-1})^l_p I^i_l I^n_m (h^{-1})^j_b h^m_r h^p_o (h^{-1})^a_i (dg^{-1})^o_n \wedge dh^b_a \\
&= -(I^{-1})^l_p I^i_l I^n_m (h^{-1})^j_b h^m_j h^p_o (h^{-1})^a_i (dg^{-1})^o_n \wedge dh^b_a \\
&= -(I^{-1})^l_p I^i_l I^n_m \delta^m_b h^p_o (h^{-1})^a_i (dg^{-1})^o_n \wedge dh^b_a \\
&= -\delta^i_p I^n_m h^p_o (h^{-1})^a_i (dg^{-1})^o_n \wedge dh^m_a \\
&= -I^n_m h^i_o (h^{-1})^a_i (dg^{-1})^o_n \wedge dh^m_a \\
&= -\delta^a_o I^n_m (dg^{-1})^o_n \wedge dh^m_a \\
&= -\delta^r_j I^i_l (dg^{-1})^j_i \wedge dh^l_r \\
&= \omega.
\end{aligned}
$$

Clearly, Proposition 3.3 saves a lot of time by allowing us to avoid such computations. We can work out the Poisson bracket for this system using the generators $(g^{-1})^j_i$ and h^l_k as follows: Consider the generators of vector fields $\partial_{(g^{-1})^j_i}$ and $\partial_{h^l_k}$. Using the symplectic structure above, we can define the map $\alpha\colon \mathrm{Der}(A \otimes A) \to \Omega(A \otimes A)$ and determine what it does to these generators:

$$
\alpha(\partial_{(g^{-1})^j_i}) = \delta^k_j I^i_l dh^l_k, \qquad \alpha(\partial_{h^j_i}) = -\delta^i_l I^k_j (dg^{-1})^l_k,
$$

so that α^{-1} acts on the corresponding generators of 1-forms $d(g^{-1})^l_k$ and dh^l_k by:

$$
\alpha^{-1}(d(g^{-1})^j_i) = -\delta^m_j (I^{-1})^i_n \partial_{h^n_m}, \qquad \alpha^{-1}(dh^j_i) = \delta^n_i (I^{-1})^j_n \partial_{(g^{-1})^n_m}.
$$

One can also show

$$\begin{aligned}\alpha^{-1}(dg^l_k) &= g^l_m(gI^{-1})^n_k\partial_{h^n_m}\\ \alpha^{-1}(d(h^{-1})^l_k) &= -(h^{-1})^n_k(I^{-1}h^{-1})^l_m\partial_{(g^{-1})^n_m}.\end{aligned}$$

Using these facts, one obtains the following table of Poisson brackets, which allows one to compute all Poisson brackets of algebraic functions on the phase space of the discrete rigid rotating body:

$$\begin{aligned}
\{(g^{-1})^l_k,(g^{-1})^j_i\} &= 0 & \{h^l_k,h^j_i\} &= 0\\
\{g^l_k,g^j_i\} &= 0 & \{(g^{-1})^j_i,h^l_k\} &= -\delta^j_k(I^{-1})^l_i\\
\{(h^{-1})^l_k,(h^{-1})^j_i\} &= 0 & \{((g^{-1})^j_i,(h^{-1})^l_k\} &= (h^{-1})^j_k(I^{-1}h^{-1})^l_i\\
\{(g^{-1})^l_k,g^j_i\} &= 0 & \{h^l_k,g^j_i\} &= -g^j_k(gI^{-1})^l_i\\
\{h^l_k,(h^{-1})^j_i\} &= 0 & \{g^j_i,(h^{-1})^l_k\} &= -(h^{-1}g)^j_k(gI^{-1}h^{-1})^l_i
\end{aligned}$$

By Proposition 2.3, this Poisson bracket turns the algebraic functions on $GL(N)\times GL(N)$ into a Poisson algebra.

It is interesting to seek infinitesimal symmetries of the Lagrangian for the discrete rigid rotating body, since they will yield conserved quantities via Noether's theorem. Note that if $k=\mathbb{R}$ we have

$$\begin{aligned}
\frac{d}{dt}\left[\mathcal{L}(e^{vt}g,h)+\mathcal{L}(g,e^{vt}h)\right]\Big|_{t=0} &= \frac{d}{dt}\left[\mathrm{tr}[hI(e^{vt}g)^{-1}]+\mathrm{tr}[(e^{vt}h)Ig^{-1}]\right]\Big|_{t=0}\\
&= \frac{d}{dt}\left[\mathrm{tr}[hIge^{-vt}]+\mathrm{tr}[(e^{vt}h)Ig^{-1}]\right]\Big|_{t=0}\\
&= -\mathrm{tr}(hIg^{-1}v)+\mathrm{tr}(vhIg^{-1})\\
&= 0
\end{aligned}$$

so any $v\in\mathfrak{gl}(N,\mathbb{R})$ gives an infinitesimal symmetry. In general, we do not have exponentials, but there is an analogous concept. For any $v\in\mathfrak{gl}(N)=$

$\mathfrak{gl}(N,k)$, we can define two derivations L_v and R_v:

$$(L_v f)(g,h) = \frac{d}{dt} f((1+tv)g, (1+tv)h)\Big|_{t=0}$$

and

$$(R_v f)(g,h) = \frac{d}{dt} f(g(1+tv), h(1+tv))\Big|_{t=0}.$$

Now, notice

$$\begin{aligned}
(1 \otimes L_v + L_v \otimes 1)\mathcal{L}(g,h) &= \frac{d}{dt}[\mathcal{L}(g,(1+tv)h) + \mathcal{L}((1+tv)g,h)]\Big|_{t=0} \\
&= \frac{d}{dt}[\mathrm{tr}((1+tv)hIg^{-1}) + \mathrm{tr}(hI((1+tv)g)^{-1})]\Big|_{t=0} \\
&= \mathrm{tr}(vhIg^{-1}) - \mathrm{tr}(hIg^{-1}v) \\
&= 0.
\end{aligned}$$

Therefore, for any $v \in \mathfrak{gl}(N) = \mathfrak{gl}(N,k)$, L_v is an infinitesimal symmetry of the Lagrangian for the discrete rigid rotating body. Using Noether's theorem, the corresponding conserved quantity F_v is given by $-(L_v \otimes 1)\mathcal{L}(g,h) = \mathrm{tr}(hIg^{-1}v)$. Henceforth, we will denote this quantity by C_v.

Proposition 5.1. *The quantities C_v, $v \in \mathfrak{gl}(N)$, are the infinitesimal generators of infinitesimal left translations. That is, for any $f \in A \otimes A$,*

$$\{C_v, f\} = (1 \otimes L_v + L_v \otimes 1)f$$

Proof - Since C_v is the conserved quantity obtained from L_v, Theorem 3.3 yields

$$\alpha^{-1}(dC_v) = 1 \otimes L_v + L_v \otimes 1.$$

Hence

$$\{C_v, f\} = \alpha^{-1}(dC_v)f = (1 \otimes L_v + L_v \otimes 1)f.$$

$\square$

We will now describe this system in a different way. Instead of using consecutive positions as our data, we use initial position g and initial "velocity" $w = g^{-1}h$. Time evolution can now be determined in terms of position and velocity.

Recall the time evolution for discrete rigid rotating body is given by

$$(g, h) \mapsto (g', h') = (h, hIg^{-1}hI^{-1}).$$

This is the same as

$$(g, h) \mapsto (gw, hw'),$$

where $w = g^{-1}h$ and $w' = (g')^{-1}h'$. Hence, to see how the velocity w evolves, we solve

$$hw' = hIg^{-1}hI^{-1}$$

and see

$$w' = IwI^{-1}.$$

Thus, the time evolution in terms of position and velocity is

$$(g, w) \mapsto (gw, IwI^{-1}).$$

We will call the map $(g, h) \mapsto (gw, hIwI^{-1})$ a "twisted right translation" since the amount of translation depends also on the time, i.e. at time t, g gets translated as usual, but h gets translated by the velocity at the next time step. Also, note

$$\mathcal{L} = -\mathrm{tr}(hIg^{-1}) \mapsto -\mathrm{tr}(hIwI^{-1}I(gw)^{-1}) = -\mathrm{tr}(hIww^{-1}g^{-1}) = \mathcal{L},$$

so the Lagrangian is invariant under this map. Our version of Noether's theorem does not apply to this type of symmetry of the Lagrangian, so we cannot use it to calculate the generators of twisted right translations. Nevertheless, we can find quantities acting to generate twisted right translations.

Proposition 5.2. *For $v \in \mathfrak{gl}(n)$, define*

$$D_v = -\mathrm{tr}(g^{-1}hIv).$$

These quantities D_v are the infinitesimal generators of twisted right translations. That is, for any $f \in A \otimes A$,

$$\{D_v, f\} = (R_v \otimes 1 + 1 \otimes R_{IvI^{-1}})f$$

Proof - We show that the Poisson bracket of D_v, with the basic elements $(g^{-1})_i^j$ and h_i^j correspond to twisted right translations by v. First

$$\begin{aligned}
\{D_v, (g^{-1})_i^j\} &= \{-\mathrm{tr}(g^{-1}hIv), (g^{-1})_i^j\} \\
&= -(Ivg^{-1})_k^l \{h_l^k, (g^{-1})_i^j\} \\
&= -(Ivg^{-1})_k^l \delta_l^j (I^{-1})_i^k \\
&= -(vg^{-1})_i^j
\end{aligned}$$

which corresponds to the $(i,\, j)$ entry of the twisted right translate of g^{-1}. Also,

$$\begin{aligned}
\{D_v, h_i^j\} &= \{-\mathrm{tr}(g^{-1}hIv), h_i^j\} \\
&= -(hIv)_l^k \{(g^{-1})_k^l, h_i^j\} \\
&= (hIv)_l^k \delta_i^l (I^{-1})_k^j \\
&= (hIvI^{-1})_i^j
\end{aligned}$$

which corresponds to the $(i,\, j)$ entry of the twisted right translate of h. Thus, the D_v's are the infinitesimal generators of twisted right translations.

□

The D_v's themselves are not conserved quantities, but they do have nice commutation relations with themselves and the C_v's.

Proposition 5.3. *Suppose D_v and C_v are defined as above. Then*

a.) $\{C_v, C_w\} = C_{[v,w]}$

b.) $\{D_v, D_w\} = D_{[v,w]}$

c.) $\{C_v, D_w\} = 0.$

Proof - For a), just apply Proposition 3.3. To see b), by 5.2

$$\begin{aligned}
\{D_v, D_w\} &= (R_v \otimes 1 + 1 \otimes R_{IvI^{-1}})D_w \\
&= (R_v \otimes 1)D_w + (1 \otimes R_{IvI^{-1}})D_w \\
&= -(R_v \otimes 1)\mathrm{tr}(g^{-1}hIw) - (1 \otimes R_{IvI^{-1}})\mathrm{tr}(g^{-1}hIw) \\
&= \mathrm{tr}(vg^{-1}hIw) - \mathrm{tr}(g^{-1}hIvw) \\
&= -\mathrm{tr}(g^{-1}hI[v,w]) \\
&= D_{[v,w]}
\end{aligned}$$

To see c), note

$$-\mathrm{tr}((wg)^{-1}whIv) = -\mathrm{tr}(g^{-1}w^{-1}whIv) = -\mathrm{tr}(g^{-1}hIv),$$

so D_v is invariant under left translations. Since the C_v's generate left translations, we see $\{C_v, D_w\} = 0$.

□

5.2 Relationship to the continuous model

Recall the time evolution for the discrete rigid rotating body in terms of position and "velocity" is given by

$$(g,h) \mapsto (gw, hw')$$

where $w = g^{-1}h$ and $w' = IwI^{-1}$.

Here, we describe the relationship between this discrete model and the continuous model of the rigid rotating body. Let I be a symmetric matrix and take initial data in $SO(N)$. Note, the system will not necessarily stay in $SO(N)$, but it will stay in $SL(N)$. Following [1], we define the "moment in the body"

$$M = g^{-1}hI - [g^{-1}hI]^T$$

Note that M is skew-symmetric. Also,

$$\begin{aligned}
(L_v M)(g,h) &= \frac{d}{dt} M((1+tv)g,(1+tv)h)\Big|_{t=0} \\
&= \frac{d}{dt}[((1+tv)g)^{-1}(1+tv)hI - [((1+tv)g)^{-1}(1+tv)hI]]\Big|_{t=0} \\
&= \frac{d}{dt}[g^{-1}hI - [g^{-1}hI]^T]\Big|_{t=0} \\
&= 0,
\end{aligned}$$

hence M is invariant under left translation. We define our angular velocity $w = g^{-1}h = e^v$ for some $v \in \mathfrak{gl}(N)$. Using this, we can determine our version of the "discrete Euler equation" in terms of the quantity M defined above. Pushing M forward one time step, we get

$$\begin{aligned}
M' &= w'I - I(w')^T \\
&= IwI^{-1}I - I(IwI^{-1})^T \\
&= Iw - w^{-1}I.
\end{aligned}$$

So

$$M' = w^{-1}Mw,$$

and, computing the "time derivative" of M,

$$\dot{M} = \frac{M' - M}{\epsilon} = \frac{w^{-1}Mw - M}{\epsilon}.$$

Approximating e^v linearly, we have $w = 1 + v\epsilon$ up to terms of order ϵ^2. Hence,

$$M = (1 + v\epsilon)I - I(1 - v\epsilon) = \epsilon(vI + Iv)$$

and (up to terms of order ϵ^2)

$$\begin{aligned} \dot{M} &= \frac{M' - M}{\epsilon} \\ &= \frac{(1 - v\epsilon)M(1 + v\epsilon) - M}{\epsilon} \\ &= \frac{\epsilon(-vM + Mv) - \epsilon^2 vMv}{\epsilon} \\ &= Mv - vM - \epsilon vMv. \end{aligned}$$

Letting $\epsilon \to 0$, we see

$$\dot{M} = [M, v]$$

which corresponds to the continuous Euler equation

$$\frac{dM}{dt} = [M, v], \quad M = Iv + vI, \quad v = g^{-1}\dot{g}.$$

Hence, given initial data in $SO(N)$, we have that our discrete system converges in $SO(N)$ and we recover the continuous equations of motion for the rigid rotating body in $SO(N)$. We now show the solution of the discrete rigid rotating body on $\mathbb{R}$ converges to the corresponding solution of the continuous rigid rotating body.

Theorem 5.1. *Consider $SL(N,\mathbb{R})$ as a submanifold of $\mathbb{R}^{N^2}$. Let $A(t)$ be a solution to the discrete rigid rotating body on the interval $[a,b]$, i.e. $A(t)$ is a solution to*

$$\begin{aligned} \dot{g}(t) &= (g(t))^{-1}v(t) \\ \dot{M}(t) &= [M(t),v(t)], \qquad M(t) = Iv(t)+v(t)I. \end{aligned}$$

Let $C(t)$ be a solution to the continuous rigid rotating body on the interval $[a,b]$, i.e. $C(t)$ is a solution to

$$\begin{aligned} g'(t) &= (g(t))^{-1}v(t) \\ M'(t) &= [M(t),v(t)], \qquad M(t) = Iv(t)+v(t)I. \end{aligned}$$

Suppose $A(a) = C(a)$. Then as $\epsilon \to 0$, $A(t)$ converges uniformly to $C(t)$.

Proof - First, the evolution of the discrete rigid rotating body is $A(a+2\epsilon) = A(a+\epsilon)I(A(a))^{-1}A(a+\epsilon)I^{-1}$. Given initial data $A(a)$ and $A(a+\epsilon)$ in $SL(N)$, we see $\det(A(a+2\epsilon)) = 1$. Hence, the discrete evolution stays in $SL(N)$. The solution to the continuous rigid rotating body always stays in $SL(N)$. Thus, both the discrete and continuous solutions stay in the submanifold $SL(N)$ of $\mathbb{R}^{N^2}$. The hypotheses of Theorem 4.1 are satisfied and therefore $A(t)$ converges uniformly to $C(t)$.

$\square$

5.3 Discrete Complete Integrability

Recall there are several conditions sufficient for a continuous system to be completely integrable. We cite [1] for the following result.

Let $F_1, \ldots, F_n$ be n smooth functions in involution on a $2n$-dimensional symplectic manifold M:

$$\{F_i, F_j\} = 0, \quad i, j = 1, \ldots, n.$$

Suppose the functions f_i, are independent on $M_a = \{x \in M | F_j(x) = a_j, j = 1, \ldots, n\}$ and the Hamiltonian fields $v_j = v_{F_j}$, $j = 1, \ldots n$ are complete on M_a. Then every component of M_a is diffeomorphic to $T^k \times \mathbb{R}^{n-k}$. In addition, there exist a choice of coordinates $\varphi_1, \ldots, \varphi_k, y_1, \ldots, y_{n-k}$ on M_a such that Hamilton's equations on M_a take the form

$$\dot{\varphi}_m = \omega_{mj}, \quad \dot{y}_s = c_{sj} \quad (\omega, c = \text{const}).$$

For now, we will assume we have a smooth connected affine algebraic variety over the real numbers, i.e. a subspace of $\mathbb{R}^n$ defined by some number of polynomial equations.

Definition 5.1. *Let A be the algebraic functions over a smooth real affine algebraic variety and let $F_1, \ldots, F_n$ be elements of A. Then $F_1, \ldots, F_n$ are independent provided their differentials $dF_1, \ldots, dF_n$ are linearly independent, that is,*

$$\sum_{i=1}^{n} a_i dF_i = 0 \;\Rightarrow\; a_i = 0 \;\forall i = 1, \ldots n.$$

where the a_i's are elements of A.

Recall that if a Lagrangian on $A \times A$ yields a non-degenerate symplectic structure, this symplectic structure may be used to turn $A \otimes A$ into a Poisson algebra. We now define discrete complete integrability.

Definition 5.2. *Let A be the algebraic functions over a smooth real affine algebraic variety. Suppose A is of dimension n over k. Then a discrete system on $A \times A$ given by a Lagrangian $\mathcal{L} \in A \otimes A$ is completely integrable provided there exists n Poisson-commuting conserved quantities $v_1, \ldots, v_n$ which are independent in the sense of Definition 5.1.*

We point out that in [10], a definition of complete integrability was introduced. This definition does not address the necessity of the independence of the conserved quantities, and, as such, does not generalize the concept of complete integrability in full to the discrete setting.

Verifying that n Poisson commuting conserved quantities are independent is in general very difficult. Here, have a result which makes proving independence easier [3].

Proposition 5.4. *If two functions on a smooth connected affine algebraic variety are independent at a point, then they are independent on an open dense set in the induced topology from $\mathbb{R}^n$.*

What this means is that if we can find any point at which the two functions are independent, then the functions are independent on most of $A \otimes A$. Thus, we have a useful criterion for proving the complete integrability of a discrete mechanical system on a smooth real affine algebraic variety.

Proposition 5.5. *Let A be the set of algebraic functions on a smooth connected algebraic variety of dimension n over the reals. Then a discrete system given by a Lagrangian $\mathcal{L}$ on $A \times A$ is completely integrable if there are n Poisson commuting conserved quantities which are all independent at some point of $A \times A$.*

5.4 Complete Integrability of Rotating Bodies in $SO(N,\mathbb{R})$

We now consider the complete integrability of our discrete system in $SO(N,\mathbb{R})$. For the case $N=3$, consider the generators of infinitesimal symmetries along the "coordinate" axes. Using Noether's theorem, these will yield conserved quantities J_x, J_y, and J_z. Using the result above, we see $\{J_x, J_y\} = J_z$ and similar results for cyclic permutations. Define $J^2 = J_x^2 + J_y^2 + J_z^2$. We now can see

$$\begin{aligned}
\{\mathcal{L}, J_z\} &= \{\mathrm{tr}(hIg^{-1}), \mathrm{tr}(hIg^{-1}e_z)\} = \mathrm{tr}(hIg^{-1}[id, e_z]) = 0 \\
\{J^2, J_z\} &= \{J_x^2 + J_y^2 + J_z^2, J_z\} = \{J_x^2, J_z\} + \{J_y^2, J_z\} + \{J_z^2, J_z\} \\
&= 2J_x\{J_x, J_z\} + 2J_y\{J_y, J_z\} + 2J_z\{J_z, J_z\} \\
&= -2J_xJ_y + 2J_yJ_x = 0 \\
\{\mathcal{L}, J^2\} &= \{\mathcal{L}, J_x^2 + J_y^2 + J_z^2\} = \{\mathcal{L}, J_x^2\} + \{\mathcal{L}, J_y^2\} + \{\mathcal{L}, J_z^2\} \\
&= 2J_x\{\mathcal{L}, J_x\} + 2J_y\{\mathcal{L}, J_y\} + 2J_z\{\mathcal{L}, J_z\} = 0.
\end{aligned}$$

The differentials of these functions vanish at the identity. So there are 3 Poisson-commuting conserved quantities for the case $N=3$, which is sufficient to show the three dimensional discrete rigid rotating body is completely integrable.

For the case $N=4$, denote the standard generators of $so(4)$ by T_i^j, $i<j$. Let J_i be the conserved quantities obtained from the infinitesimal left translations by $T_i^4 + T_j^k$, where ijk is a cyclic permutation of 1, 2, 3, and let K_i be the conserved quantities obtained from the infinitesimal left translations by $T_i^4 - T_j^k$. Let $J^2 = J_1^2 + J_2^2 + J_3^2$ and $K^2 = K_1^2 + K_2^2 + K_3^2$. Note that $[J_i, K_j] = 0$ $i,j = 1,2,3$. Also $[J_1, J_2] = J_3$, $[K_1, K_2] = K_3$, and similarly

for cyclic permutations. Also, consider the Lagrangian $\mathcal{L}$ and the conserved quantity $F_1 = \mathrm{tr}(IM^2)$. Note that F_1 is invariant under infinitesimal left translation. Hence, we have

$$\begin{array}{rclrclrcl}
\{J_1, K_1\} &=& 0 & \{J_1, J^2\} &=& 0 & \{J_1, K^2\} &=& 0 \\
\{J_1, \mathcal{L}\} &=& 0 & \{J_1, F_1\} &=& 0 & \{K_1, J^2\} &=& 0 \\
\{K_1, K^2\} &=& 0 & \{K_1, \mathcal{L}\} &=& 0 & \{K_1, F_1\} &=& 0 \\
\{J^2, K^2\} &=& 0 & \{J^2, \mathcal{L}\} &=& 0 & \{J^2, F_1\} &=& 0 \\
\{K^2, \mathcal{L}\} &=& 0 & \{K^2, F_1\} &=& 0 & \{\mathcal{L}, F_1\} &=& 0.
\end{array}$$

Therefore, there are six Poisson commuting conserved quantities, which is sufficient for the integrability of the discrete rigid rotating body in the $SO(4, \mathbb{R})$ case.

5.5 Discrete WZW Model

We now turn our attention to a system similar to a discrete WZW model. Recall that the continuous WZW model corresponds to the wave equation for a field with values in a Lie group. So where the evolution of the discrete wave equation takes the form

$$\varphi(x, t+1) = \varphi(x+1, t) - \varphi(x, t-1) + \varphi(x-1, t),$$

the evolution our discrete WZW model takes the form

$$\varphi(x, t+1) = \varphi(x+1, t)[\varphi(x, t-1)]^{-1}\varphi(x-1, t).$$

We would like to find such an evolution equation in our formalism starting from a Lagrangian. A surprising result is that we can get an evolution equation very similar to this from the Lagrangian of the rigid rotating body model on $GL(N)$ by thinking about the system in a particular way.

The idea here is to think of the configuration at each site of the model as an element of $GL(N)$. Then we can summarize the state of the entire system as a block diagonal matrix with each block being an element of $GL(N)$. This will give us an element of $GL(NM)$, where M is the number of sites in the model. Now consider the matrix $I \in SO(NM)$ of the following form

$$I = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \\ & & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & & 0 \end{pmatrix}$$

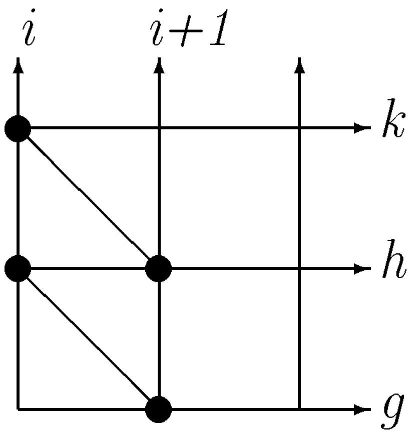
where 0 denotes the $N \times N$ zero matrix and 1 denotes the $N \times N$ identity matrix. Now if we let $g = \mathrm{diag}(g_1, \ldots, g_m)$ and $h = \mathrm{diag}(h_1, \ldots, h_m)$ be the elements in $GL(NM)$ representing the states of the system at consecutive times, we can then use the rigid rotating body model equations of motion to get the matrix which represents the configuration at the next time step. First, note that conjugating a block diagonal matrix $\mathrm{diag}(a_1, a_2, \ldots, a_m)$ by I will yield the block diagonal matrix $\mathrm{diag}(a_2, \ldots, a_m, a_1)$. Next, if g is a block diagonal matrix in $GL(NM)$, then $g^{-1} = \mathrm{diag}(g_1^{-1}, \ldots, g_m^{-1})$. Now, recall the equations of motion for the rigid rotating body model:

$$k = hIg^{-1}hI^{-1}$$

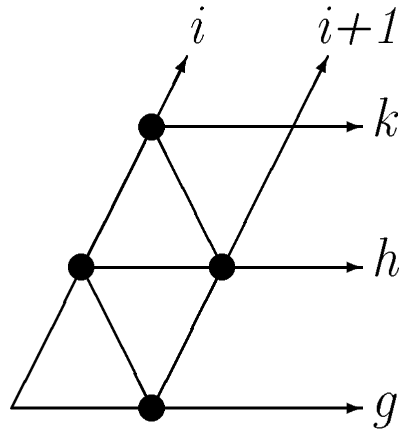
Computing k based on the initial data above, we see

$$k = \mathrm{diag}(h_1 g_2^{-1} h_2, \ldots, h_{m-1} g_m^{-1} h_m, h_m, g_1^{-1} h_1)$$

so $k_i = h_i g_{i+1}^{-1} h_{i+1}$. We now show how this evolution corresponds in some way to the evolution for a WZW model. We show graphically what this evolution looks like in terms of the sites of a WZW model:



Now, if we rotate the time axis clockwise 30°, we get



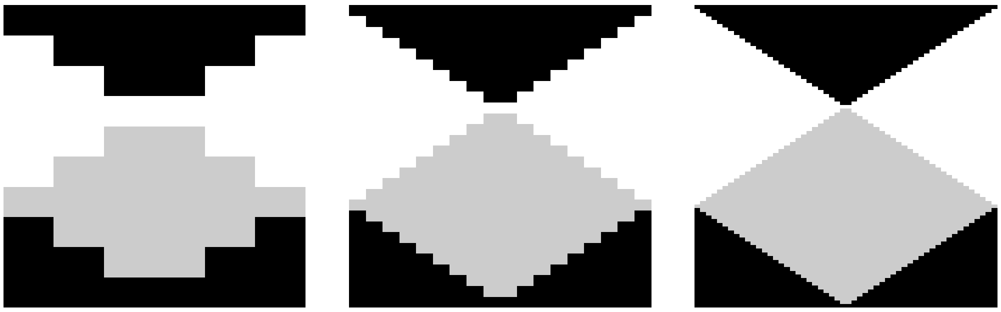
which is the evolution we wanted. So the evolution for this special case of the discrete rigid rotating body model is related to the evolution of a discrete WZW model by a smooth transformation. We now discuss some of the properties of solutions to this discrete WZW model.

First, note if $\varphi(t,x) = f(t+x)g(t-x)$, then

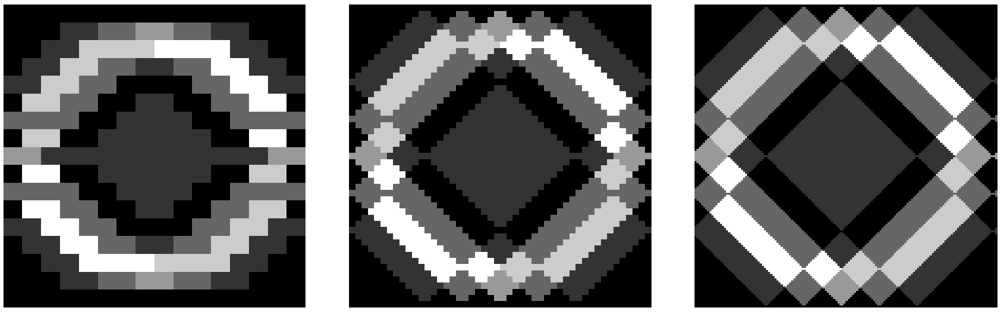
$$\begin{aligned}
\varphi(t+1,x) &= \varphi(t,x+1)[\varphi(t-1,x)]^{-1}\varphi(t,x-1) \\
&= f(t+x+1)g(t-x-1)[f(t+x-1)g(t-x-1)]^{-1}\cdots \\
&\quad \cdots f(t+x-1)g(t-x+1)
\end{aligned}$$

$$
\begin{aligned}
&= f(t+x+1)g(t-x-1)[g(t-x-1)]^{-1}[f(t+x-1)]^{-1}\cdots \\
&\quad \cdots f(t+x-1)g(t-x+1) \\
&= f((t+1)+x)g((t+1)-x).
\end{aligned}
$$

Hence, any function of the form $\varphi(t,x) = f(t+x)g(t-x)$ is a solution to this discrete WZW model. Thus, a solution can be made up of left and right moving waves. We demonstrate this below with an example. Here, we work with sites taking values in $S_3 \cong GL(2,2)$ and identify the end sites, thus working on a cylinder. In the first series below, we see that as we refine the data by adding more sites and allowing the evolution to run accordingly longer, the discrete WZW model "smoothes" out and approaches a continuous model. Also, we can observe the right and left moving waves in the solution interacting in a non-linear manner.



The next series shows the non-linear interactions can be quite complex.



Now, since we can think of this discrete WZW model as a special case of the discrete rigid rotating body, the integrability of the discrete WZW model would be given by the integrability of the discrete rigid rotating body. It thus is worthwhile to consider the integrability of the discrete rigid rotating body in higher dimensions.

Bibliography

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1989.

[2] J. Baez and J. Gilliam, An algebraic approach to discrete mechanics, Lett. Math. Phys. **31** (1994), 205-212.

[3] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[4] T. Itoh and K. Abe, Discrete Lagrange's equations and canonical equations based on the principle of least action, Appl. Math. Comput. **29** (1989), 161-183.

[5] K. Kaneko, Symplectic cellular automata, Phys. Lett. A **129** (1988) 9-16.

[6] S. Lang, *Algebra*, Addison-Wesley Publ. Co., Reading, Mass, 1993.

[7] N. Margolus, Physics and computation, MIT Technical Report MIT/LCS/TR-415.

Physics-like models of computation, Physica **10D** (1984), 81 - 95.

[8] S. V. Manakov, Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body, Funct. Anal. Appl. **10** (1976) 328-329.

[9] H. Matsumura, Commutative algebra, Benjamin, New York, 1970.

[10] J. Moser and A. P. Veselov, Discrete Versions of some Classical Integrable Systems and the Factorization of Matrix Polynomials, Comm. Math. Phys. **139** (1991) 227-243.

[11] D. Mumford, An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related non-linear equations, Intl. Symp. on Algebraic Geometry, Kyoto, 1977, pp. 115-153.

[12] Y. Nambu, Field theory of Galois' fields, in *Quantum Field Theory and Quantum Statistics*, edited by I. A. Batalin, C. J. Isham, and G. A. Vilkovisky, Adam Hilger, Bristol, 1987.

[13] *Integrable Systems*, edited by S. Novikov, London Mathematical Lecture Note Series 60, Cambridge U. P., Cambridge, 1981.

[14] T. Toffoli and N. Margolus, *Cellular Automata Machines: a New Environment for Modeling*, MIT Press, Cambridge, Mass., 1987.

[15] E. Thiram, D. Verstegen and J. Weyers, p-Adic dynamics, Jour. Stat. Phys. **54** 893-913 (1989).

[16] S. Wolfram, personal communication.

[17] A. P. Veselov, What is an integrable mapping?, in *What Is Integrability?*, V. E. Zakharov, ed., Springer-Verlag, New York, 1991, pp. 251-272.