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2-Tangles as a Free Braided Monoidal 2-Category with Duals

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ABSTRACT OF THE DISSERTATION

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The algebraic characterization of tangles by Freyd, Turaev and Yetter has led to the discovery of new invariants for links. In this dissertation, we prove an analogous result one dimension higher: that the 2-category of unframed, unoriented 2-tangles is the free semistrict braided monoidal 2-category with duals on one unframed self dual object. We give appropriate definitions of the 2-category of 2-tangles, and of duality for monoidal and braided monoidal 2-categories. We use the movie moves of Carter, Rieger and Saito, to show that there is a 2-functor from this 2-category to any braided monoidal 2-category with duals containing an unframed self dual object. Knotted surfaces in 4-space are naturally included in this characterization, since they are 2-tangles without boundary.

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1 Introduction

The recent work on quantum invariants of knots and links in 3-dimensional space is based on an algebraic characterization. The work of Freyd, Shum, Turaev and Yetter shows that the category of framed oriented tangles in 3 dimensions is the free braided monoidal category with duals on one object [10, 14, 15, 13]. This characterization makes it possible to get functors from the category of framed oriented tangles to any braided monoidal category with duals, and, in particular, to the categories of representations of quantum groups. Any such functor naturally gives an invariant of framed oriented tangles.

In higher dimensions we look for analogous algebraic structures. The tangle hypothesis proposed by Baez and Dolan [2] gives a characterization of n-dimensional surfaces embedded in $(n+k)$-dimensional space as a k-tuply monoidal n-category for all n and k . An n-category is a generalization of a category for which there are objects, morphisms between objects, 2-morphisms between morphisms, and so on up to n-morphisms. A *k-tuply monoidal $(n+k)$-category* is an $(n+k)$-category whose j-morphisms are trivial for all $j < k$. A k-tuply monoidal $(n+k)$-category can also be described as an n-category with some added conditions. The tangle hypothesis says that the n-category of framed oriented n-tangles in $n+k$ dimensions is the free weak k-tuply monoidal $(n+k)$-category with duals on one object. In the corresponding n-category, objects correspond to collections of points embedded in $[0,1]^k$, and j-morphisms correspond to j-dimensional manifolds in $[0,1]^{k+j}$ with boundary in $[0,1]^k \times \partial([0,1]^j)$ going from one $(j-1)$-morphism to another (in $[0,1]^{k+j} \times \partial[0,1]$)

for $j = 1, 2, \ldots, n-1$. For the highest dimension, n-morphisms correspond to ambient isotopy classes of n-manifolds with similar boundary conditions; we refer to these n-manifolds as n-tangles in $n+k$ dimensions. An n-dimensional knot or link is an n-manifold without boundary, so it is also an n-tangle; it corresponds to an n-morphism with trivial source and target.

The tangle hypothesis, as stated above, involves concepts from topology and n-category theory that are well understood only in certain low-dimensional cases. The version that we prove here treats the case of unframed, unoriented smooth 2-tangles in 4-space, which allows us to use the combinatorial characterization given in the recent work of Carter, Rieger and Saito [6]. There is, as yet, no general definition of precisely what the k-tuply monoidal n-category of n-tangles should be, so we give an explicit construction for the case $n = k = 2$. This construction is consistent with the structures used by Carter, Saito and Rieger, and indeed the basic ideas are suggested in their paper. The corresponding doubly monoidal 2-category is equivalent to a braided monoidal 2-category (a 'braided' n-category being a doubly monoidal $(n+2)$-category).

We use the 2-tangle decompoitions of Carter, Rieger and Saito to give a combinatorial description of the 2-category associated with unframed, unoriented 2-tangles. In particular, as suggested in this work, we give a description wherein the 2-morphism generators correspond to elementary string interactions (ESIs). The work of Fischer [9], which used Carter and Saito's earlier description [7] of 2-tangles to give a 2-category structure for 2-tangles, is flawed by the lack of a height function: Kharlamov and Turaev have shown that if the order of heights of maxima and minima are specified for tangles (representing 1-morphisms), then the corresponding 2-tangles form a 2-category, but without such a condition 2-morphism composition is not well defined [12]. The height function restriction in Carter, Rieger and Saito's more recent description of 2-tangles [6] is somewhat stronger, ordering heights of maxima, minima and crossings of a specified projection. In the following section, we rely on the structure of this height function to show that 2-morphism composition is well defined.

Our characterization of 2-tangles is given in terms of semistrict rather than weak k-tuply monoidal $(n+k)$-categories, since these are better understood for the case $n = k = 2$. A weak n-category is the most general n-category, in the sense that there are relations specified only between n-morphisms. Baez and Dolan give a definition of weak n-category for all n, from which one may define weak k-tuply monoidal $(n+k)$-categories [3]. In practice, however, it can be quite difficult to work with the full generality of weak monoidal n-categories, so formulations have been given of semistrict monoidal and braided monoidal categories for $n \leq 2$. The idea motivating these semistrict categories is to give definitions which are easier to work with, but which are sufficiently general to be useful in most problems. The first definitions of monoidal and braided monoidal 2-categories were given by Kapranov and Voevodsky [11]. More recently Day and Street [8] and Baez and Neuchl [4] have given equivalent definitions of monoidal and braided monoidal 2-categories, which are slightly modified from those given by Kapranov and Voevodsky. In the following we use the formulations given by Baez and Neuchl with the addition of the relations noted in (1) of the Conclusions of that paper.

The study of duality and of universal properties is just beginning, so we must define what we mean by a 2-category 'with duals' and by the 'free' braided monoidal 2-category with duals on one unframed self dual object. Our definition of duality for monoidal and braided monoidal 2-categories is given in section two, and is largely analogous to the appropriate definitions for categories.

In addition to the expected structures, the unframed 2-tangle structure requires that we introduce an unframed self dual object that is associated with a 2-morphism corresponding to the Reidemeister I move. This 2-morphism is somewhat analogous to the 'balancing' morphism $b_A: A \to A$, which corresponds to a twist for framed tangles. The twist, in the 2-dimensional case, is not equal to the identity, instead there is a 2-isomorphism (the Reidemeister I 2-morphism) between it and the identity. The method proposed by Fischer to produce a 2-category description of oriented manifolds (by discarding the ESI that corresponds to the first Reidemeister move and the relations that contain it) has the topological flaw that, while any oriented 2-knot can be described without this generator, two such descriptions of ambient isotopic 2-knots are not necessarily equal in the resulting combinatorial equivalence.

By the 'free' braided monoidal 2-category with duals on one unframed self dual object, we mean that for any braided monoidal 2-category with duals containing an unframed self dual object, there is a particular type of 2-functor mapping the 'free' 2-category into the other 2-category. In section four we define precisely what properties we require the 2-functor to have.

In section two we define the 2-category of 2-tangles. We define duality for monoidal and braided monoidal 2-categories, and show that the 2-category of 2-tangles is a braided monoidal 2-category with duals with an unframed self dual object. In section three we give a combinatorial description for 2-tangles, relying heavily on the work of Carter, Rieger and Saito, and show that the resulting 2-categories are isomorphic. In section four we use this combinatorial structure to show that 2-tangles are generated, as a braided monoidal 2-category with duals, by one unframed self dual object. We give a definition of a strict monoidal 2-functor which preserves braiding and duals strictly on an unframed self dual abject, and show the existence of such 2-functors for the 2-category of 2-tangles. We conclude that 2-tangles form the free braided monoidal 2-category with duals on one unframed self dual object.

2 Topological 2-Tangles

In this section we define a 2-category $\mathcal{T}$ whose objects, morphisms and 2-morphisms are smooth isotopy classes of finite point sets, smooth tangles (with a height function) and 2-tangles. In particular, 2-tangles are smooth 2-manifolds embedded in $D^2 \times D^2$, with boundary in $\partial(D^2) \times D^2$, where the source and target of the associated 2-morphism are represented by opposing tangles in the boundary, with trivial (straight) strands joining them. A morphism is associated with a smooth tangle diagram with an associated height function, and the source and target of the morphism are represented by the point set at the top and bottom of the tangle.

We define $\mathcal{T}$ in terms of smooth manifolds, and show that this forms a braided monoidal 2-category in the sense of Baez and Neuchl [4]. We define duality for monoidal and braided monoidal 2-categories, and define an unframed self dual object. Finally, we show that the 2-category of 2-tangles is a braided monoidal 2-category with duals with one unframed self dual object. Throughout, 2-categories (monoidal, braided, or with duals) will mean semistrict 2-categories (monoidal, braided or with duals)

2.1 The 2-Category $\mathcal{T}$

Objects, 1-morphisms and 2-morphisms should be thought of as lying in a 4-disk: $I_1 \times I_2 \times I_3 \times I_4$, where $I_i = [0, 1]$. In describing the representative sets, tangles and surfaces, we will assume tangles are embedded in a 3-cube with axes x, y, z, and we will often consider a generic projection of a surface into the 3-cube with axes y, z, t

(see fig 1). We will refer to points as being behind or in front if they have greater or smaller x vaules, to the left and right if they have smaller or greater y values, above and below for smaller and greater z values and before and after for smaller and greater t values. Sometimes a 2-morphism will be illustrated by a series of tangles which are cross-sections at successive values of t in the 2-morphism (similar to the movies of a surface used by Carter Rieger and Saito in [6]).

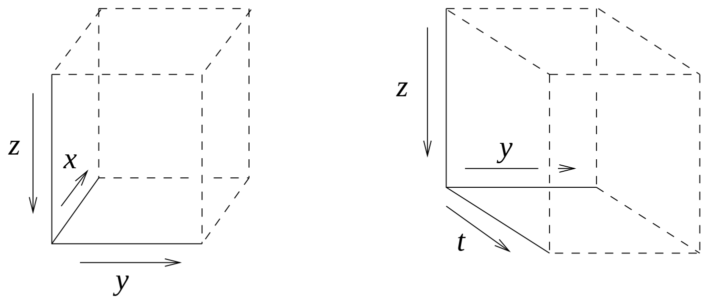


Figure 1: Orientation of illustrations

2.1.1 Objects

Objects of $\mathcal{T}$ are in one-to-one correspondence with whole numbers. We define objects to be equivalence classes of finite point sets lying in the interior of $I_1 \times I_2$ that are ordered by their second coordinate, where we require that no two points in the set have the same second coordinate.

Two such sets are equivalent if there is a level preserving smooth ambient isotopy between them (where level preserving means that it is a fiber isotopy for the bundle $I_1 \times I_2$, with projection $p: I_1 \times I_2 \to I_2$) such that $H(A, 0) = A$ and $H(A, 1) = B$. This defines an equivalence relation under which any two such sets that have the same number of elements are equivalent.

2.1.2 1-Morphisms

The set of 1-morphisms of $\mathcal{T}$ consists of equivalence classes of smooth tangles with specified diagrams, and a height function. We may think of these as diagrams rather than tangles because the equivalence relation preserves the diagram structure (crossings maxima and minima) as well as the order of a height function on the crossings and extrema.

Specifically, we consider tangles lying in $\mathrm{int}(I_1 \times I_2) \times I_3$, with boundary points in $\mathrm{int}(I_1 \times I_2) \times \{0,1\}$, such that the projection to $I_2 \times I_3$ is generic, and crossings, maxima and minima in the projection occur at distinct heights, where the height function is given by z; we further require that the tangles be vertical near the top and bottom; that is, for each tangle T there exists $\epsilon > 0$ such that if $(x,y,z) \in T$ then $(x,y,z') \in T$ if both z and z' are within ϵ of either 0 or 1. The object corresponding to the set $T \cap (I_1 \times I_2 \times \{0\}/\{1\})$ is the source/target of the 1-morphism given by the tangle T.

Two tangles are equivalent if there is a smooth isotopy

$$H(x,y,z,T)\colon I_1 \times I_2 \times I_3 \times [0,1] \to I_1 \times I_2 \times I_3$$

between them such that

1. H is level preserving in the sense that it is a fiber isotopy on the bundle $I_1 \times I_2 \times I_3$, with projection $p\colon I_1 \times I_2 \times I_3 \to I_2 \times I_3$
2. H is also level preserving in the sense that it is a fiber isotopy on the bundle $I_1 \times I_2 \times I_3$, with projection $\pi\colon I_1 \times I_2 \times I_3 \to I_3$
3. H has a product structure in a neighborhood of $I_1 \times I_2 \times \partial I_3$, specifically

$$H(x,y,z,T) = (X_H(x,y,n,T), Y_H(x,y,n,T), z)$$

if z is within ϵ of $n \in \{0,1\}$, for some $\epsilon > 0$.

The level preserving properties of a 1-morphism equivalence are precisely the conditions that specify that tangles that differ by Reidemeister moves or by exchanging heights of extrema or crossings are not equivalent.

2.1.3 2-Morphisms

A 2-morphism in $\mathcal{T}$ is an equivalence class of smooth 2-tangles with generic projections. A smooth 2-manifold, F, embedded in $I_1 \times I_2 \times I_3 \times I_4$ is a 2-tangle with generic projections if:

1. The projections $p(F)$, and $\pi(p(F))$ are generic, for projections $p: I_1 \times I_2 \times I_3 \times I_4 \to I_2 \times I_3 \times I_4$, and $\pi: I_2 \times I_3 \times I_4 \to I_3 \times I_4$.
2. The boundary of F is embedded in $I_1 \times I_2 \times \partial(I_3 \times I_4)$, $F \cap (I_1 \times I_2 \times I_3 \times \partial I_4)$ is a pair of 1-morphism representatives, and $F \cap (I_1 \times I_2 \times \partial I_3 \times I_4)$ consists of finitely many straight lines of the form $(x, y, z) \times I_4$.
3. The surface F has a product structure near the boundary. That is, there is an $\epsilon > 0$ such that, for any point (x, y, z, t) in F if z is within ϵ of z_0, then (x, y, z_0, t) is also in F for $z_0 = 0$ or 1, and if t is within ϵ of t_0 then (x, y, z, t_0) is in F for $t_0 = 0$ or 1.

The tangles $F \cap (I_1 \times I_2 \times I_3 \times \{0\}/\{1\})$ represent the source/target of the 2-morphism (see Fig.2).

Two 2-tangles are equivalent (that is, they represent the same 2-morphism) if there is a smooth ambient isotopy $H(x, y, z, t, T)$ between them, such that

1. The restriction of H to $I_1 \times I_2 \times I_3 \times t_0$ for $t_0 = 0$ or 1 is a 1-morphism equivalence.
2. The restriction of H to $I_1 \times I_2 \times z_0 \times I_4$ for $z_0 = 0$ or 1 is level preserving in the sense that it is a fiber isotopy on the bundle $I_1 \times I_2 \times I_4$ with fibers given by the projection $p': I_1 \times I_2 \times I_4 \to I_2 \times I_4$, and also in the sense that it is

a fiber isotopy on the bundle $I_1 \times I_2 \times I_4$ with fibers given by the projection $\pi' : I_1 \times I_2 \times I_4 \to I_4$.

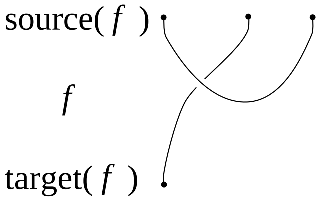
3. H has a product structure near $I_1 \times I_2 \times \partial(I_3 \times I_4)$; specifically, there is an $\epsilon > 0$ such that

$$H(x, y, z, t, T) = (X_H(x, y, z, n, T), Y_H(x, y, z, n, T), Z_H(x, y, z, n, T), t)$$

if t is within ϵ of $n \in \{0, 1\}$; and

$$H(x, y, z, t, T) = (X_H(x, y, z, n, T), Y_H(x, y, z, n, T), z, t_H(x, y, z, n, T))$$

if z is within ϵ of $n \in \{0, 1\}$.



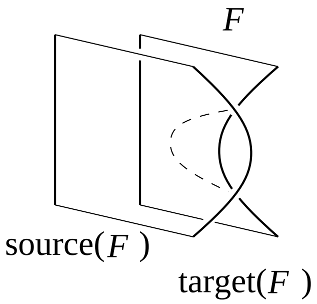


Figure 2: Morphisms and 2-morphisms

2.1.4 Composition of morphisms and 2-morphisms

We may think of composition of morphisms as being a gluing of smooth tangles along their source and target sets. Since we require that tangles be straight near their source and target, the resulting tangle is also smooth. Specifically, let f and $g \in \mathcal{T}$ be morphisms such that $f : A \to B$ and $g : B \to C$. Then the composition $fg : A \to C$ will be the tangle obtained by gluing f to g along B as follows. Choose representatives f' and g' of f and g such that the ending set B' of f' is the beginning set of g', and so that f' has a product structure in z in $I_1 \times I_2 \times [3/8, 1]$ (that is

$f' \cap I_1 \times I_2 \times \{s\} \times \{t\} = B'$ for $s \in [3/8, 1]$; and let g' have a product structure on z in $i_1 \times I_2 \times [0, 5/8]$. Then fg will be the 1-morphism given by the tangle

$$(f' \cap I_1 \times I_2 \times [0, 1/2] \times \{t\}) \cup (g' \cap I_1 \times I_2 \times [1/2, 1] \times \{t\}).$$

The tangles f' and g' agree on $I_1 \times I_2 \times [3/8, 5/8]$, so this gives an appropriate smooth tangle.

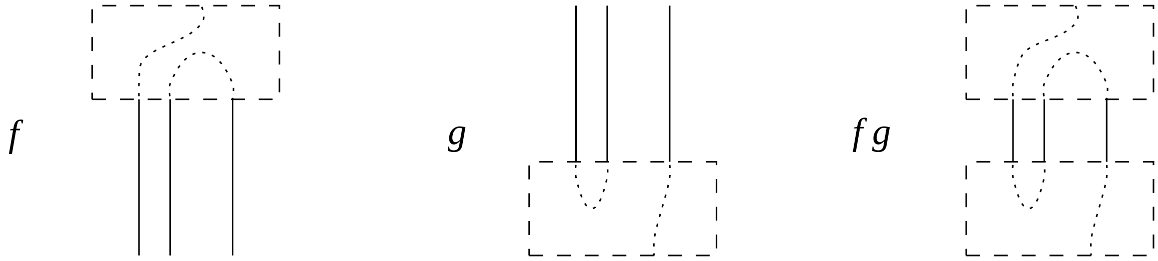


Figure 3: Composition of 1-morphisms

We may think of a horizontal composition of 2-morphisms as being a gluing along the strands at the top and bottom of 2-tangles. Since we require 2-tangles to be flat near the top and bottom, this gluing results in a smooth 2-tangle, and since the top and bottom of a 2-tangle consists of finitely many straight strands, there is a unique way to paste these together. Specifically, given 2-morphisms $F: f \to f'$,and $G: g \to g' \in \mathcal{T}$, such that target$(f)$ = target(f') = source(g) = source(g') = A, we define the horizontal composition $F \circ G$ as follows: we choose representatives F' and G' of F and G so that

$$F \cap (I_1 \times I_2 \times \{1\} \times I_4) = G \cap (I_1 \times I_2 \times \{0\} \times I_4),$$

and F' has a product structure in z in $I_1 \times I_2 \times [3/8, 1] \times I_4$, and G' has a product structure in z in $I_1 \times I_2 \times [0, 5/8] \times I_4$. Then $F \circ G$ is the 2-morphism represented by

$$F' \cap (I_1 \times I_2 \times [0, 1/2] \times I_4)) \cup (G' \cap (I_1 \times I_2 \times [1/2, 1] \times I_4)$$

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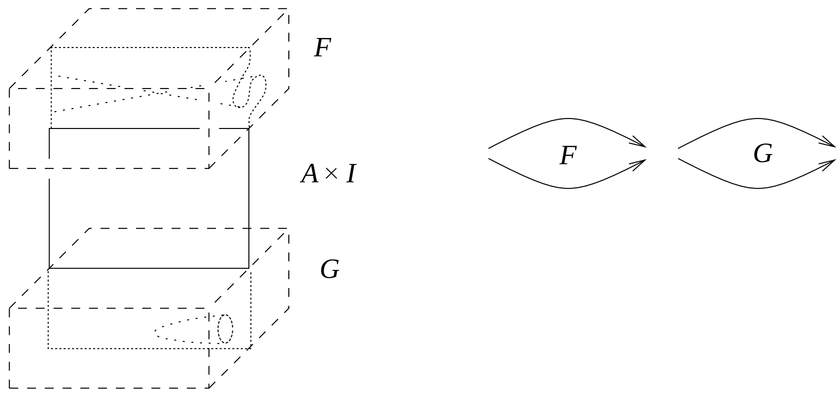

Figure 4: Horizontal composition of 2-morphisms

We may think of a vertical composition of 2-morphisms as being a gluing along the beginning and ending tangles of 2-tangles. Since we require 2-tangles to be flat near source and target, this gluing results in a smooth 2-tangle, and since we do not allow extrema and crossings to exchange heights in tangles, there a unique way to glue these together. Specifically, given 2-morphisms $F\colon f \to g$ and $G\colon g \to h \in \mathcal{T}$ we define the vertical composition $F \cdot G$ as follows: we choose representatives F' and G' of F and G so that

$$F' \cap I_1 \times I_2 \times I_3 \times \{1\} = G' \cap I_1 \times I_2 \times I_3 \times \{0\},$$

and F' has a product structure in t in $I_1 \times I_2 \times I_3 \times [3/8, 1]$, and G' has a product structure in t in $I_1 \times I_2 \times I_3 \times [0, 5/8]$. Then $F \cdot G$ is the 2-morphism represented by

$$F' \cap I_1 \times I_2 \times I_3 \times [0, 1/2]) \cup (G' \cap I_1 \times I_2 \times I_3 \times [1/2, 1].$$

Lemma 1. *Composition of 1-morphisms, and horizontal and vertical composition of 2-morphisms are well defined.*

Proof - Let $f : A \to B$ and $g : B \to C$ be 1-morphisms. To check that fg is well defined we choose two pairs of representatives f', g' and f'', g'' of f and g that satisfy the conditions given in the definition of fg. There exists an equivalence isotopy

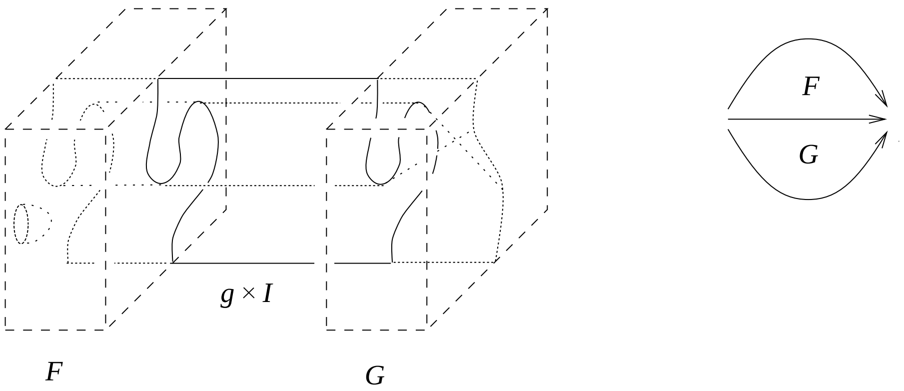

Figure 5: Vertical composition of 2-morphisms

$H(x, y, z, T)$ between f' and f'' that has a product structure in z on $I_1 \times I_2 \times [3/8, 1]$ (that is, $H(x, y, z, T) = (x_T, y_T, z)$ for $z \in [3/8, 1]$ where x_T and y_T do not depend on z). Let $J(x, y, z, T)$ be an equivalence isotopy between g' and g'' that has a product structure in z on $I_1 \times I_2 \times [0, 5/8]$. We paste these in the following manner, let:

$$K(x, y, z, T) = (1 - \phi(z))H(x, y, z, T) + \phi(z)J(x, y, z, T)$$

where $\phi : [0, 1] \to [0, 1]$ is a smooth monotone function such that $\phi([0, 3/8 + \epsilon]) = 0$ and $\phi([5/8 - \epsilon, 1]) = 1$ for some $\epsilon > 0$.

This function $K$is a homotopy between the tangles

$$(f' \cap I_1 \times I_2 \times [0, 1/2]) \cup (g' \cap I_1 \times I_2 \times [1/2, 1])$$

and

$$(f'' \cap I_1 \times I_2 \times [0, 1/2]) \cup (g'' \cap I_1 \times I_2 \times [1/2, 1]).$$

We proceed to show that K is an equivalence isotopy.

The property that K has a product structure in a neighborhood of $I_1 \times I_2 \times \partial(I_3)$ follows trivially from the same properties for H and J. Likewise, it is clear that K is level preserving in both senses on $I_1 \times I_2 \times ([0, 3/8] \cup [5/8, 1])$. The product structure requirements on H and K imply that both H and J preserve the value of the third coordinate on $I_1 \times I_2 \times [3/8, 5/8]$, so K must also preserve the third coordinate value on $I_1 \times I_2 \times [3/8, 5/8]$, hence it is a fiber isotopy on the bundle $I_1 \times I_2 \times I_3$, with projection $\pi : I_1 \times I_2 \times I_3 \to I_3$. Further, any linear combination of level preserving maps (in this case H and J with fixed z) will also be level preserving; and since ϕ depends only on z, and H and J preserve z, $K|_{I_1 \times I_2 \times z_0}$ is a fiber isotopy on the bundle $I_1 \times I_2$ with projection $p' \colon I_1 \times I_2 \to I_2$, hence K is a fiber isotopy on the bundle $I_1 \times I_2 \times I_3$, with projection $p : I_1 \times I_2 \times I_3 \to I_2 \times I_3$.

It is also clear that the homotopy K is smooth, since H, J and ϕ are smooth. The property that K is bijective follows from the product structure on H and J that preserves z on $I_1 \times I_2 \times [3/8, 5/8]$ (every fiber of π maps to itself), and the fiber structure of p gives us monotone increasing, bijective functions on the second coordinate from each of H and J for fixed z and T. When we take a linear combination of such functions (recall that ϕ depends only on z) we again get an bijective function, and similarly, (from the same level preserving condition) we get bijective functions of x for fixed y, z, T.

We can construct similar equivalence isotopies to show that horizontal and vertical composition of 2-morphisms are well defined. Since this is the point where the height function for tangles is used, we will give a fairly complete sketch of the proof the more interesting case of vertical composition of 2-morphisms.

To show that $\alpha \cdot \beta$ is well defined, we choose pairs of representatives α', β' and α'', β'' of α and β that are of the appropriate form to paste together to get representatives of α and β, specifically, α' and α'' should be flat on $t \in [3/8, 1]$, and β and β' should be flat on $t \in [0, 5/8]$. We specify equivalence isotopies H between α' and α'' and J between β' and β'' that have a product structure on $t \in [3/8, 1]$ and $t \in [0, 5/8]$ respectively; specifically, $H(x, y, z, t, T) = (x_{H,T}, y_{H,T}, z_{H,T}, t)$ for $t \in [3/8, 1]$, and $J(x, y, z, t, T) = (x_{J,T}, y_{J,T}, z_{J,T}, t)$ for $t \in [0, 5/8]$. Then we glue H and J to get a homotopy

$$K(x, y, z, t, T)(1 - \phi(t))H(x, y, z, T) + \phi(t)J(x, y, z, T)$$

between the representatives of $\alpha \cdot \beta$. Again, $\phi : [0, 1] \to [0, 1]$ is a smooth monotone function such that $\phi([0, 3/8 + \epsilon]) = 0$ and $\phi([5/8 - \epsilon, 1]) = 1$ for some $\epsilon > 0$.

In showing that K is an equivalence isotopy, the conditions are trivially satisfied outside $3/8 \leq t \leq 5/8$. In the region $3/8 \leq t \leq 5/8$, both H and J fix values of t, so we may consider homotopies K_{t_0} that are the restriction of K to a specific t_0. Any such homotopy K_{t_0} is a linear combination of the isotopies H_{t_0} and J_{t_0}, since ϕ depends only on t. Since we require a product structure on H and J in this region, and $H_{t=1'}$ and $J_{t=0}$ are 1-morphism equivalences, the isotopies H_{t_0} and J_{t_0} are also 1-morphism equivalences, and in particular they are level preserving on z and on y, z (corresponding to the projections $\pi(p)$ and p respectively). Since K_{t_0} is a linear combination, it is also level preserving in both senses, which, together with the condition that H and J are isotopies implies that K_{t_0} is an isotopy, hence K is an isotopy. The fact that the restriction of K to $I_1 \times I_2 \times 0/1 \times I_4$ is level preserving in the appropriate senses follows from the product structure in t, and the level preserving properties of H and J.

To show that $\alpha \circ \beta$ is well defined, we define a similar isotopy, pasting together isotopies of representatives of α and β along $3/8 \leq z \leq 5/8$. The structure and level preserving properties of these isotopies imply that the resulting map is an equivalence isotopy, using a similar argument to those previous. $\square$

Lemma 2. *$\mathcal{T}$ is a 2-category.*

Proof - The isotopies in the above definitions of objects, 1-morphisms and 2-morphisms are equivalence relations, so objects, morphisms and 2-morphisms are well defined. The restriction of a 2-morphism equivalence isotopy to $I_1 \times I_2 \times I_3 \times \{0\}/\{1\}$ gives equivalence isotopies on 1-morphisms, so the source and target maps on 2-morphisms are well defined; likewise the restriction of a 1-morphism equivalence isotopy to $I_1 \times I_2 \times \{0\}/\{1\}$ is an equivalence isotopy for objects, so the source and target maps are well defined on 1-morphisms. By lemma 1 the composition of 1-morphisms and horizontal and vertical composition of 2-morphisms are well defined.

For each object A and 1-morphism f, we define identity morphisms $1_A\colon A \to A$ and $1_f\colon f \to f$ as follows. For every object A there is a 1-morphism $1_A\colon A \to A$ that is given by a tangle with no crossings, maxima or minima; if A' is a representative of A then $A' \times I_3$ is a representative of 1_A. For any 1-morphism f, there is a 2-morphism $1_f\colon f \to f$ represented by $f' \times I_4$, where f' is a representative of f. Equivalence isotopies for representatives of A and f extend by the product structure to equivalence isotopies on 1_A and 1_f, so these identity morphisms are well defined. We write $f\alpha$ for $1_f \circ \alpha$ and αg for $\alpha \circ 1_g$.

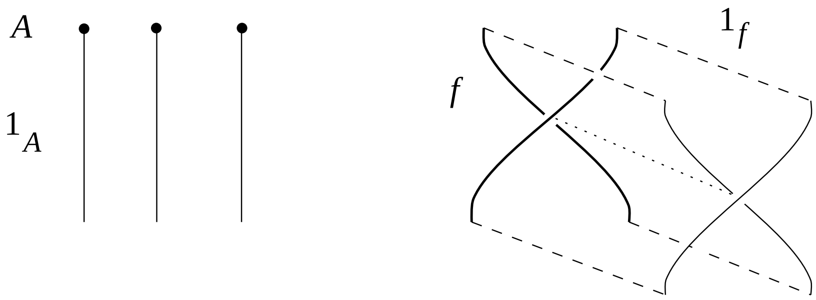

Figure 6: Identity morphisms and 2-morphisms

The identity 1-morphisms satisfy the condition that, for $f\colon A \to B$,

$$1_A f = f 1_B = f.$$

To see this, notice that we can find representatives of f that are straight on specified intervals: $[0, \epsilon]$ or $[\epsilon, 1]$ for any $\epsilon \in (0, 1)$; such a representative that is straight on $[0, 5/8]$ would also clearly be a representative of 1_f, and a representative that is straight on $[3/8, 1]$ is, likewise, clearly also a representative of $f1_B$. Similarly, the identity 2-morphisms satisfy the conditions: $1_f \cdot \alpha = \alpha \cdot 1_g$, and $1_{1_A} \circ \alpha = \alpha \circ 1_{1_B}$, for $\alpha\colon f \to g$ and $f, g\colon A \to B$.

The identity 2-morphisms are also compatible with composition of 1-morphisms and horizontal composition of 2-morphisms in the sense that if $\mathrm{target}(f) = \mathrm{source}(g)$ then $1_{fg} = 1_f \circ 1_g$. To see this, recall that in the definition of composition of 1-morphisms, we take representatives of f and g that are straight on intervals in the

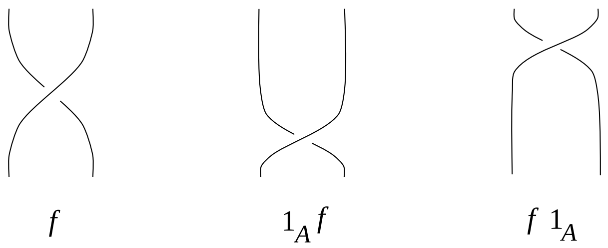

Figure 7: Relations between identity morphisms

third coordinate; likewise, in the definition of horizontal composition of 2-morphisms we take representatives of 1_f and 1_g that are straight on the same intervals in the third coordinate. We can, in fact use the representatives of f and g to get representatives of the 2-morphisms, then the 2-morphisms 1_{fg} and $1_f \circ 1_g$ are represented by the same surface $(fg)' \times I_4$, where $(fg)'$ is a representative of fg.

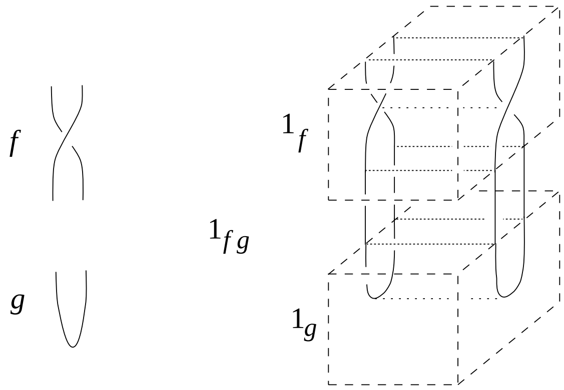

Figure 8: $1_{fg} = 1_f \circ 1_g$

The property that a 2-morphism representative is straight on $I_1 \times I_2 \times \partial I_3 \times I_4$ implies that

$$\text{target}(\text{source}(\alpha)) = \text{target}(\text{target}(\alpha)),$$

and

$$\text{source}(\text{target}(\alpha)) = \text{source}(\text{source}(\alpha))$$

for any 2-morphism α.

The operations of composition of 1-morphisms and horizontal and vertical composition of 2-morphisms are associative. This is true in each case because the equivalence relations for 1-morphisms and 2-morphisms allow for smooth changes in the coordinate directions, if small neighborhoods of the source and target are fixed. This is sufficient to transform a naive (follow the definitions) representative of one composition to a naive representative of the other.

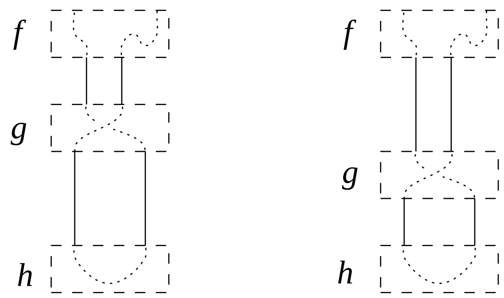

Figure 9: $(fg)h = f(gh)$

Let α, β, γ and δ be 2-morphisms such that the compostions below are defined. To see that

$$(\alpha \cdot \beta) \circ (\gamma \cdot \delta) = (\alpha \circ \gamma) \cdot (\beta \circ \delta),$$

consider representatives of the 2-morphism $(\alpha \cdot \beta) \circ (\gamma \cdot \delta)$ that is flat (ie. it is given by a product in t) outside small regions representingα, β, γ and δ, as in Fig. 10. Splitting this horizontally, it is clear that this is a representative of $(\alpha \cdot \beta) \circ (\gamma \cdot \delta)$, but splitting it vertically, it is also clear that this is a representative of $(\alpha \circ \gamma) \cdot (\beta \circ \delta)$. $\square$

2.2 $\mathcal{T}$ is a Monoidal 2-Category

We take as our definition of monoidal and braided monoidal 2-category the characterization given in lemmas 4 and 7 in Baez and Neuchl's paper [4] with the additional relations $R_{A,I} = R_{I,A} = 1_A$, and $R_{I,f} = R_{f,I} = 1_f$ mentioned in the Conclusions of that paper.

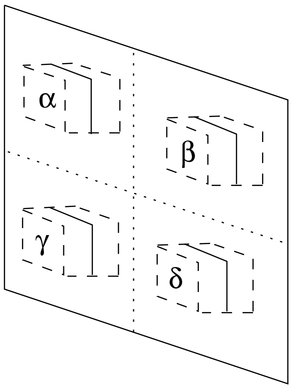

Figure 10: $\alpha \circ \beta \ = (\alpha g) \cdot (f' \beta)$

2.2.1 Tensor products with an object

Given objects A and B, choose representatives A' and B', such that the y coordinates of all points in A' precede the y coordinates of all points in B', then $A \otimes B$ is the object determined by $A' \cup B'$. To simplify notation, we will sometimes suppress the $\otimes$ in writing tensor products of objects, that is $A \otimes B$ may be written as AB.

Let A be an object, and $f: B \to C$ be a 1-morphism, then $A \otimes f: A \otimes B \to A \otimes C$ is represented by a tangle with f on the right and 1_A on the left. Specifically, take representatives A' and f' of A and f such that the y coordinate of all points in A' precedes the y coordinates of all points in f', and let $A' \times I$ be the straight tangle from A' to A', then $A \otimes f$ is the 1-morphism given by $(A' \times I) \cup f'$. Clearly the beginning and ending sets of this tangle are representatives of the source and target as above. The product $f \otimes A: B \otimes A \to C \otimes A$ is defined similarly from a tangle with f on the left and 1_A on the right.

Let $\alpha: f \to g$ be a 2-morphism, and let A be an object, then $A \otimes \alpha: A \otimes f \to A \otimes g$ is the 2-morphism with 1_{1_A} to the left of α. Specifically, in $I_1 \times I_2 \times I_3 \times I_4$, let α' and A' be representatives of α and A, such that the y coordinate of every point of A' precedes the y coordinate of every point of α', then $A \otimes \alpha$ is the 2-morphism determined by $(A' \times I_3 \times I_4) \cup \alpha'$. Similarly, we define $\alpha \otimes A$ as the 2-morphism with 1_{1_A} to the right of α. Clearly the beginning and ending tangles of this surface are representatives of the source and target as above.

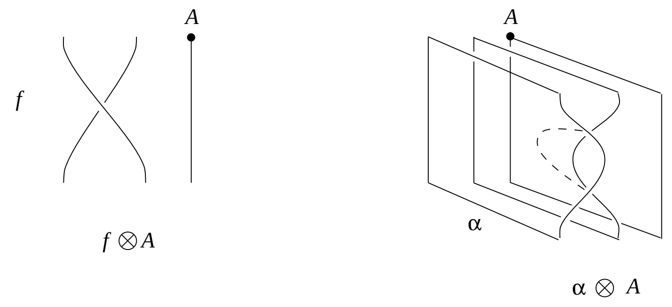

Figure 11: Tensor product of an object with morphisms and 2-morphisms respectively

Lemma 3. *The tensor products $A \otimes B$, $A \otimes f$, $f \otimes A$, $\alpha \otimes A$ and $A \otimes \alpha$ are well defined for objects A and B, 1-morphism f and 2-morphism α.*

We will outline the proof for $A \otimes \alpha$, the other results follow from similar arguments: Let A' and A'' be representatives of A and let α' and α'' be representatives of α such that the y coordinate of every element of A' is less than some number n', and the y coordinate of every point of α' is greater than n', and similarly for A'', α'', and n''. Without loss in generality we may assume that $n' < n''$. There exists an equivalence isotopy $H(x, y, T)$ from A'' to A' that is fixed for $y > n''$. We extend H to an isotopy from $A'' \times I_3 \times I_4$ to $A' \times I_3 \times I_4$ by the product structure; this function is also an isotopy from $(A'' \times I_3 \times I_4) \cup \alpha''$ to $(A' \times I_3 \times I_4) \cup \alpha''$. Then take an equivalence isotopy $J(x, y, z, t, T)$ from α'' to α' that is fixed for $y < n'$; this isotopy is also an equivalence isotopy between $(A' \times I_3 \times I_4) \cup \alpha''$ and $(A' \times I_3 \times I_4) \cup \alpha'$. Following H by J we get an appropriate isotopy from $(A'' \times I_3 \times I_4) \cup \alpha''$ to $(A' \times I_3 \times I_4) \cup \alpha'$, hence $A \otimes \alpha$ is well defined. □

2.2.2 The tensor product of two 1-morphisms

For any 1-morphisms $f \colon A \to A'$ and $g \colon B \to B'$ we define the 2-morphism

$$\bigotimes\nolimits_{f,g} \colon (A \otimes g)(f \otimes B') \Rightarrow (f \otimes B)(A' \otimes g)$$

as follows: Take a representative $[(A \otimes g)(f \otimes B')]'$ of the source of $\bigotimes_{f,g}$ that is straight outside small regions containing representatives of f and g. Then let $H\colon I_1 \times I_2 \times I_3 \times I \to I_1 \times I_2 \times I_3$ be a smooth isotopy that slides the region containing f up, and the region containing g down, as in Fig. 12, so that f is above g. We further require that H be trivial near $T = 0, 1$. Clearly, we can find such an isotopy H for which the surface

$$F = \{(H(x, y, z, T), T) | (x, y, z) \in [(A \otimes g)(f \otimes B')]'\}$$

has generic projections $p(F)$ and $\pi(p(F))$, hence F represents a 2-morphism. Let the 2-morphism represented by this surface be $\bigotimes_{f,g}$.

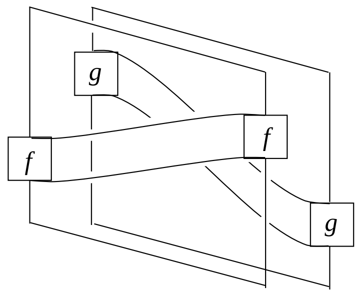

Figure 12: The tensor product $\bigotimes_{f,g}$ of two morphisms

Any two isotopies of the type above, for a pair of representatives f', g', will give equivalent surfaces (since they act by shifting only z values). Also, any two representatives of f (or g) that satisfy the above conditions will be isotopic by an equivalence isotopy that is constant outside the regions specified for f and g, so the isotopy can be extended in a natural way to an equivalence isotopy of the surface. Hence $\bigotimes_{f,g}$ is well defined. Since H is an isotopy, $\bigotimes_{f,g}$ is a 2-isomorphism, where $\bigotimes_{f,g}^{-1}$ is the 2-morphism represented by

$$\{(H(x, y, z, T), 1 - T) | (x, y, z) \in [(A \otimes g)(f \otimes B')]'\}.$$

Lemma 4. *$\mathcal{T}$ is a monoidal 2-category*

Proof - we will show that $\mathcal{T}$ satisfies the conditions to be a monoidal 2-category. As shown above, $\mathcal{T}$ is a 2-category.

Let I be the object represented by the empty set. The object, morphisms and 2-morphisms $A \otimes B$, $A \otimes f$, $f \otimes A$, $A \otimes \alpha$, and $\alpha \otimes A$ for objects A, B, morphism f and 2-morphism α are defined above, and by lemma 3 they are well defined. The 2-morphism $\bigotimes_{f,g}$ is a well defined isomorphism as shown above. We proceed to show that these data satisfy the properties of a monoidal 2-category.

(i) For any object $A \in \mathcal{T}$, the maps $A \otimes -\colon \mathcal{T} \to \mathcal{T}$ and $- \otimes A\colon \mathcal{T} \to \mathcal{T}$ are 2-functors.

The functor $A \otimes -$, maps objects (B) 1-morphisms (f) and 2-morphisms (α) to $A \otimes B$, $A \otimes f$, and $A \otimes \alpha$ as defined above. Since $A \otimes f$ is represented by $(A' \times I_3 \times I_4) \cup (f' \times I_4) = ((A' \times I_3) \cup f) \times I_4$, which also represents $1_A \otimes f$, for appropriate representatives A', f of A and f, we have $A \otimes 1_f = 1_{A \otimes f}$. Similarly, $A \otimes 1_B = 1_{A \otimes B}$. The result for B follows from a similar argument.

We can also show that $A \otimes fg = (A \otimes f)(A \otimes g)$, $A \otimes (\alpha \circ \beta) = (A \otimes \alpha) \circ (A \otimes \beta)$ and $A \otimes (\alpha \cdot \beta) = (A \otimes \alpha) \cdot (A \otimes \beta)$. These arguments for all three cases are similar; we demonstrate this for the case $A \otimes (\alpha \cdot \beta) = (A \otimes \alpha) \cdot (A \otimes \beta)$. Choose representatives α' and β' of α and β that are compatible (source and target tangles are equal, α' is straight on $t \in [3/8, 1]$ and β' is straight on $t \in [0, 5/8]$), and that lie to the right of a representative A' of A. Then, since $A' \times I_3 \times I_4$ is flat, the representative

$$(A' \times I_3 \times I_4) \cup ((\alpha' \cap I_1 \times I_2 \times I_3 \times [0, 1/2]) \cup (\beta' \cap I_1 \times I_2 \times I_3 \times [1/2, 1]))$$

of $A \otimes (\alpha \cdot \beta)$, and the representative

$$((A' \times I_3 \times I_4 \cup \alpha') \cap I_1 \times I_2 \times I_3 \times [0, 1/2]) \cup ((A' \times I_3 \times I_4 \cup \beta') \cap I_1 \times I_2 \times I_3 \times [1/2, 1]))$$

of $(A \otimes \alpha) \cdot (A \otimes \beta)$ are equal, hence $A \otimes (\alpha \cdot \beta) = (A \otimes \alpha) \cdot (A \otimes \beta)$.

(ii) For x any object, morphism or 2-morphism of $\mathcal{C}$, $x \otimes I = I \otimes x = x$.

Since I is represented by the empty set, its product with the intervals I_3 and I_4 is also empty, hence $x \otimes I = I \otimes x = x$ for any object, morphism of 2-morphism x.

(iii) For x any object, morphism or 2-morphism of $\mathcal{C}$, and for all objects $A, B \in \mathcal{C}$ we have $A \otimes (B \otimes x) = (A \otimes B) \otimes x$, $A \otimes (x \otimes B) = (A \otimes x) \otimes B$ and $x \otimes (A \otimes B) = (x \otimes A) \otimes B$.

This follows from the property that equivalence isomorphisms of objects, morphisms and 2-morphisms all allow shifts in y.

(iv) For any 1-morphisms $f: A \to A'$, $g: B \to B'$ and $h: C \to C'$ in $\mathcal{C}$ we have $\bigotimes_{A \otimes g,h} = A \otimes \bigotimes_{g,h}$, $\bigotimes_{f \otimes B,h} = \bigotimes_{f,B \otimes h}$ and $\bigotimes_{f,g \otimes C} = \bigotimes_{f,g} \otimes C$.

This follows from the fact that we may choose a representative surface, for example, for $\bigotimes_{A \otimes g,h}$ for which the tangle representing 1_A is straight, and the sheet containing it is flat and disjoint from the surface containing the representative of g. Arguments for the other cases are similar.

(v) For any objects $A, B \in \mathcal{T}$ we have $1_A \otimes B = A \otimes 1_B = 1_{A \otimes B}$, and for any 1-morphisms $f: A \to A'$, $g: B \to B'$ in $\mathcal{C}$ we have $\bigotimes_{1_A,g} = 1_{A \otimes g}$ and $\bigotimes_{f,1_B} = 1_{f \otimes B}$.

The properties of objects are clear from the definitions. The properties for 1-morphisms follow from the definition given of $\bigotimes_{f,g}$: sliding an identity morphism doesn't change the tangle, so the only effect of isotopy defining $\bigotimes_{1_A,g}$ or $\bigotimes_{f,1_B}$ is to change z coordinates for g or f. This sliding may be given by an equivalence isotopy, so the resulting surface is isotopic to a surface which is flat in t, and is therefore an identity 2-morphism.

(vi) For any morphisms $f: A \to A'$, $g, h: B \to B'$, and any 2-morphism $\beta: g \Rightarrow g'$,

$$((A \otimes \beta)(f \otimes B')) \cdot \bigotimes_{f,h} = \bigotimes_{f,g} \cdot ((f \otimes B)(A' \otimes \beta)),$$

and *(vii) For any morphisms $f, g: A \to A'$, $h: B \to B'$ and any 2-morphism $\alpha: f \Rightarrow g$,*

$$\bigotimes_{f,h} \cdot ((\alpha \otimes B)(A' \otimes h)) = ((A \otimes 1_h)(\alpha \otimes B')) \cdot \bigotimes_{g,h}.$$

The argument is similar in the two cases. A picture of the proof in the second case is shown in Fig. 13 below. This consists of sliding a piece of the surface that represents α in a naive representative of $((A \otimes 1_h)(\alpha \otimes B')) \cdot \otimes_{g,h}$ along the path specified by $\bigotimes_{g,h}$, resulting in a surface in which α follows $\bigotimes_{f,h}$, and which represents $\bigotimes_{f,h} \cdot ((\alpha \otimes B)(A' \otimes h))$. Clearly, this sliding can be accomplished by an equivalence isotopy.

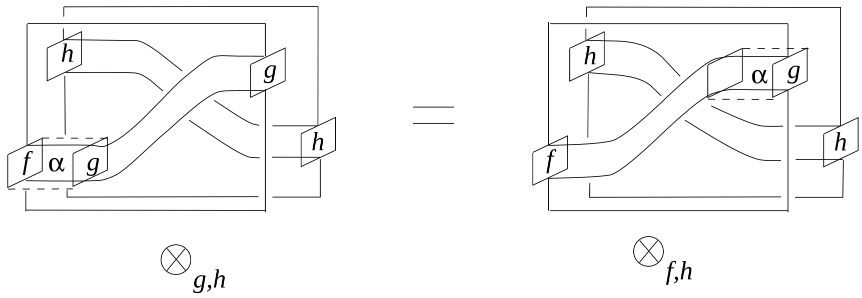

Figure 13: $A \otimes \beta \circ 1_{f \otimes B'} \cdot \bigotimes_{f,g'} = \bigotimes_{f,g} \cdot (1_{f \otimes B} \circ A' \otimes \beta)$

(viii) For any 1-morphisms $f\colon A \to A'$, $g\colon B \to B'$ and $g'\colon B' \to B''$,

$$\bigotimes_{f,gg'} = ((A \otimes g) \bigotimes_{f,g'}) \cdot (\bigotimes_{f,g}(A' \otimes g')).$$

For any 1-morphisms $f\colon A \to A'$, $f'\colon A' \to A''$ and $g\colon B \to B'$,

$$\bigotimes_{ff',g} = (\bigotimes_{f,g}(f' \otimes B')) \cdot ((f \otimes B) \bigotimes_{f',g}).$$

A picture of $((A \otimes g) \bigotimes_{f,g'}) \cdot (\bigotimes_{f,g}(A' \otimes g'))$ and $\bigotimes_{f,gg'}$ are shown in Fig. 14, for which the representatives of f, g and g' are straight outside small regions.

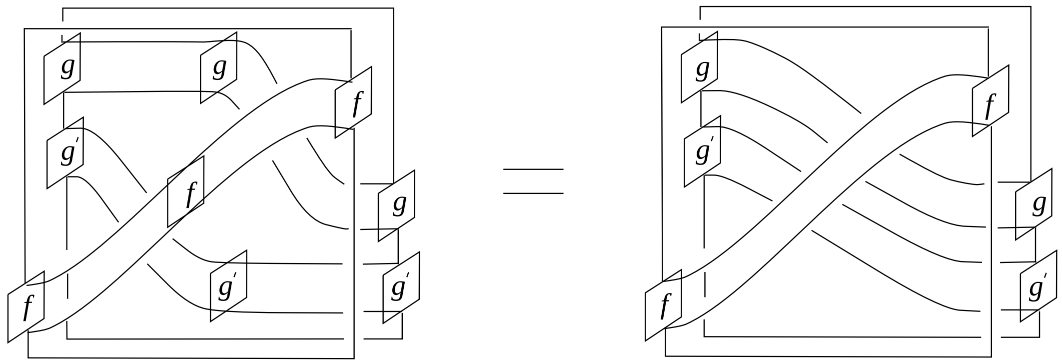

Figure 14: $(1_{A \otimes g} \circ \bigotimes_{f,g'}) \cdot (\bigotimes_{f,g} \circ 1_{A' \otimes gg'})$ and $\bigotimes_{f,gg'}$

Clearly there is an equivalence isotopy between these surfaces. $\square$

2.3 $\mathcal{T}$ is a braided monoidal 2-category

2.3.1 Braiding of objects

For any objects A and B, we define the braiding 1-morphism $R_{A,B}\colon A \otimes B \to B \otimes A$ to be the morphism represented by a tangle consisting of only positive crossings, such that each strand beginning at a point of A crosses all strands beginning at points of B before the next strand of A crosses any strands of B, as in Fig. 15. We represent the morphism $\bar{R}_{A,B}$ by the reflection in t of a tangle representing $R_{A,B}$. Clearly, there are surfaces (which can be defined in terms of repeated Reidemeister 2 moves) that represent 2-isomorphisms between $R_{A,B}\bar{R}_{A,B}$ and $1_{A\otimes B}$, and between $\bar{R}_{A,B}R_{A,B}$ and $1_{B\otimes A}$, so that $R_{A,B}$ is an equivalence.

2.3.2 Braiding of an object and a morphism

Let $f\colon A \to A'$ be a 1-morphism and B be an object then we define the 2-isomorphism $R_{f,B}\colon (f \otimes B)R_{A',B} \Rightarrow R_{A,B}(B \otimes f)$ as follows. Consider a representative of the source $(f \otimes B)R_{A',B}$, for which f lies behind (has greater x values than) the strands beginning at B. Then there is an isotopy that moves f past the strands of B, as in Fig. 15. If $H(x,y,z,t)$ is such an isotopy, the surface representing $R_{f,B}$ is given by $\{(H(x,y,z,t),t)|(x,y,z) \in (f \otimes B)R_{A',B}\}$. Since the surface is defined by the trace of an isotopy, $R_{A,f}$ is a 2-isomorphism, and $R^{-1}_{A,f}$ is represented by $\{(H(x,y,z,1-t),t)|(x,y,z) \in (f \otimes B)R_{A',B}\}$. We define similarly the 2-isomorphism $R_{A,g}\colon (A \otimes g)R_{A,B'} \Rightarrow R_{A,B}(g \otimes A)$ for $g\colon B \to B'$.

2.3.3 Yang-Baxter triangles

For any objects $A, B, C \in \mathcal{T}$, the 2-isomorphism

$$\tilde{R}_{(A|B,C)}\colon (R_{A,B} \otimes C)(B \otimes R_{A,C}) \Rightarrow R_{A,B\otimes C}$$

is represented by the surface determined by an isotopy which changes only the order of distant crossings (crossings that have no strands in common) in a representative

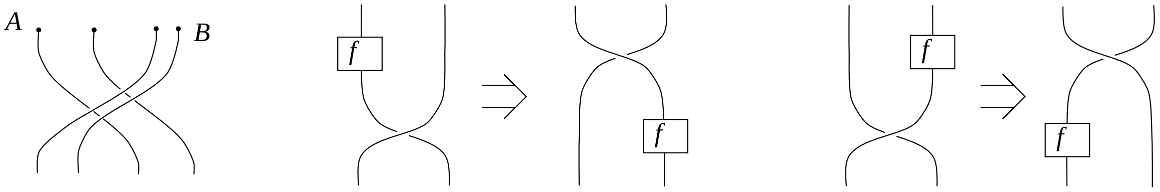

Figure 15: $R_{A,B}$, $R_{f,A}$ and $R_{A,f}$

of $(R_{A,B} \otimes C)(B \otimes R_{A,C})$ to give a representative of $R_{A,B\otimes C}$. Since this 2-morphism is defined in terms of an isotopy, we can construct an inverse using the isotopy with t reversed, so $\tilde{R}_{(A|B,C)}$ is a 2-isomorphism. Note that if Z is the object represented by a single point, $R_{Z,B\otimes C}$ is an identity 2-morphism. We illustrate a non-trivial case of $R_{A,B\otimes C}$ in Fig. 16. Since $(A \otimes R_{B,C})(R_{A,C} \otimes B) = R_{A\otimes B,C}$, we may define $\tilde{R}_{A,B|C}$ to be an identity 2-morphism

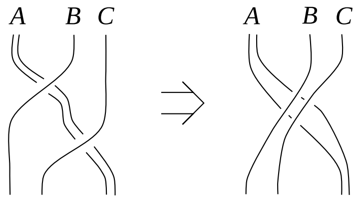

Figure 16: $\tilde{R}_{A|B,C}$

Lemma 5. *$\mathcal{T}$ is a braided monoidal 2-category.*

Proof - By the previous lemma, $\mathcal{T}$ is a monoidal category, and we have defined above the morphisms and 2-morphisms $R_{\cdot,\cdot}$, $\tilde{R}_{\cdot|\cdot,\cdot}$ and $\tilde{R}_{\cdot,\cdot|\cdot}$. Further, we have shown that $R_{A,B}$ is a well defined equivalence, and that $R_{f,B}$, $R_{A,g}$, $\tilde{R}_{(-|-,-)}$ and $\tilde{R}_{(-,-|-)}$ are well defined 2-isomorphisms. We proceed to show that they satisfy the necessary conditions:

($\rightarrow \otimes \rightarrow$) The relation

$$((f \otimes B)R_{A',g}) \cdot (R_{f,B}(g \otimes A')) \cdot (R_{A,B} \bigotimes{}_{g,f})$$

$$= (\bigotimes{}_{f,g}^{-1}(R_{A',B'})) \cdot ((A \otimes g)R_{f,B'}) \cdot (R_{A,g}(B' \otimes f))$$

corresponds to the equivalence of the surfaces given by the isotopies illustrated in Fig. 17. Since the action of the isotopies on the tangles representing f and g can be restricted in x, they can be shifted independently, and it is clear that these surfaces are equivalent 2-morphism representatives.

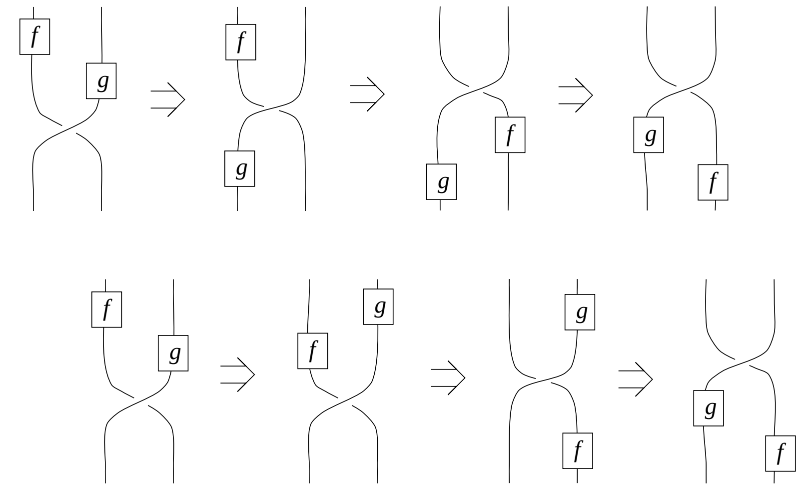

Figure 17: ($\rightarrow \otimes \rightarrow$)

($\Downarrow \otimes \bullet$) The relation

$$((\beta \otimes A)R_{B',A}) \cdot R_{f',A} = R_{f,A} \cdot (R_{B,A}(A \otimes \beta))$$

corresponds to the equivalence of the surfaces illustrated in Fig. 18. The isotopy between these surfaces slides the 2-tangle representing β in along the path determined by the representative of $R_{f',A}$. Since the x values of β and A can be separated (β lies behind A), it is clear that the surfaces are equivalent 2-morphism representatives.

The relation ($\bullet \otimes \Downarrow$) holds by a similar argument.

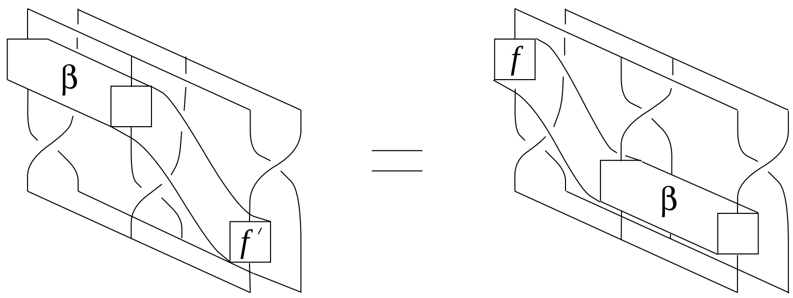

Figure 18: $(\bullet\otimes \Downarrow)$

$(\rightarrow\rightarrow \otimes\bullet)$ The relation

$$((f \otimes B)R_{f',B}) \cdot (R_{f,B}(B \otimes f')) = R_{ff',B}$$

corresponds to the equivalence of the surfaces illustrated in Fig. 19. Since the 2-tangles representing f and f' may be restricted to large x values, and the braided strand(s) at B may be restricted to small x values, it is clear that there is an isotopy between these surfaces.

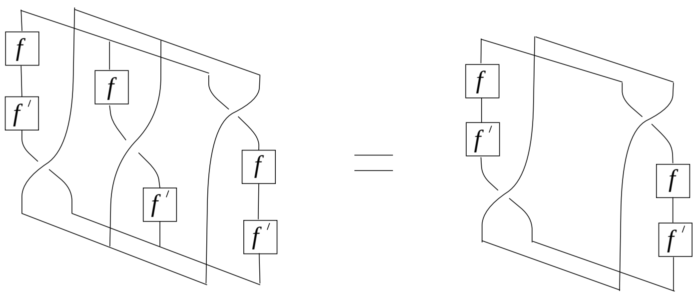

Figure 19: $(\rightarrow\rightarrow \otimes\bullet)$

Similarly, $(\bullet\otimes \rightarrow\rightarrow)$ holds.

$(\bullet \otimes \bullet)\otimes \rightarrow))$ The relation

$$((AB \otimes f)\tilde{R}_{(A|B,C')}) \cdot R_{A,B\otimes f} =$$

$$((\bigotimes{}_{R_{A,B},f}(B \otimes R_{A,C'})) \cdot ((R_{A,B} \otimes C)(B \otimes R_{A,f})) \cdot (\tilde{R}_{(A|B,C)}(B \otimes f \otimes A))$$

corresponds to the equivalence of the surfaces illustrated in Fig. 20. The other prism relations $(\bullet\otimes\rightarrow\rightarrow)$, $((\bullet\otimes\bullet)\otimes\rightarrow)$, $(\rightarrow\otimes(\bullet\otimes\bullet))$, $((\rightarrow\otimes\bullet)\otimes\bullet)$, $((\bullet\otimes\rightarrow)\otimes\bullet)$,$(\bullet\otimes(\rightarrow\otimes\bullet))$ and $(\bullet\otimes(\bullet\otimes\rightarrow))$ correspond to equivalences of similar surfaces.

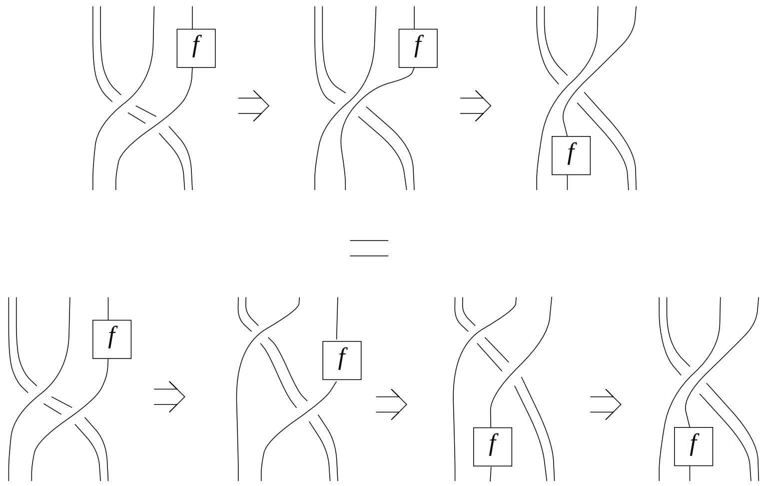

Figure 20: $(\bullet\otimes(\bullet\otimes\rightarrow))$

The relation $((\bullet\otimes\bullet\otimes\bullet)\otimes\bullet)$ holds trivially, since all of the 2-morphisms involved are of the form $\tilde{R}_{(\cdot,\cdot|\cdot)}$, which are identity morphisms.

$(\bullet\otimes(\bullet\otimes\bullet\otimes\bullet))$ corresponds to the relation

$$B\otimes\tilde{R}_{(A|C,D)}\cdot\tilde{R}_{(A|B,C\otimes D)}=\tilde{R}_{(A|B,C)}\otimes D\cdot\tilde{R}_{(A|B\otimes C,D)}$$

which corresponds to the equivalence of surfaces in Fig. 21.

$((\bullet\otimes\bullet)\otimes(\bullet\otimes\bullet))$ corresponds to the relation

$$((A\otimes R_{B,C}\otimes D)\bigotimes{}_{R_{A,C},R_{B,D}}(C\otimes R_{A,D}\otimes B))\cdot((\tilde{R}_{(A,B|C)}\otimes D)\circ(C\otimes\tilde{R}_{(A,B|D)}))\cdot\tilde{R}_{(A\otimes B|C,D)}$$

$$=((A\otimes\tilde{R}_{(B|C,D)})\circ(\tilde{R}_{(A|C,D)}\otimes B))\cdot\tilde{R}_{(A,B|C\otimes D)}.$$

When we eliminate the trivial 2-morphisms of the form $\tilde{R}_{(\cdot,\cdot|\cdot)}$, this simplifies to

$$(A\otimes R_{B,C}\otimes D\bigotimes{}_{R_{A,C},R_{B,D}}C\otimes R_{A,D}\otimes B)\cdot\tilde{R}_{(A\otimes B|C,D)}$$

$$=(A\otimes\tilde{R}_{(B|C,D)})\circ(\tilde{R}_{(A|C,D)}\otimes B).$$

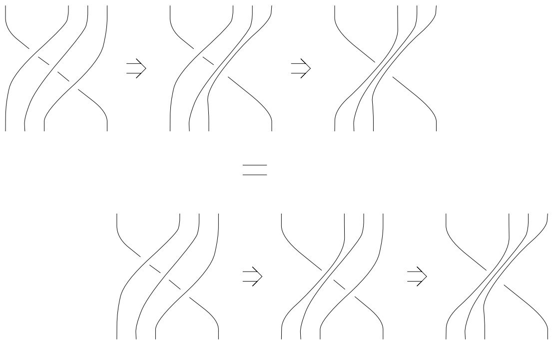

Figure 21: $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$

In the case where A, B are represented a single point, this further reduces to the fact that the change of crossing order in $\tilde{R}_{(A\otimes B|C,D)}$ is the same as the crossing order change given by $(A\otimes R_{B,C}\otimes D \bigotimes^{-1}_{R_{A,C},R_{B,D}} C\otimes R_{A,D}\otimes B)$, as in Fig. 15. The result for other objects A, B follows similarly, and indeed follows inductively from the simpler result.

$S^{+}_{A,B,C} = S^{-}_{A,B,C}$ It is clear that surfaces for to this relation are equivalent, as illustrated in Fig. 22

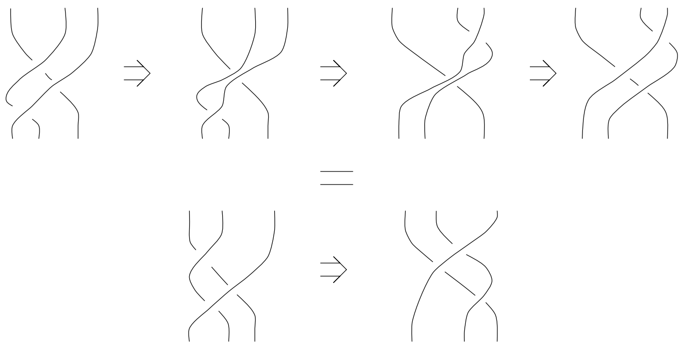

Figure 22: $S^{+}_{A,B,C} = S^{-}_{A,B,C}$

Clearly, since I is represented by the empty set, the extra relations $R_{A,I} = R_{I,A} = 1_A$, and $R_{I,f} = R_{f,I} = 1_f$ also hold in $\mathcal{T}$. $\square$

2.4 $\mathcal{T}$ is a braided monoidal 2-category with duals.

Definition 6. *A* monoidal 2-category with duals *is a monoidal 2-category equipped with the following structures:*

1. For every 2-morphism $\alpha\colon f \to g$ there is a 2-morphism $\alpha^*\colon g \to f$ called the *dual* of α.

2. For every morphism $f\colon A \to B$ there is a morphism $f^*\colon B \to A$ called the *dual* of f, and 2-morphisms $i_f\colon 1_A \to ff^*$ and $e_f\colon f^*f \to 1_B$, called the *unit* and *counit* of f, respectively.

3. For any object A, there is a object A^* called the *dual* of A, 1-morphisms $i_A\colon I \to A \otimes A^*$ and $e_A\colon A^* \otimes A \to I$ called the *unit* and *counit* of A, respectively, and a 2-morphism $T_A\colon (i_A \otimes A)(A \otimes e_A) \Rightarrow 1_A$ called the *triangulator* of A.

We say that a 2-morphism α is unitary *if it is invertible and $\alpha^{-1} = \alpha^*$. Given a 2-morphism $\alpha\colon f \Rightarrow g$, we define the* adjoint $\alpha^\dagger\colon g^* \Rightarrow f^*$ *by*

$$\alpha^\dagger = (g^* i_f) \cdot (g^* \alpha f^*) \cdot (e_g f^*).$$

In addition, the structures above are also required to satisfy the following conditions:

1. $X^{**} = X$ for any object, morphism or 2-morphism X.

2. $1_X^* = 1_X$ for any object or morphism X.

3. For all objects A, B, morphisms f, g, and 2-morphisms α, β for which both sides of the following equations are well-defined, we have

$$(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*,$$

$$(\alpha \circ \beta)^* = \alpha^* \circ \beta^*,$$

$$(fg)^* = g^* f^*,$$

$$(A \otimes \alpha)^* = A \otimes \alpha^*,$$

$$(A \otimes f)^* = A \otimes f^*,$$

$$(\alpha \otimes A)^* = \alpha^* \otimes A,$$

$$(f \otimes A)^* = f^* \otimes A,$$

and

$$(A \otimes B)^* = B^* \otimes A^*.$$

4. For all morphisms f and g, the 2-morphism $\bigotimes_{f,g}$ is unitary.

5. For any object or morphism X we have $i_{X^*} = e_X^*$ and $e_{X^*} = i_X^*$.

6. For any object A, the 2-morphism T_A is unitary.

7. For any objects A and B we have

$$i_{A \otimes B} = i_A \cdot (A \otimes i_B \otimes A^*),$$

$$e_{A \otimes B} = (B^* \otimes e_A \otimes B) \cdot e_B,$$

and

$$T_{A \otimes B} = [(i_A \otimes A \otimes B)(A \otimes \bigotimes_{i_B, e_A}^{-1} \otimes B)(A \otimes B \otimes e_B)] \cdot [(T_A \otimes B) \circ (A \otimes T_B)].$$

8. $T_I = 1_I$

9. For any object A and morphism f, we have

$$i_{A \otimes f} = A \otimes i_f,$$

$$i_{f \otimes A} = i_f \otimes A,$$

$$e_{A \otimes f} = A \otimes e_f,$$

and

$$e_{f \otimes A} = e_f \otimes A$$

10. For any morphisms f and g, $i_{fg} = i_f \cdot (f i_g f^*)$ and $e_{fg} = (g^* e_f g) \cdot e_g$.

11. For any 1-morphism f, $i_f f \cdot f e_f = 1_f$ and $f^* i_f \cdot e_f f^* = 1_{f^*}$.

12. For any 2-morphism α, $\alpha^{\dagger *} = \alpha^{*\dagger}$.

13. For any object A we have

$$[i_A(A \otimes T^{\dagger}_{A^*})] \cdot [\bigotimes^{-1}_{i_A, i_A}(A \otimes e_A \otimes A^*)] \cdot [i_A(T_A \otimes A^*)] = 1_{i_A}$$

Definition 7. *A* braided monoidal 2-category with duals *is a monoidal 2-category with duals that is also a braided monoidal 2-category for which the braiding is unitary in the sense that:*

1. *For any objects A, B, the 2-morphisms $i_{R_{A,B}}$ and $e_{R_{A,B}}$ are unitary.*

2. *For any object A and morphism f, the 2-morphisms $R_{A,f}$ and $R_{f,A}$ are unitary.*

3. *For any objects A, B, C, the 2-morphisms $\tilde{R}_{(A,B|C)}$ and $\tilde{R}_{(A|B,C)}$ are unitary.*

Definition 8. *An* unframed object *in a braided monoidal 2-category with duals is an object A equipped with a unitary 2-morphism*

$$W_A \colon i_A \Rightarrow i_{A^*} R_{A^*,A}$$

called the writhing *of A, satisfying the equation:*

$$T^{\dagger}_A \cdot ((A \otimes W_{A^*})(e_{A^*} \otimes A)) \cdot ((A \otimes (i_{A^*} R_{A^*,A})) R^{\dagger}_{A,i_A}) \cdot$$

$$((A \otimes (i_{A^*} R_{A^*,A})) \tilde{R}^{\dagger}_{A|A,A^*}(A \otimes e_{A^*})) \cdot ((A \otimes (i_{A^*} i^*_{R_{A,A^*}}))(R^*_{A^*,A} \otimes A)(A \otimes e_{A^*}))$$

$$= T_A \cdot ((i_A \otimes A)(A \otimes (i_{R_{A^*,A}} e_A))) \cdot ((i_A \otimes A)(A \otimes (R_{A^*,A} W^{\dagger}_A))) \cdot$$

$$(R^{-1}_{e_{A^*},A}(A \otimes (R_{A^*,A} e_{A^*}))) \cdot ((A \otimes i_A) \tilde{R}^{\dagger}_{A,A^*|A}(A \otimes (R_{A^*,A} e_{A^*}))) \cdot$$

$$((A \otimes i_A)(R^*_{A,A} \otimes A^*)(A \otimes (e_{R_{A^*,A}} e_{A^*})))$$

Definition 9. *A* self-dual object *in a braided monoidal 2-category with duals is an object A with $A^* = A$.*

Lemma 10. *$\mathcal{T}$ is a braided monoidal 2-category with duals with one unframed self-dual object.*

Proof - We begin by describing the duals of objects, morphisms and 2-morphisms, units and counits of objects and morphisms, and the triangulator.

For any object $A \in \mathcal{T}$, let the dual object $A^* = A$. When considering a specific representative of an object in $A \in \mathcal{T}$, we will picture the dual object as the reflection of the representative in $\{y = 1/2\}$ (y-coordinates reversed); note that since this set has the same number of points, it represents the same object in $\mathcal{T}$.

A representative of the unit $i_A: I \to A \otimes A^*$ can be obtained by rotating a representative of A^* that lies in $\{y > 1/2\}$ around the axis $\{y = 1/2, z = 1\}$ through $180°$; the tangle traced out by this rotation is a disjoint union of semicircles with endpoints in the target plane, $\{z = 1\}$, with the right endpoints representing $A^* = A$, and the left endpoints representing A; we alter this by a small isotopy to get a smooth tangle that is straight near $\{z = 1\}$. Any such representative of $A*$ will give an equivalent representative of i_A because we can restrict the equivalence isotopy to act within $\{y > 1/2\}$, and such an isomorphism can be extended to the tangle representing i_A. A representative of the counit $e_A: A^* \otimes A \to I$ may be obtained similarly from a rotation of a representative of A that lies in $\{y < 1/2\}$ around the axis $\{y = 1/2, z = 0\}$ through $180°$ and smoothed appropriately; the tangle traced out by this rotation is a disjoint union of semicircles with endpoints in the source plane, $\{z = 0\}$, with the right endpoints representing A, and the left endpoints representing $A* = A$.

Clearly there is a 2-morphism T_Z that will straighten a standard representative of $(i_Z \otimes Z)(Z \otimes e_Z)$, where Z is the object represented by a single element set, and a representative of T_Z can be described by an isotopy between representatives of the source and target. Clearly an equivalence isotopy between representatives of Z can be extended to T_Z, so T_Z is well defined. We use the fact that for any object A, $A = Z \otimes A'$, and we define T_A inductively by the relation:

$$T_A = [(i_Z \otimes Z \otimes A')(Z \otimes \bigotimes^{-1}_{i'_A, e_Z} \otimes A')(Z \otimes A' \otimes e'_A)] \cdot [(T_Z \otimes A') \circ (Z \otimes T'_A)].$$

Let $f: A \to B$, and let f' be a representative of f. Then we let the dual morphism $f^*: B \to A$ be the morphism represented by the tangle that is the reflection of f' in $\{z = 1/2\}$. Any equivalence between representatives of f can be similarly reflected; hence f^* is well defined.

To define i_f, take a representative of $f \otimes f^*$ that lies in $t = 1$, and is symmetric in $z = 1/2$. Then take the trace of the rotation of this tangle around $z = 1/2, t = 1$, and continue the surface trivially (by the product structure in t) to the boundary. By straightening this surface near $t = 1$ we obtain a surface that represents a 2-morphism, let i_f be the 2-morphism represented by this surface. Similarly we define e_f by rotating an appropriate representative of $f^* \otimes f$ around $z = 1/2, t = 0$.

Let $\alpha: f \to g$ be a 2-morphism, and let α' be a representative of α. Then we define the dual 2-morphism $\alpha^*: g \to f$ to be the 2-morphism represented by the 2-tangle that is the reflection of α' in $\{t = 1/2\}$. Any equivalence isotopy between representatives of α can be likewise reflected, so α^* is well defined.

We proceed to check that the objects, morphisms and 2-morphisms as defined above satisfy the conditions for a monoidal 2-category with duals:

For objects, $A = A^*$, hence $A = A^{**}$. For morphisms and 2-morphisms, duals are defined in terms of reflections of representatives. Since reflecting a representative of a morphism or 2-morphism twice is an identity map, the representatives are the same, so $X = X^{**}$ for X any morphism or 2-morphism.

A standard representative of 1_A for any object A is straight in the z direction. Since the reflection that defines 1_A^* is in a plane perpendicular to the z axis, the representative is not changed by the reflection, so $1_A = 1_A^*$. A standard representative for 1_f for any 1-morphism f is straight in the t direction, and so, by a similar argument, $1_f = 1_f^*$.

In a standard representative of the vertical composition $\alpha \cdot \beta$, β follows α in the t direction, so when the representative is reflected (changing t coordinates), not only are the β and α components reflected, but also, the reflected β now precedes the reflected α, hence $(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*$.

In a standard representative of the horizontal composition $\alpha \circ \beta$, β follows α in the z direction; this order is not changed by the reflection that changes t coordinates, so $(\alpha \circ \beta)^* = \alpha^* \circ \beta^*$.

In a standard representative of fg, g follows f in the z direction. Since the 1-morphism dual is obtained by a reflection that changes z coordinates, the order of the reflected f and g is reversed, so $(fg)^* = g^* f^*$.

Tensor products of any object A with a morphism or 2-morphism X are represented by a disjoint union of X with 1_A or 1_{1_A} on the right or left (y direction). Since the reflections defining duals for morphisms and 2-morphisms do not change y-coordinates, the order of the tensor product remains the same after reflection. Since $1_X^* = 1_X$, as above, the reflection does not change A; hence $(A \otimes X)^* = A \otimes X^*$ and $(X \otimes A)^* = X^* \otimes A$.

Note that representatives of $A \otimes B$ and $B \otimes A$ have the same number of elements, so $A \otimes B = B \otimes A$. It then follows from $A = A^*$ that $(A \otimes B)^* = B^* \otimes A^*$.

A standard representative of $\bigotimes_{f,g}$ is generated by an isotopy that moves f and g past each other (in z), and the inverse is obtained from using the same isotopy with t reversed. Reversing t in the isotopy gives exactly the same surface as reflecting the surface representing $\bigotimes_{f,g}$ to reverse the t coordinates; hence $\bigotimes_{f,g}^{-1} = \bigotimes_{f,g}^*$.

Recall that a standard representative of i_A is obtained by rotating by $180°$ (in the positive direction) a representative A' of A^* lying in the right ($y > 1/2$) half of $\{z = 1\}$. Then i_A^* can be represented by the trace of A' (translated to the right half of $\{z = 0\}$) rotated by $180°$ in the negative direction. Now, to obtain a representative of e_{A^*}, choose as a representative of $A = A^{**}$ the reflection of A' that lies in the left half of $\{z = 0\}$, and take the trace when rotated through $180°$ in the positive direction. Clearly, this is the same tangle as the one exhibited for i_A^*, hence $e_{A^*} = i_A^*$. It follows from the properties $i_A^{**} = i_A$ and $A = A^*$ that $i_{A^*} = e_A^*$. Similarly, if we begin with appropriate representatives of $f \otimes f^*$ and $f^* \otimes f$, the surfaces obtained by rotating and reflecting are the same, implying $i_{f^*} = e_f^*$ and $e_{f^*} = i_f^*$

A representative of T_Z can be described by an isotopy between source and target tangles, so the 2-morphism T_Z is unitary (it has an inverse that is given by reversing t in the isotopy). The general case that T_A is unitary follows from the fact that T_Z and $\bigotimes_{\cdot,\cdot}$ are unitary, and the relation that defines T_A.

A standard representative of $i_{A\otimes B}$ is defined by rotating by 180° (in the positive direction) a representative of $B^* \otimes A^*$ lying in the right ($y > 1/2$) half of $\{z = 1\}$ (B^* lies to the left of A^*). This produces a tangle consisting of nested semi-circles (with no intersections when projected to the $y-z$ plane), and the maximum points of the semi-circles with boundary $A \cup A^*$ are above (smaller z) the maxima of the semi-circles with boundary $B \cup B^*$. If we take a small isotopy that straightens strands for a small interval just above the maxima of the semi-circles with boundary $B \cup B^*$, we get an equivalent tangle which represents $i_A \cdot (A \otimes i_B \otimes A^*)$; hence $i_{A\otimes B} = i_A \cdot (A \otimes i_B \otimes A^*)$. Similarly $e_{A\otimes B} = (A^* \otimes e_B \otimes A) \cdot e_A$.

The relation

$$T_{A\otimes B} = [(i_A \otimes A \otimes B)(A \otimes \bigotimes_{i_B, e_A}^{-1} \otimes B)(A \otimes B \otimes e_B)] \cdot [(T_A \otimes B) \circ (A \otimes T_B)]$$

is a consequence of the relation defining T_A, and in fact is identical to that relation for $A = Z$

Representatives of the 2-morphisms i_f, e_f are generated by rotating a representative of the 1-morphism around an axis perpendicular to the z, t plane; hence the x and y coordinates of f are preserved. In particular, in the case of $A \otimes f$ or $f \otimes A$, a standard representative consists of straight strands for A disjoint from a tangle representing f. When rotated, the strands associated with A become flat planes disjoint from the 2-tangle (representing i_f) that results from rotating the representative of f. We can make a similar argument for $e_{A\otimes f}$ and $e_{f\otimes A}$. Hence

$$i_{A\otimes f} = A \otimes i_f,$$

$$i_{f\otimes A} = i_f \otimes A,$$

$$e_{A\otimes f} = A \otimes e_f,$$

and

$$e_{f\otimes A} = e_f \otimes A$$

for any object A and morphism f.

Recall that i_{fg} is represented by the trace of a symmetric representative of $fg \otimes (fg)^*$, rotated around $z = 1/2, t = 1$. Since there is a height function specified for this tangle, there is a value ϵ so that f is represented by the tangle in $z < 1/2 - \epsilon$, and g is represented by the tangle in $1/2 - \epsilon < z < 1/2$. Then, in the 2-tangle produced by rotating, the subtangle in $t < 1 - \epsilon$ represents $i_{f\otimes f^*}$ (after smoothing and straightening near $t = 1/2 - \epsilon$, and the subtangle in $t > 1 - \epsilon$ represents $f i_g f^*$, so $i_{fg} = i_f \cdot (f i_g f^*)$. Similarly, $e_f g = (g * e_f g) \cdot e_f$.

We specify a representative of $i_f f \cdot f e_f$ as follows. Choose a representative f' of f that is straight outside $z < 1/3 - \epsilon$ for some $\epsilon > 0$. We will begin with a surface that includes the trace of $f' \in \{z \in [0, 1/3]\} \times 1/2$ rotated $180°$ around $z = 1/3, t = 1/2$ (so that the trace lies in $t > 1/2$; let f'' be the resulting tangle in $z \in [1/3, 2/3], t = 1/2$), the trace of f'' rotated $180°$ around $z = 2/3, t = 1/2$, (so that the trace lies in $t < 1/2$). continue this to the boundary by the product structure in t, and smooth locally. Since this surface is flat near the boundary, and near the axes we specified for the rotations above, the rotations can be straightened by an isotopy to give a representative of 1_f (see Fig. 23). Hence we conclude that the triangle relations $i_f f \cdot f e_f = 1_f$, and similarly, $f e_f \cdot i_f f = 1_f$. hold in $\mathcal{T}$.

We can find a representative of $\alpha^{*\dagger}$ that is straight (in t) outside the representatives of α^* in $\{1/ < t < 2/3, 1/3 < z < 2/3\}$, i_g in $\{t < 1/3, z > 1/3\}$ and e_f in $\{z < 2/3, t > 2/3\}$, and the surfaces representing i_g and e_f are rotations of the appropriate tangles around $z = 2/3, t = 1/3$ and $z = 1/3, t = 2/3$. We can get an equivalence isotopy of this surface by sliding the surface representing α^* as shown in Fig. 24. After following this by an isotopy pushing the surface representing e_f to the left (to lie in $1/3 < t < 2/3$), and smoothing where necessary, this clearly yields a

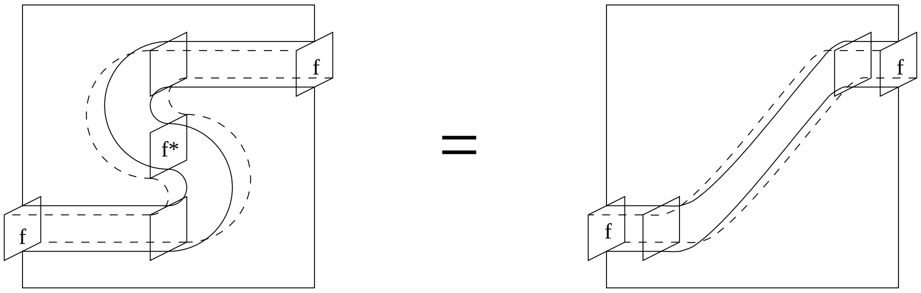

Figure 23: The triangle relation for morphisms

representative of $f^* i_f \cdot e_f f^* \cdot \alpha^*_r$, where α^*_r is the 2-morphism represented by a surface representing α^* that has been rotated as above. It follows from $f e_f \cdot i_f f = 1_f$, $i_{f^*} = e^*_f$ and $e_{f^*} = i^*_f$ that this 2-morphism reduces to α^*_r. similarly we can find an isotopy of a representative of $\alpha^{\dagger *}$ to α^*_r. Thus we conclude $\alpha^{\dagger *} = \alpha^{* \dagger}$.

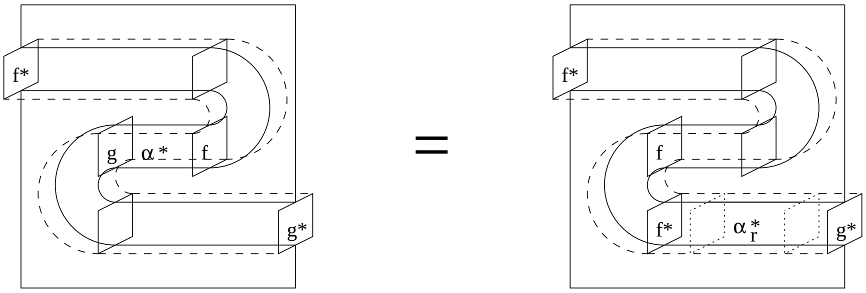

Figure 24: Representatives of $\alpha^{*\dagger}$

Clearly the relation

$$[i_Z(Z \otimes T^\dagger_{Z^*})] \cdot [\bigotimes^{-1}_{i_Z, i_Z}(Z \otimes e_Z \otimes Z^*)] \cdot [i_Z(T_Z \otimes Z^*)] = 1_{i_Z}$$

holds for the single element set Z, see Fig. 25; this relation corresponds to the swallowtail singularity. For A represented by an n-element set, we can inductively straighten sheets beginning either at the outside or inside, and get the result

$$(i_A)(A \otimes T^\dagger_{A^*}) \cdot (\bigotimes^{*}_{i_A, i_A})(A \otimes e_A \otimes A^*) \cdot (i_A)(T_A \otimes A^*) = 1_{i_A}$$

for any object A.

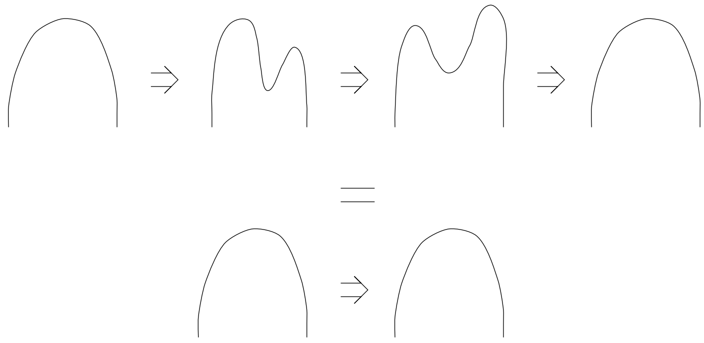

Figure 25: The swallowtail relation for Z

Next we check that $\mathcal{T}$ satisfies the further conditions to be a braided monoidal 2-category with duals.

A representative of $i_{R_{A,B}} \cdot i^*_{R_{A,B}}$ can be generated by rotating a representative of $f \otimes f^*$ that lies in $t = 1/2$ and is symmetric in $z = 1/2$ around $z = 1/2, t = 1/2$ (continuing the surface by the product structure in t to the boundary and smoothing). This particular surface consists of disjoint sheets, one for each point in a representative of $A \otimes B$, and the boundary of each sheet is a trivial loop $((x_0, y_0) \times \partial(I_3 \times I_4))$. Further, these sheets can be straightened one at a time (moving from left to right or vice versa) without moving the remaining sheets by an equivalence isotopy, so this is also a representative of the identity 2-morphism $1_{1_{A\otimes B}}$. We can similarly straighten representatives of $i^*_{R_{A,B}} \cdot i_{R_{A,B}}$, $e_{R_{A,B}} \cdot e^*_{R_{A,B}}$, and $e^*_{R_{A,B}} \cdot e_{R_{A,B}}$, which yields the result that $i_{R_{A,B}}$ and $e_{R_{A,B}}$ are unitary.

Since the 2-morphisms $R_{A,f}$, $R_{f,A}$, $\tilde{R}_{(A,B|C)}$ and $\tilde{R}_{(A|B,C)}$ are defined in terms of the trace of an isotopy, and their inverses are defined by the isotopy with time reversed, it follows that the 2-morphisms are unitary.

Finally, we show that the object Z represented by a single object is an unframed self dual object.

The writhing $W_Z \colon i_Z \Rightarrow i_{Z^*} R_{Z^*,Z}$ of Z can be represented by the trace of the first Reidemeister move, which is an isotopy in I^3, so the 2-morphism W_Z is unitary.

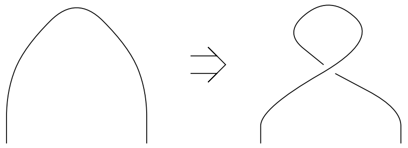

Figure 26: The writhing W_Z

The relation

$$T_A^\dagger \cdot ((A \otimes W_{A^*})(e_{A^*} \otimes A)) \cdot ((A \otimes (i_{A^*} R_{A^*,A}))R^\dagger_{A,i_A})\cdot$$

$$((A \otimes (i_{A^*} R_{A^*,A}))\tilde{R}^\dagger_{A|A,A^*}(A \otimes e_{A^*})) \cdot ((A \otimes (i_{A^*} i^*_{R_{A,A^*}}))(R^*_{A^*,A} \otimes A)(A \otimes e_{A^*}))$$

$$= T_A \cdot ((i_A \otimes A)(A \otimes (i_{R_{A^*,A}} e_A))) \cdot ((i_A \otimes A)(A \otimes (R_{A^*,A} W_A^\dagger)))\cdot$$

$$(R^{-1}_{e_{A^*},A}(A \otimes (R_{A^*,A} e_{A^*}))) \cdot ((A \otimes i_A)\tilde{R}^\dagger_{A,A^*|A}(A \otimes (R_{A^*,A} e_{A^*})))\cdot$$

$$((A \otimes i_A)(R^*_{A,A} \otimes A^*)(A \otimes (e_{R_{A^*,A}} e_{A^*})))$$

corresponds the movie move in Fig. 27, which holds in $\mathcal{T}$.

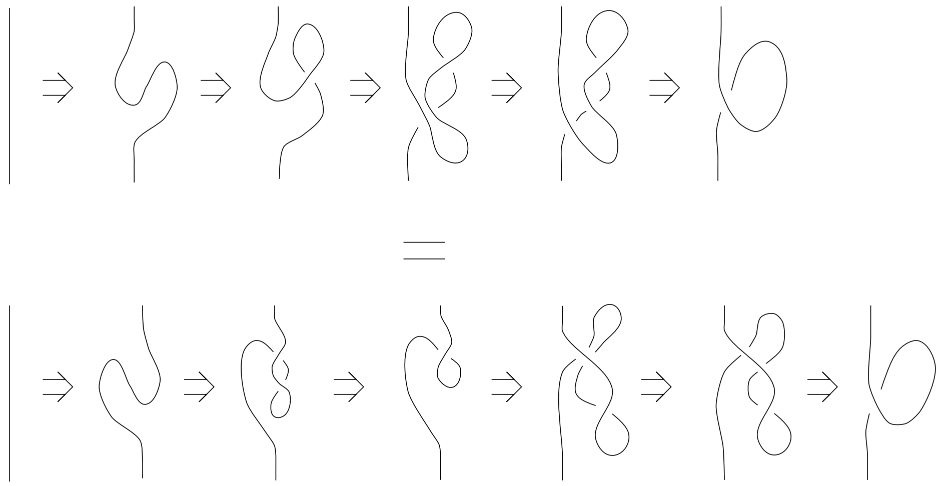

Figure 27: The relation for W_Z

$A = A^*$ for every object in $\mathcal{T}$, and so in particular $Z = Z^*$, so Z is an unframed self-dual object. $\square$

3 A Combinatorial Description of 2-Tangles

In this section we define a combinatorially described 2-category, that corresponds to the description of 2-dimensional surfaces given by Carter, Rieger and Saito in [6]. These correspond to the same surfaces, topologically, as the representative surfaces used to define 2-tangles in $\mathcal{T}$. We use this fact to show that $\mathcal{C}$ and $\mathcal{T}$ are isomorphic 2-categories. We use the isomorphism, along with the definition of $\mathcal{C}$, to show that $\mathcal{T}$ is generated by the object Z.

3.1 The combinatorial 2-category $\mathcal{C}$

3.1.1 Objects

Objects in $\mathcal{C}$ correspond to whole numbers, and are denoted by A_n, where n is a whole number. These correspond to finite point sets, where n is the number of points.

3.1.2 1-Morphisms

We may think of a 1-morphism as the diagram of a tangle, with distinct heights for maxima, minima, and crossings. A combinatorial description of such tangles (following the notation used by Carter Rieger and Saito) is given as follows: 1-morphisms are generated by $X_{i,j}, \bar{X}_{i,j}, \cup_{i,j}, \cap_{i,j}$ and 1_n, where X denotes a positive crossing, $\bar{X}$ a negative crossing, $\cup$ is a minimum, $\cap$ is a maximum, and 1_n is n strands with no crossings, maxima or minima. The subscripts i, j denote the number of strands to the left and right respectively of the crossing, maximum or minimum. We define the source and target of a tangle by the sets at the top (beginning) and bottom (end) respectively of the tangle. In particular the sources and targets of the generators are as follows: the sources and targets of both $X_{i,j}$ and $\bar{X}_{i,j}$ are A_{i+j+2}, source$(\cap_{i,j}) = A_{i+j} =$ target$(\cup_{i+j})$, and target$(\cap_{i,j}) = A_{i+j+2} =$ source$(\cup_{i,j})$.

Two 1-morphisms f and g may be composed as fg if target$(f) =$ source(g). The source of a composition of 1-morphisms fg is source(f), and target$(fg) =$ target(g). The relations are:

- the composition is associative
- $f1_m = 1_n f = f$ for any $f : A_n \to A_m$.

The 1-morphisms correspond to the words in Carter Rieger and Saito's paper [6], and the source and target correspond to the initial and terminal numbers of a word. Note that these relations do not include equivalence under Reidemeister moves, nor changing the order of maxima, minima and crossings in the corresponding tangles. We will sometimes refer to a presentation of a 1-morphism in terms of the generators as a word.

3.2 2-Morphisms

The following description of 2-morphisms is compatible with that given by Carter Rieger and Saito [6], with the notation changed to better fit a 2-category structure (describing surfaces in terms of 2-morphisms explicitly, rather than their sources and targets). Definitions of objects and 1-morphisms have been chosen to be compatible with the 2-morphism definition, and with the 2-category structure. 2-morphisms correspond to fragments of sentences in the notation used by Carter Rieger and Saito [6], where a complete sentence is required to begin and end at an empty word. If the source and target of a 2-morphism are empty (A_0), then it corresponds to a sentence, and therefore to a knotted surface in I^4 (without boundary). If the source and target are not empty, then it corresponds to a knotted surface in I^4 with boundary, which we refer to as a 2-tangle. (We assume that the boundary of the surface is contained in the boundary of I^4).

The generators correspond to the elementary string interactions (or ESIs). We add to the standard list, the 2-morphism 1_f, which corresponds to the trvial 2-tangle, or adding a word to the sentence that is identical to the previous word:

1. $I_{m,n} : 1_{m+n} \Rightarrow \cap_{m,n}\cup_{m,n}$

2. $E_{m,n} : \cup_{m,n}\cap_{m,n} \Rightarrow 1_{m+n+2}$

3. $W_{m,n} : \cap_{m,n} \Rightarrow \cap_{m,n}X_{m,n}$ and $W^{\dagger}_{m,n} : \bar{X}_{m,n}\cup_{m,n} \Rightarrow \cup_{m,n}$

4. $II_{m,n} : 1_{m+n+2} \Rightarrow X_{m,n}\bar{X}_{m,n}$

5. $S_{0;m,n} : X_{m,n+1}X_{m+1,n}X_{m,n+1} \Rightarrow X_{m+1,n}X_{m,n+1}X_{m+1,n}$

   $S_{1;m,n} : X_{m,n+1}X_{m+1,n}\bar{X}_{m,n+1} \Rightarrow \bar{X}_{m+1,n}X_{m,n+1}X_{m+1,n}$

   $S_{2;m,n} : X_{m,n+1}\bar{X}_{m+1,n}\bar{X}_{m,n+1} \Rightarrow \bar{X}_{m+1,n}\bar{X}_{m,n+1}X_{m+1,n}$

6. $H_{m,n} : \cap_{m,n+1}X_{m+1,n} \Rightarrow \cap_{m+1,n}\bar{X}_{m,n+1}$ and

   $H^{\dagger}_{m,n} : X_{m,n+1}\cup_{m+1,n} \Rightarrow \bar{X}_{m+1,n}\cup_{m,n+1}$

7. $T_{m,n} : \cap_{m,n+1}\cup_{m+1,n} \Rightarrow 1_{m+n+1}$, and $T^{\dagger}_{m,n} : 1_{m+n+1} \Rightarrow \cap_{m+1,n}\cup_{m,n+1}$

8. $N_{Y_{m,n},Z_{i,j}}: Z_{i',j}Y_{m,n'} \;\Rightarrow\; Y_{m,n}Z_{i,j}$ where Y and Z are $X, \bar{X}, \cup$ or $\cap$, and $i \geq m+2$ if $Y \neq \cup$, and $i \geq m$ if $Y = \cup$, and i, j, m, n, i' and n' are chosen so that the source and target 1-morphisms are defined (i.e. that the source and target of Y and Z satisfy the conditions for the compositions to be defined).

9. $1_f : f \Rightarrow f$, where f is a 1-morphism.

10. For each generating 2-morphism $\alpha : f \Rightarrow g$, there is a 2-morphism $\alpha^* : g \Rightarrow f$; and for each generating 2-morphism $\alpha : f \Rightarrow g$ there is a 2-morphism $\bar{\alpha} : \bar{f} \Rightarrow \bar{g}$, where $\bar{f}$, is the tangle with all crossings reversed, i.e. with X replaced by $\bar{X}$ and vice versa in a presentation of f.

Note that the subscripts on 2-morphisms, as with those of the 1-morphisms, denote the number of strands to the left and right that are not affected by the move.

These generators do not precisely correspond to the generators given by Carter Rieger and Saito, because they do not act on a whole word. To extend these generators we allow composition with 1-morphisms as follows:

For any 2-morphism α, objects $A = \text{source}(\text{source}(\alpha)) = \text{source}(\text{target}(\alpha))$ and $B = \text{target}(\text{source}(\alpha)) = \text{target}(\text{target}(\alpha))$, and 1-morphisms $f\colon A' \to A$ and $g\colon B \to B'$, the 2-morphism $f\alpha g$ is defined, and corresponds to the sentence fragment

$$(f\text{source}(\alpha)g, f\text{target}(\alpha)g).$$

The sequence of moving from one word to the next is equivalent to vertical composition:

The vertical composition of two 2-morphisms α and β , $\alpha \cdot \beta$ is defined when $\text{target}(\alpha) = \text{source}(\beta)$.

The source and target of compositions are defined as follows: $\text{source}(\alpha \cdot \beta) = \text{source}(\alpha)$, $\text{target}(\alpha \cdot \beta) = \text{target}(\beta)$ $\text{source}(f\alpha g) = f\text{source}(\alpha)g$, and $\text{target}(f\alpha g) = f\text{target}(\alpha)g$.

We will say that two allowed expressions represent the same 2-morphism if they can be related by the following list of relations. This list is parallel to the list of relations given by Carter, Rieger and Saito (Thm. 3.5.5 [6]), with the addition of the first 8, which are consistent with those moves, but which did not need to be explicitly stated in that notation:

1. $1_f \cdot \alpha = \alpha \cdot 1_f = \alpha$ for $\alpha\colon f \to g$

2. $1_{fg} = f1_g = 1_f g$.

3. $(\alpha f) \cdot (\beta f) = (\alpha \cdot \beta)f$ and $(f\alpha) \cdot (f\beta) = f(\alpha \cdot \beta)$

4. $1_f^* = 1_f$

5. $(\bar{\alpha})^* = (\bar{\alpha^*})$

6. $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$

7. $(ff')\alpha = f(f'\alpha)$, $(f\alpha)g = f(\alpha g)$ and $(\alpha g)g' = \alpha(gg')$

8. $\bar{I}_{m,n} = I_{m,n}$, $\bar{E}_{m,n} = E_{m,n}$, $\bar{T}_{m,n} = T_{m,n}$, $\bar{N}_{Y_{m,n},Z_{i,j}} = N_{\bar{Y}_{m,n},\bar{Z}_{i,j}}$, and $\bar{1}_f = 1_{\bar{f}}$

9. $W_{m,n}$ is an isomorphism and $W_{m,n}^{-1} = W_{m,n}^*$

10. $I\!I_{m,n}$ is an isomorphism and $I\!I_{m,n}^{-1} = I\!I_{m,n}^*$

11. $S_{i;m,n}$ is a 2-isomorphism for $i = 1, 2, 3$, and $S_{i;m,n}^{-1} = S_{i;m,n}^*$.

12. $$\begin{aligned}
&(Z_{i;m,n+1}X_{m+2,n}X_{m+1,n+1}X_{m,n+2}) \cdot (C_{m+1,n+1}B_{m,n+2}Z_{A;m+1,n}X_{m,n+2})\\
&\cdot(C_{m+1,n+1}N^*_{B_{m,n+2},X_{m+2,n}}X_{m+1,n+1}A_{m+2,n}X_{m,n+2})\\
&\cdot(C_{m+1,n+1}X_{m+2,n}B_{m,n+2}X_{m+1,n+1}N_{X_{m,n+2},A_{m+2,n}})\\
&\cdot(C_{m+1,n+1}X_{m+2,n}Z_{B;m,n+1}A_{m+2,n})\\
&\cdot(Z_{C;m+1,n}X_{m,n+2}B_{m+1,n+1}A_{m+2,n})\\
&\cdot(X_{m+2,n}X_{m+1,n+1}N_{X_{m,n+2},C_{m+2,n}}B_{m+1,n+1}A_{m+2,n})\\
&= (A_{m,n+2}B_{m+1,n+1}N^*_{C_{m,n+2}X_{m+2,n}}X_{m+1,n+1}X_{m,n+2})\\
&\cdot(A_{m,n+2}B_{m+1,n+1}X_{m+2,n}Z_{C;m,n+1})\\
&\cdot(A_{m,n+2}Z_{B;m+1,n}X_{m,n+2}C_{m+1,n+1})\\
&\cdot(N^*_{A_{m,n+2},X_{m+2,n}}X_{m+1,n+1}X_{m+2,n}X_{m,n+2}X_{m+1,n+1})\\
&\cdot(X_{m+2,n}A_{m,n+2}X_{m+1,n+1}N_{X_{m,n+2},B_{m+2,n}}C_{m+1,n+1})\\
&\cdot(X_{m+2,n}Z_{A;m,n+1}B_{m+2,n}C_{m+1,n+1})\\
&\cdot(X_{m+2,n}X_{m+1,n+1}X_{m,n+2}Z_{i;m+1,n})
\end{aligned}$$

where $Z = S$ or $\bar{S}$, $i \in \{1,2,3\}$, $A, B, C = X$ or $\bar{X}$, $\mathrm{source}(Z_{i;m,n+1}) = A_{m,n+2}B_{m+1,n+1}C_{m,n+2}$ and $Z_{Y;j,k} = S_{0,j,k}$ if $Y = X$ and $Z_{Y;j,k} = \bar{S}_{1,j,k}$ if $Y = \bar{X}$ for $Y = A, B, C$.

13. $(\cap_{m+1,n} II_{m,n+1} \bar{X}_{m+1,n} X_{m,n+1}) \cdot (\bar{H}^*_{m,n} \bar{X}_{m,n+1} \bar{X}_{m+1,n} X_{m,n+1})$
$\cdot(\cap_{m,n+1} \bar{X}_{m+1,n} \bar{S}_{1;m,n}) \cdot (\cap_{m,n+1} \bar{II}^*_{m+1,n} \bar{X}_{m,n+1} \bar{X}_{m+1,n})$
$\cdot(\bar{W}^*_{m,n+1} \bar{X}_{m+1,n})$
$= (\bar{W}^*_{m+1,n} X_{m,n+1}) \cdot \bar{H}^*_{m,n}$

14. $(\cap_{m+1,n} II_{m,n+1} X_{m+1,n} X_{m,n+1}) \cdot (\bar{H}^*_{m,n} \bar{X}_{m,n+1} X_{m+1,n} X_{m,n+1})$
$\cdot(\cap_{m,n+1} \bar{X}_{m+1,n} \bar{S}_{2;m,n}) \cdot (\cap_{m,n+1} \bar{II}^*_{m+1,n} X_{m,n+1} \bar{X}_{m+1,n}) \cdot (W^*_{m,n+1} \bar{X}_{m+1,n})$
$= (W^*_{m+1,n} X_{m,n+1}) \cdot \bar{H}^*_{m,n}$

15. $T_{m,n}$ is an isomorphism and $T^{-1}_{m,n} = T^*_{m,n}$

16. $(\cap_{m,n} T^{\dagger}_{m+1,n}) \cdot (N^*_{\cap_{m,n},\cap_{m+2,n}} \cup_{m+1,n+1}) \cdot (\cap_{m,n} T_{m,n+1}) = 1_{\cap_{m,n}}$

17. $H_{m,n}$ is an isomorphism, and $H^{-1}_{m,n} = H^*_{m,n}$

18. $(\cap_{m+1,n} H^{\dagger}_{m,n}) \cdot (\bar{W}^*_{m+1,n} \cup_{m,n+1}) \cdot T^{\dagger *}_{m,n} = (\bar{H}^*_{m,n} \cup_{m+1,n}) \cdot (\cap_{m,n+1} W^{\dagger}_{m+1,n}) \cdot T_{m,n}$

19. $(\cap_{m,n+1} \bar{H}^{\dagger *}_{m,n}) \cdot (\bar{W}^*_{m,n+1} \cup_{m+1,n}) \cdot T_{m,n} = (H_{m,n} \cup_{m,n+1}) \cdot (\cap_{m+1,n} W^{\dagger}_{m,n+1}) \cdot T^{\dagger *}_{m,n}$

20. $(\cap_{m+2,n} H^{\dagger *}_{m,n+1}) \cdot (N_{X_{m,n} \cap_{m+2,n}} \cup_{m+1,n+1}) \cdot (X_{m,n} T^{\dagger *}_{m+1,n})$
$= (H^*_{m+1,n} \cup_{m,n+2}) \cdot (\cap_{m+1,n+1} N_{\cup_{m,n+2}, X_{m,n}}) \cdot (T^{\dagger *}_{m,n+1} X_{m,n})$

21. $(\bar{H}_{m+1,n} A_{i;m,n+2} B_{i;m+1,n+1}) \cdot (\cap_{m+2,n} Z_{i;m,n+1})$
$\cdot(N_{B_{i;m,n},\cap_{m+2,n}} A_{i;m+1,n+1} X_{m,n+2})$
$\cdot(B_{i;m,n} J_{A;m+1,n+1} X_{m,n+2}) \cdot (B_{i;m,n} \cap_{m+1,n+1} N_{X_{m,n+2},\bar{A}_{i;m+2,n}})$
$= (\cap_{m+1,n+1} N_{A_{i;m,n+2},\bar{X}_{m+2,n}} B_{i;m+1,n+1}) \cdot (J_{A;m,n+1} \bar{X}_{m+2,n} B_{i;m+1,n+1})$

$\cdot(\cap_{m,n+2} W_{1;m+1,n}) \cdot (N^*_{\cap_{m,n+2}, B_{i;m+2,n}} \bar{X}_{m+1,n+1} \bar{A}_{i;m+2,n})$

$\cdot(B_{i;m,n} \bar{H}_{m,n+1} \bar{A}_{i;m+2,n})$

where $A_{0;j,k} = X_{j,k}, B_{0;j,k} = X_{j,k}$; $A_{1;j,k} = X_{j,k}, B_{1;j,k} = \bar{X}_{j,k}$; or $A_{2;j,k} = \bar{X}_{j,k}, B_{2;j,k} = \bar{X}_{j,k}$, $Z_{0;j,k} = S^*_{0;j,k}, Z_{1;j,k} = \bar{S}^*_{2;j,k}$, $Z_{2;j,k} = \bar{S}^*_{1;j,k}$, $W_{0;j,k} = \bar{S}_{1;j,k}$ $W_{1;j,k} = \bar{S}_{0;j,k}$, $W_{2;j,k} = S_{2;j,k}$ and $J_{X;j,k} = \bar{H}^*_{j,k}$, and $J_{\bar{X};j,k} = H^*_{j,k}$.

22. $N_{Y_{m,n}, Z_{I,j}}$ is an isomorphism for $Y, Z \in \{X, \bar{X}, \cap, \cup\}$, and
$N^{-1}_{Y_{m,n}, Z_{I,j}} = N^*_{Y_{m,n}, Z_{I,j}}$, where m, n, l, j are chosen so that $N_{Y_{m,n}, Z_{I,j}}$ is defined.

23. $(N^*_{Y_{m,n}, Y'_{i,j}} Y''_{k,l}) \cdot (Y'_{i',j} N^*_{Y_{m,n'}, Y''_{k,l}}) \cdot (N^*_{Y'_{i',j}, Y''_{k',l}} Y_{m,n''})$
$= (Y_{m,n} N^*_{Y'_{i,j}, Y''_{k,l}}) \cdot (N^*_{Y_{m,n}, Y''_{k'',l}} Y'_{i,j'}) \cdot (Y''_{k''',l} N^*_{Y_{m,n'''}, Y'_{i,j'}})$
where $Y, Y', Y'' \in \{X, \bar{X}, \cap, \cup\}$, and the subscripts are chosen so that all the above 2-morphisms and compositions are defined. The other variations of this relation follow from this one and the invertibility of $N_{\cdot,\cdot}$.

24. $(N^*_{Y_{m,n}, A_{j,k+1}} B_{j+1,k} C_{j,k+1}) \cdot (A_{j',k+1} N^*_{Y_{m,n}, B_{j+1,k}} C_{j,k+1})$
$\cdot(A_{j',k+1} B_{j'+1,k} N^*_{Y_{m,n}, C_{j,k+1}}) \cdot (Z_{A,B,C;j',k} Y_{m,n})$
$= (Y_{m,n} Z_{A,B,C;j,k}) \cdot (N^*_{Y_{m,n}, C_{j+1,k}} B_{j,k+1} A_{j+1,k})$
$\cdot(C_{j'+1,k} N^*_{Y_{m,n}, B_{j,k+1}} A_{j+1,k}) \cdot (C_{j'+1,k} B_{j',k+1} N^*_{Y_{m,n}, A_{j+1,k}})$
where $A, B, C \in \{X, \bar{X}\}$ satisfy $A_{j,k+1} B_{j+1,k} C_{j,k+1} = \text{source}(Z_{A,B,C;j,k})$ for $Z_{A,B,C;j,k} = S_{i;j,k}$ or $\bar{S}_{i;j,k}$, and $m+1 < j$. Also similar relations where $N^*_{Y_{m,n}, \chi_{j,k}}$ is replaced by $N_{\chi_{j,k}, Y_{m,n}}$, $m > j+2$ for $\chi = A, B, C$, and $j' = j$ or $j \pm 2$, depending on Y.

25. $(N^*_{Y_{m,n}, \cap_{j,k+1}} \cup_{j+1,k}) \cdot (\cap_{j',k+1} N^*_{Y_{m,n+2}, \cup_{j+1,k}}) \cdot (T_{j',k} Y_{m,n}) = Y_{m,n} T_{j,k}$
where $m < j - 1$, and $j' = j$ or $j \pm 2$, depending on Y. Also a similar relation where $N^*_{Y_{m,n}, \chi_{j,k}}$ is replaced by $N_{\chi_{j,k}, Y_{m,n}}$ for $\chi = \cap, \cup$, and $n < k - 1$.

26. $(N^*_{Y_{m,n},\cap_{j,k}}) \cdot (Y_{m,n} W_{j,k}) = (W_{j',k} Y_{m,n}) \cdot (\cap_{j',k} N^*_{Y_{m,n},X_{j,k}}) \cdot (N^*_{Y_{m,n},\cap_{j,k}} X_{j,k})$ where $m < j-1$, and $j' = j$ or $j \pm 2$, depending on Y. Also a similar relation where $N^*_{Y_{m,n},\chi_{j,k}}$ is replaced by $N_{\chi_{j,k},Y_{m,n}}$ for $\chi = \cap, X$, and $n < k-1$.

27. $(N^*_{Y_{m,n},\cap_{j+1,k}} \bar{X}_{j,k+1}) \cdot (\cap_{j'+1,k} N^*_{Y_{m,n+2},\bar{X}_{j,k+1}}) \cdot (\bar{H}^*_{j',k} Y_{m,n+2})$
$= (Y_{m,n} \bar{H}^*_{j,k}) \cdot (N^*_{Y_{m,n},\cap_{j,k+1}} X_{j+1,k}) \cdot (\cap_{j',k+1} N^*_{Y_{m,n+2},X_{j+1,k}})$
where $m < j-1$, and $j' = j$ or $j \pm 2$, depending on Y. Also a similar relation, where $N^*_{Y_{m,n},\chi_{j,k}}$ is replaced by $N_{\chi_{j,k},Y_{m,n}}$ for $\chi = \cap.X, \bar{X}$, and $n < k-1$.

28. $(\cap_{m+1,n+1} N^*_{X_{m,n+2},\bar{X}_{m+2,n}} \cup_{m+1,n+1}) \cdot (\cap_{m+1,n+1} \bar{X}_{m+2,n} H^{\dagger}_{m,n+1})$
$\cdot (\bar{H}_{m+1,n} \bar{X}_{m+1,n+1} \cup_{m,n+2}) \cdot (\cap_{m+2,n} II^*_{m+1,n+1} \cup_{m,n+2}) \cdot (N_{\cup_{m,n},\cap_{m,n}})$
$= (\bar{H}^*_{m,n+1} \bar{X}_{m+2,n} \cup_{m+1,n+1}) \cdot (\cap_{m,n+2} \bar{X}_{m+1,n+1} H^{\dagger *}_{m+1,n})$
$\cdot (\cap_{m,n+2} \bar{II}^*_{m+1,n+1} \cup_{m+2,n}) \cdot (N^*_{\cap_{m,n+2},\cup_{m+2,n}})$

29. $(Y_{m,n} Z_{i,j}) \cdot (N^*_{Y_{m,n},\chi_{i,j}} \chi^*_{i,j}) = (Z_{i',j} Y_{m,n}) \cdot (\chi_{i',j} N_{Y_{m,n'},\chi^*_{i,j}})$

    where $Z \in \{II, I, E\}$, and $\mathrm{target}(Z) = \chi\chi^*$, and the subscripts are chosen to be compatible with the restrictions for defining N and the compositions.

    Also a similar equation, with $n \leq j$.

30. $(X_{m,n} \bar{II}_{m,n}) \cdot (II^*_{m,n} X_{m,n}) = 1_{X_{m,n}}$

31. $(I_{m,n} \cap_{m,n}) \cdot (\cap_{m,n} E_{m,n}) = 1_{\cap_{m,n}}$

32. $(\cap_{m+1,n} E^*_{m,n+1}) \cdot (T^{\dagger *}_{m,n} \cap_{m,n+1}) = (T^*_{m,n} \cap_{m+1,n}) \cdot (\cap_{m,n+1} E_{m+1,n})$

33. $(II_{m,n+1} A_{m+1,n} B_{m,n+1}) \cdot (X_{m,n+1} \bar{S}_{i;m,n})$
$= (A_{m+1,n} B_{m,n+1} II_{m+1,n}) \cdot (Z_{i;m,n} \bar{X}_{m+1,n})$
where $A, B = X$ or $\bar{X}$, and i satisfy $\mathrm{source}(\bar{S}_{i;m,n}) = \bar{X}_{m,n+1} A_{m+1,n} B_{m,n+1}$, and $Z_{i;m,n} = S^*_{j;m,n}$ or $\bar{S}^*_{j;m,n}$ satisfies $\mathrm{source}(Z_{i;m,n}) = A_{m+1,n} B_{m,n+1} X_{m+1,n}$.

34. $(\cap_{m,n} II_{m,n}) \cdot (W^*_{m,n} \bar{X}_{m,n}) = (\bar{W}_{m,n})$

35. $I_{m,n} \cdot (W_{m,n} \cup_{m,n}) = I_{m,n} \cdot (\cap_{m,n} \bar{W}^{\dagger *}_{m,n})$

36. $I_{m,n+1} \cdot (\cap_{m,n+1} II_{m+1,n} \cup_{m,n+1}) \cdot (H_{m,n} \bar{X}_{m+1,n} \cup_{m,n+1})$

$= I_{m+1,n} \cdot (\cap_{m+1,n} \bar{II}_{m,n+1} \cup_{m+1,n}) \cdot (\cap_{m+1,n} \bar{X}_{m,n+1} H^{\dagger}_{m,n})$

37. $E^*_{m,n} \cdot (\cup_{m,n} W_{m,n}) \cdot (E_{m,n} X_{m,n}) = E^*_{m,n} \cdot (\bar{W}^{\dagger *}_{m,n} \cap_{m,n}) \cdot (X_{m,n} E_{m,n})$

38. $(H^{\dagger *}_{m,n} \cap_{m+1,n} \bar{X}_{m,n+1}) \cdot (X_{m,n+1} E_{m+1,n} \bar{X}_{m,n+1}) \cdot II^*_{m,n+1}$

$= (\bar{X}_{m+1,n} \cup_{m,n+1} H^*_{m,n}) \cdot (\bar{X}_{m+1,n} E_{m,n+1} X_{m+1,n}) \cdot (\bar{II}^*_{m+1,n})$

39. $(\alpha \mathrm{source}(\beta)) \cdot (\mathrm{target}(\alpha)\beta) = (\mathrm{source}(\alpha)\beta) \cdot (\alpha \mathrm{target}(\beta))$

for 2-morphisms α and β.

In addition to each of the above relations $\alpha = \beta$, we include the analogous relations:

- $\alpha^* = \beta^*$, where Z^* is defined above for the generating 2-morphisms, and the others come from $Z^{**} = Z$, $(Y \cdot Z)^* = Z^* \cdot Y^*$, $(Zf)^* = Z^* f$ and $(fZ)^* = fZ^*$
- $\alpha^\dagger = \beta^\dagger$, where $Z^\dagger$ is defined above for the generating 2-morphisms P, H, and T, $I^\dagger = I$,$E^\dagger = E$, $II^\dagger = II$, $S^\dagger_{0,m,n} = \bar{S}_{0,m,n}$, $S^\dagger_{1,m,n} = S_{2,m,n}$, $N^\dagger_{f_{m,n}, g_{i,j}} = N^*_{f^*_{m,n}, g^*_{i,j}}$ and $1^\dagger_f = 1_{f^*}$ (where $X^* = \bar{X}$, $\cap^* = \cup$, $f^{**} = f$, and $(fg)^* = g^* f^*$); then one can find $\alpha^\dagger$ for any 2-morphism if we specify that: $\alpha^{\dagger\dagger} = \alpha$, $(\alpha \cdot \beta)^\dagger = \beta^\dagger \cdot \alpha^\dagger$, and $(\alpha \circ \beta)^\dagger = \alpha^\dagger \circ \beta^\dagger$
- $\bar{\alpha} = \bar{\beta}$, where $\bar{Z}$ is defined above for the generating 2-morphisms, and $\bar{\bar{Z}} = Z$, $\alpha \bar{\cdot} \beta = \bar{\alpha} \cdot \bar{\beta}$, and $\bar{f\alpha g} = \bar{f}\bar{\alpha}\bar{g}$.

  Since the actions $\bullet^*$, $\bar{\bullet}$, and $\bullet^\dagger$ all correspond to reflections, we also add the relations that $\bar{\bullet}^* = \bar{\bullet^*}$, $\bullet^{*\dagger} = \bullet^{\dagger *}$, and $\bar{\bullet}^\dagger = \bar{\bullet^\dagger}$.

Lemma 11. *$\mathcal{C}$ is a 2-category.*

The sources and targets of the 2-morphisms in each relation are equal, so the source and target are well defined.

Composition of morphisms and vertical composition of 2-morphisms are the compositions defined above. We define the horizontal composition of two 2-morphisms α and β such that target(source(α)) = source(source(β)) to be

$$\alpha \circ \beta = (\alpha \mathrm{source}(\beta)) \cdot (\mathrm{target}(\alpha)\beta) = (\mathrm{source}(\alpha)\beta) \cdot (\alpha \mathrm{target}(\beta)).$$

It follows from relation 39 that these formulations are consistent.

We define $1_{A_n} = 1_n$, and 1_f is defined above; then the relations: $1_f = f1_B = f$, $1_f \cdot \alpha = \alpha \cdot 1_g = \alpha$, and $1_{1_A} \circ \alpha = \alpha \circ 1_{1_B} = \alpha$, for $\alpha\colon f \to g$ and $f, g : A \to B$, and $1_{fg} = 1_f \circ 1_g$, and the associativity of and vertical composition are included in the relations above. The associativity of the horizontal composition follows from $(ff')\alpha = f(f'\alpha)$,and $(\alpha g)g' = \alpha(gg')$. Note that for any generating 2-morphism

$$\mathrm{target}(\mathrm{source}(\alpha)) = \mathrm{target}(\mathrm{target}(\alpha)),$$

and

$$\mathrm{source}(\mathrm{target}(\alpha)) = \mathrm{source}(\mathrm{source}(\alpha)).$$

Further, it is easy to show that

$$\mathrm{target}(\mathrm{source}(\alpha \cdot \beta)) = \mathrm{target}(\mathrm{target}(\alpha \cdot \beta))$$

$$\mathrm{source}(\mathrm{target}(\alpha \cdot \beta)) = \mathrm{source}(\mathrm{source}(\alpha \circ \beta))$$

and similarly for $f\alpha g$, hence the equations

$$\mathrm{target}(\mathrm{source}(\alpha)) = \mathrm{target}(\mathrm{target}(\alpha)),$$

and

$$\mathrm{source}(\mathrm{target}(\alpha)) = \mathrm{source}(\mathrm{source}(\alpha))$$

hold for any 2-morphism in $\mathcal{C}$.

The final equation

$$(\alpha \circ \beta) \cdot (\alpha' \circ \beta') = (\alpha \cdot \alpha') \circ (\beta \cdot \beta')$$

follows from

$$(\alpha f) \cdot (\beta f) = (\alpha \cdot \beta) f$$

$$(f \alpha) \cdot (f \beta) = f(\alpha \cdot \beta)$$

and

$$(\alpha \circ \mathrm{source}(\beta)) \cdot (\mathrm{target}(\alpha) \circ \beta) = (\mathrm{source}(\alpha) \circ \beta) \cdot (\alpha \circ \mathrm{target}(\beta)) = \alpha \circ \beta.$$

□

3.3 $\mathcal{T}$ and $\mathcal{C}$ are isomorphic 2-categories

The main idea behind showing that $\mathcal{T}$ and $\mathcal{C}$ are isomorphic 2-categories comes from realizing that they are two different formulations of the structure Carter, Rieger and Saito use to describe 2-tangles and 2-knots. The surfaces they consider have generic projections into R^3 and a retinal plane; the singularities of the projection into the retinal plane correspond to the elementary string interactions which are the generators for the combinatorial description. In $\mathcal{T}$, we consider surfaces in a 4-cube with generic projections into a 3-cube and a square, which are analogous to the projections into R^3 and the retinal plane. These surfaces can be thought of as a piece cut out of such a 2-knot, where there are no (retinal plane) singularities on the boundary. The combinatorial description of these 2-tangles corresponds to the definition of $\mathcal{C}$.

Definition 12. *Two 2-categories $\mathcal{M}, \mathcal{N}$ are isomorphic if there is a 2-functor $F: \mathcal{M} \to \mathcal{N}$ with an inverse 2-functor $G: \mathcal{N} \to \mathcal{M}$.*

Theorem 13. *$\mathcal{T}$ and $\mathcal{C}$ are isomorphic 2-categories.*

We define a 2-functor $F\colon \mathcal{T} \to \mathcal{C}$ as follows:

Each object $A \in \mathcal{T}$ is determined by the number of points in a representative of A. We define $F(A) = A_n$, where n is the number of points in a representative of A. Clearly, F is well defined and bijective on objects, since objects in both $\mathcal{T}$ and $\mathcal{C}$ are uniquely determined by a whole number n.

Each 1-morphism $f \in \mathcal{T}$ is represented by a tangle in $I_1 \times I_2 \times I_3$ whose diagram (under the projection to $I_1 \times I_2$) has crossings, maxima, and minima at distinct heights. This tangle diagram corresponds to a 1-morphism in $\mathcal{C}$. Let $F(f_{\mathcal{T}}) = f_{\mathcal{C}}$ where $f_{\mathcal{T}} \in \mathcal{T}$ and $f_{\mathcal{C}}$ is the combinatorial description of the diagram of a tangle representing f_T. This description is an ordered listing of crossing and critical point data. The equivalence relation for 1-morphisms in $\mathcal{T}$ does not allow heights of crossings, maxima and minima to change order, hence F is well defined on 1-morphisms. Further, any two tangles that have the same combinatorial description can be related by an equivalence isotopy, and therefore represent the same 1-morphism in $\mathcal{T}$. Every allowed combinatorial tangle can be realized as a smooth geometric tangle, and is therefore the image of some 1-morphism. Thus, we may conclude that F is bijective on 1-morphisms. Note that $F(1_{F^{-1}(A_n)}) = 1_n$, so $F(1_A) = 1_{F(A)}$.

The source and target of a 1-morphism are determined by the sets at the top and bottom of a representative tangle of a 1-morphism both $\mathcal{T}$, and by a similar tangle given by a combiunatorial 1-morphim in $\mathcal{C}$. Since F is defined on 1-morphisms by such tangles, it is clear that $\text{source}(F(f)) = F(\text{source}(f))$, and $\text{target}(F(f)) = F(\text{target}(f))$.

In the notation we use to describe tangles, the crossings and critical points are listed in order from top to bottom. The composition of 1-morphisms is consistent with this in both $\mathcal{T}$ and $\mathcal{C}$: in $\mathcal{T}$, the 1-morphism fg is a gluing with f above g, and in $\mathcal{C}$ the $f'g'$ is the juxtaposition with f' preceding g'. Hence, $F(fg) = F(f)F(g)$.

Each 2-morphism $\alpha_{\mathcal{T}} \in \mathcal{T}$ has a representative α' that is a surface in $I_1 \times I_2 \times I_3 \times I_4$ whose projections to $I_2 \times I_3 \times I_4$, $I_3 \times I_4$, and I_4 are generic. The singularities of the projection of the surface to $I_3 \times I_4$ are local singularities, and they correspond to the singularities of the projection of a 2-knot to the retinal plane; these singularities are associated with ESI's which correspond to 2-morphism generators in $\mathcal{C}$, hence the surface has a combinatorial description which is a 2-morphism in $\mathcal{C}$. We let $F(\alpha_{\mathcal{T}}) = \alpha_{\mathcal{C}}$, where $\alpha_{\mathcal{C}}$ is the combinatorial description of such a representative. Any two such representatives of α are related by an equivalence isotopy; we may assume that this isotopy has only generic singularities. The proof of thm 4.5 in [6] shows that a singularity in the isotopy corresponds to one of the movie moves. The movie moves are equivalent to the relations that define 2-morphisms in $\mathcal{C}$, so the combinatorial descriptions of the representatives define the same 2-morphism in $\mathcal{C}$. Thus, $F(\alpha)$ does not depend on the representative of α, and F is well defined on 2-morphisms.

Clearly F is surjective on 2-morphisms, since every combinatorial 2-morphism, when considered as a surface, is a representative of some 2-morphism in $\mathcal{T}$. To see that F is also injective on 2-morphisms we consider two such combinatorial words that give the same 2-morphism in $\mathcal{C}$. There is a series of relations whereby one can get from one to the other. Each relation is associated with, and indeed comes from, an isotopy between two surfaces. We may assume that each of these isotopies fixes the tangles that represent the source and target, and is a smooth ambient isotopy; thus each relation can be associated with an isotopy that is a 2-morphism equivalence between the corresponding surfaces in $I_1 \times I_2 \times I_3 \times I_4$, and the surfaces represent the same 2-morphism in $\mathcal{T}$. Note that $F(1_{F^{-1}(f)}) = 1_f$, which implies $F(1_f) = 1_{F(f)}$.

The source and target of a 2-morphism are determined by the tangle at the beginning ($t = 0$) and ending ($t = 1$) of a representing surface of a 2-morphism in $\mathcal{T}$, and by the tangle at the beginning and ending of a surface determined by the combinatorial description of a 2-morphism in $\mathcal{C}$. Since we may assume that the surface is the same for both α and $F(\alpha)$, it is clear that $\text{source}(F(\alpha)) = F(\text{source}(\alpha))$, and $\text{target}(F(\alpha)) = F(\text{target}(\alpha))$.

Vertical composition $\alpha \cdot \beta$ in $\mathcal{T}$ is represented by a gluing of representatives where α precedes β (in the t direction). When we write a combinatorial description of a surface, it is ordered by t, where 2-morphism generators of $\mathcal{C}$ corresponding to earlier singularities precede those corresponding to later singularities. The composition $\alpha' \cdot \beta'$ thus is consistent in order to the composition in $\mathcal{T}$. Hence, $F(\alpha \cdot \beta) = F(\alpha) \cdot F(\beta)$.

In both $\mathcal{T}$ and $\mathcal{C}$, $f\alpha g$ corresponds to a the same gluing of 2-tangles, so we conclude that $F(f\alpha g) = F(f)F(\alpha)F(g)$.

Let $\alpha: f \to g$ and $\beta: f' \to g'$ be 2-morphisms in $\mathcal{T}$ or $\mathcal{C}$ for which $\alpha \circ \beta$ is defined. We may apply 2-category axioms to get $\alpha \circ \beta = (\alpha f') \cdot (g\beta)$, then $F(\alpha \cdot \beta) = F(\alpha) \cdot F(\beta)$ implies that $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$. $\square$

Corollary 14. *$\mathcal{C}$ is a braided monoidal 2-category with duals with an unframed self-dual object A_1.*

Since $\mathcal{T}$ and $\mathcal{C}$ are isomorphic 2-categories, and since $\mathcal{T}$ is a braided monoidal 2-category with duals, with an unframed self-dual object, we may also conclude that $\mathcal{C}$ is a braided monoidal 2-category with duals, with an unframed self dual object, where the tensor product, braiding, and other special morphisms and structures are determined by their action in $\mathcal{T}$, for example $R^{\mathcal{C}}_{A,B} = F(R^{\mathcal{T}}_{F^{-1}(A),F^{-1}(B)})$. We list these below:

1. $I = A_0$

2. $A_m \otimes A_n = A_{m+n}$

3. $A_n \otimes Y_{i,j} = Y_{n+i,j}$ and $Y_{i,j} \otimes A_n = Y_{i,j+n}$ for $Y = X, \bar{X}, \cap$ or $\cup$; and $A_n \otimes 1_m = 1_n \otimes A_m = 1_{n+m}$; $A_n \otimes f$ is determined for any 1-morphism f by the monoidal relation $A \otimes (fg) = (A \otimes f)(A \otimes g)$.

4. $\otimes_{Y_{i,j},Z_{m,n}} = N_{Y_{i,j+z},Z_{y+m,n}}$, $\otimes_{Y_{i,j},1_n} = 1_{Y_{i,j+n}}$ and $\otimes_{A_n,Y_{i,j}} = 1_{Y_{n+i,j}}$ where Y and Z are $X, \bar{X}, \cup$ or $\cap$, and source$(Z_{m,n}) = z$, and target$(Y_{i,j}) = y$; $\otimes_{f,g}$

for any 1-morphisms f, g is determined by the monoidal relations: $\bigotimes_{ff',g} = (\bigotimes_{f,g} f')(f \bigotimes_{f',g})$, and $\bigotimes_{f,gg'} = (g \bigotimes_{f,g'})(\bigotimes_{f,g} g')$.

5. $A_n \otimes Y_{i,j} = Y_{n+i,j}$ and $Y_{i,j} \otimes A_n = Y_{i,j+n}$ for $Y = I, E, W, II, S_{i,}, T, H$, and $\bar{\bullet}$, $\bullet^*$, and $\bullet^\dagger$ of any of those; $A_n \otimes N_{Y_{i,j},Z_{k,l}} = N_{Y_{n+i,j},Z_{n+k,l}}$ and $N_{Y_{i,j},Z_{k,l}} \otimes A_n = N_{Y_{i,j+n},Z_{k,l+n}}$; $A_n \otimes 1_f = 1_{A_n \otimes f}$ and $1_f \otimes A_n = 1_{f \otimes A_n}$; $A_n \otimes \alpha$, and $\alpha \otimes A_n$ are determined by these from the 2-functor structure properties of $\otimes$.

6. $R_{A_1,A_1} = X_{0,0}$; R_{A_n,A_m} can be expressed in terms of X and the tensor product from the equations: $R_{A_1,A_{m+n}} = (R_{A_1,A_n} \otimes A_n)(A_m \otimes R_{A_1,A_n})$, and $R_{A_{m+n},A_i} = (A_m \otimes R_{A_n,A_i})(R_{A_m,A_i} \otimes A_n)$.

7. $\tilde{R}_{(A,B|C)} = 1_{(A \otimes R_{B,C})(R_{A,C} \otimes B)}$.

8. $\tilde{R}_{(A_1|B,C)} = 1_{(R_{A_1,B} \otimes C)(B \otimes R_{A_1,C})}$; $\tilde{R}_{(A_n|B,C)}$ is determined by the above, together with the braiding relation $((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$.

9. $R_{A_1,X_{0,0}} = S^*_{0;0,0}$, $R_{X_{0,0},A_1} = S_{0;0,0}$, $R_{A_1,\bar{X}_{0,0}} = S^*_{1;0,0}$, $R_{\bar{X}_{0,0},A_1} = \bar{S}_{2;0,0}$,

   $R_{A_1,\cap_{0,0}} = (\bar{H}^* X_{1,0}) \cdot (\cap_{0,1} \bar{II}^*_{1,0})$, $R_{\cap_{0,0},A_1} = (H_{0,0} X_{0,1}) \cdot (\cap_{1,0} \bar{II}^*_{0,1})$,

   $R_{A_1,\cup_{0,0}} = (II^*_{0,1} \cup_{1,0})(\bar{H}^\dagger_{0,0})$, $R_{\cup_{0,0},A_1} = (II_{1,0} \cup_{0,1})(X_{1,0} H^{\dagger *})$,

   $R_{A_1,1_n} = 1_{R_{A_1,A_n}}$, and $R_{1_n,A_1} = 1_{R_{A_n,A_1}}$; $R_{A,f}$, and $R_{f,A}$ for any 1-morphism f and object A are determined from the above by the prism braiding relations $((\bullet \otimes \bullet) \otimes \rightarrow)$, $(\rightarrow \otimes (\bullet \otimes \bullet))$, $((\rightarrow \otimes \bullet) \otimes \bullet)$, $((\bullet \otimes \rightarrow) \otimes \bullet)$, $\bullet \otimes (\rightarrow \otimes \bullet))$, and $(\bullet \otimes (\bullet \otimes \rightarrow))$.

10. $i_{A_1} = \cap_{0,0}$; i_{A_n} is determined by the duals relation: $i_{A \otimes B} = i_A \cdot (A \otimes i_B \otimes A^*)$.

11. $e_{A_1} = \cup_{0,0}$, e_{A_n} is determined by the duals relation: $e_{A \otimes B} = (A^* \otimes e_B \otimes A) \cdot e_A$

12. $i_I = e_I = 1_I$

13. $i_{\cup_{0,0}} = E^*_{0,0}$, $i_{\cap_{0,0}} = I_{0,0}$, $i_{X_{0,0}} = II_{0,0}$, $i_{\bar{X}_{0,0}} = \bar{II}_{0,0}$, $i_{1_n} = 1_{1_n}$. Then i_f and e_f are determined from the above by the duals relations $i_{fg} = i_f \cdot f \circ i_g \circ f^*$ and $e_f = i^*_{f^*}$.

14. $T_{A_1} = T_{0,0}$; T_{A_n} is determined from this by the duals relation:

$$T_{A\otimes B} = ((i_A \otimes A \otimes B)(A \otimes \bigotimes_{i_B, e_A}^{-1} \otimes B)(A \otimes B \otimes e_B)) \cdot ((T_A \otimes B) \circ (A \otimes T_B)).$$

15. A_1 is the unframed self dual object in $\mathcal{C}$, and $W_{A_1} = W_{0,0}$.

16. the dual map * on 1- and 2-morphisms corresponds to the action * as defined for $\mathcal{C}$.

17. We can show from the above, and the relations defining † that $\alpha^\dagger$ as defined for 2-morphisms in $\mathcal{T}$ corresponds to the $\alpha^\dagger$ as defined for $\mathcal{C}$. For example, for the generator H, this follows from relations 30, 36, 39. □

We use the isomorphism between $\mathcal{T}$ and $\mathcal{C}$ in the following theorems, and sometimes use the notation and formulation of $\mathcal{C}$ to prove properties of $\mathcal{T}$.

4 The Free Category Structure of $\mathcal{T}$

In this section we define what we mean for a braided monoidal 2-category with duals to be generated by an unframed self-dual object, and show that $\mathcal{T}$ is generated by Z in this sense. We define a strict monoidal functor which preserves braiding and duals on an unframed self-dual object. We then show what that there is such a functor from $\mathcal{T}$ to any braided monoidal 2-category with duals containing an unframed self-dual object.

Definition 15. *We say a braided monoidal 2-category is* generated *by an unframed self-dual object Z if:*

1. Every object is a tensor product of copies of Z.
2. Every morphism can be obtained by composition from:
   - 1_Z,
   - i_Z,
   - $R_{Z,Z}$,
   - tensor products of arbitrary objects with the above morphisms,
   - duals of the above morphisms.
3. Every 2-morphism can be obtained by horizontal and vertical composition from:
   - 2-morphisms 1_f for arbitrary morphisms f,
   - 2-morphisms $\bigotimes_{f,g}$ for arbitrary morphisms f and g,
   - 2-morphisms $R_{Z,f}$ and $R_{f,Z}$ for generating morphisms f,
   - 2-morphisms i_f for arbitrary morphisms f,
   - T_Z,
   - W_Z,

- tensor products of arbitrary objects with the above 2-morphisms, and
- duals of the above 2-morphisms.

Theorem 16. *$\mathcal{T}$ is a braided monoidal 2-category generated by the unframed self-dual object Z.*

Proof - Since any object is determined by the number of points in a representative, and since the tensor product adds the number of points in such a representative, any object can be written as a tensor product of copies of Z. We also know that any 1-morphism can be determined by a list of the maxima, minima, and crossings of a representative together with their order and the number of strands to the right and left of each such point. Maxima and minima can be written as i_Z and i_Z^*, positive and negative crossings can be written as $R_{Z,Z}$, and $R^*_{Z,Z}$; the number of strands to the right and left can be determined from the tensor product of objects with the above morphisms; a tangle with no crossings, maxima or minima can be written as a tensor product of an appropriate object with 1_Z.

Any 2-morphism in $\mathcal{T}$ can be written as a composition of the 2-morphism generators described for $\mathcal{C}$: $1, I, E, S, T, W, II, H, N$. Since all 1-morphisms are generated by Z, so is 1_f. Descriptions for the others are listed below. Note that it is sufficient to give a description of $Y_{0,0}$ for $Y = I, E, S, T, W, II, H$, since $Y_{m,n} = A_m \otimes Y_{0,0} \otimes A_n$. Further, for any 2-morphism generator Y, the corresponding 2-morphisms Y^* and $Y^\dagger$ as given combinatorially are the same as the dual and adjoint in the 2-category sense, and so those generators follow from the generating operations, so we need only give descriptions for the variant $\bar{Y}$ of Y for the case where $\bar{Y} \neq Y$. We may also omit a separate definition for $\bar{N}_{Y,Z}$, since $\bar{N}_{Y,Z} = N_{\bar{Y},\bar{Z}}$

- $I_{0,0} = i_{i_Z}$,
- $E_{0,0} = e_{i_Z}$
- $W_{0,0} = W_Z$

- $\Pi_{0,0} = i_{R_{Z,Z}}$

- $S_{0;0,0} = R^{-1}_{Z,R_{Z,Z}} = R_{R_{Z,Z},Z}$

- $S_{1;0,0} = R^{-1}_{Z,R^*_{Z,Z}}$,

- $S_{2;0,0} = R^{\dagger}_{Z,R^*_{Z,Z}}$,

- $H_{0,0} = ((i_Z \otimes Z)(Z \otimes R_{Z,Z})(i_{R_{Z,Z}} \otimes Z)) \cdot ((R_{i_Z,Z} \otimes Z)(R^*_{Z,Z} \otimes Z))$.

- $T_{0,0} = T_Z$.

- $N_{Y_{m,n},Z_{i,j}} = \bigotimes_{Y_{m,n''},Z_{0,j}}$ where $n'' = \max\{n-j-2, n'-j-2\}$,

- $\bar{W}_{0,0} = (i_Z i_{R_{Z,Z}}) \cdot (W_Z^{-1} R^*_{Z,Z})$

- $\bar{\Pi}_{0,0} = i_{R^*_{Z,Z}}$.

- $\bar{S}_{0;0,0} = R^{\dagger}_{Z,R_{Z,Z}} = R^{\dagger -1}_{R_{Z,Z},Z}$,

- $\bar{S}_{1;0,0} = R^{\dagger -1}_{R^*_{Z,Z},Z}$,

- $\bar{S}_{2;0,0} = R_{R^*_{Z,Z},Z}$,

- $\bar{H}_{0,0} = ((i_Z \otimes Z)(Z \otimes R^*_{Z,Z})(i_{R^*_{Z,Z}} \otimes Z)) \cdot ((R^{\dagger}_{Z,i_Z})(R_{Z,Z} \otimes Z))$

All the above are specified in the list of 2-morphisms generated by the object Z, or are specified products of 2-morphisms listed. Applying $Y_{m,n} = A_m \otimes Y_{0,0} \otimes A_n$, and taking duals and adjoints (that are determined by specified compositions) of the above complete the list of generating 2-morphisms, all of which are therefore generated by Z. Any 2-morphism is a composition of such 2-morphisms, so all 2-morphisms in $\mathcal{T}$ are generated by Z. $\square$

4.1 An Algebraic Description of $\mathcal{T}$

Definition 17. *For monoidal 2-categories $\mathcal{C}$ and $\mathcal{D}$, a* strict monoidal 2-functor *$F: \mathcal{C} \to \mathcal{D}$ is a 2-functor such that $F(I) = I$, $F(A \otimes X) = F(A) \otimes F(X)$, $F(X \otimes A) = F(X) \otimes F(A)$ and $F(\bigotimes_{f,g}) = \bigotimes_{F(f),F(g)}$, for any object A, object, morphism or 2-morphism X, and morphisms f and g.*

Definition 18. *If $\mathcal{C}, \mathcal{D}$ are braided monoidal 2-categories with duals and $\mathcal{C}$ is generated by the unframed self-dual object Z, we say a monoidal 2-functor $F: \mathcal{C} \to \mathcal{D}$ mapping Z to an unframed self-dual object in $\mathcal{D}$* preserves braiding and duals strictly on the generator *if:*

1. $F(X^*) = F(X)^*$ for every object, morphism, or 2-morphism X,

2. $F(i_f) = i_{F(f)}$ for every morphism f,

3. $F(i_Z) = i_{F(Z)}$,

4. $F(T_Z) = T_{F(Z)}$,

5. $F(W_Z) = W_{F(Z)}$

6. $F(R_{Z,Z}) = R_{F(Z),F(Z)}$,

7. $F(R_{Z,f}) = ((F(Z) \otimes F(f))\tilde{R}_{(F(Z)|F(Z),F(Z))})$

   $\cdot R_{F(Z),F(f)} \cdot (\tilde{R}^{-1}_{(F(Z)|F(Z),F(Z))}(F(f) \otimes F(Z)))$

   and

   $F(R_{f,Z}) = ((F(f) \otimes F(Z))\tilde{R}_{(F(Z),F(Z)|F(Z))})$

   $\cdot R_{F(f),F(Z)} \cdot (\tilde{R}^{-1}_{(F(Z),F(Z)|F(Z))}(F(Z) \otimes F(f)))$

   for $f = R_{Z,Z}, R^*_{Z,Z}$.

8. $F(R_{Z,i_Z}) = ((F(Z) \otimes F(i_Z))\tilde{R}_{(F(Z)|F(Z),F(Z))}) \cdot R_{F(Z),F(i_Z)}$

   and

$F(R_{i_Z,Z}) = ((F(i_Z) \otimes F(Z))\tilde{R}_{(F(Z),F(Z)|F(Z))}) \cdot R_{F(i_Z),F(Z)}$

9. $F(R_{Z,e_Z}) = R_{F(Z),F(e_Z)} \cdot (\tilde{R}^{-1}_{(F(Z)|F(Z),F(Z))}(F(e_Z) \otimes F(Z)))$

   and

   $F(R_{e_Z,Z}) = R_{F(e_Z),F(Z)}(\tilde{R}^{-1}_{(F(Z),F(Z)|F(Z))}(F(Z) \otimes F(e_Z)))$

Theorem 19. *For any braided monoidal 2-category with duals $\mathcal{A}$ and unframed self-dual object $V \in \mathcal{A}$ there is a unique strict monoidal 2-functor $F: \mathcal{T} \to \mathcal{A}$ with $F(Z) = V$ that preserves braiding and duals strictly on the generator.*

Proof - We define a monoidal 2-functor $F: \mathcal{T} \to \mathcal{A}$ by using the properties of a strict monoidal 2-functor that preserves braiding and duals strictly on the generator, and the data $F(Z) = V$ to define F on the combinatorial generators (from the definition of $\mathcal{C}$). The 2-functor F then naturally extends to all objects morphisms and 2-morphisms by their combinatorial descriptions. We note that, since $\otimes$ is associative, and $A\otimes-$, $-\otimes A$ are 2-functors, the properties $F(A\otimes X) = F(A)\otimes F(X)$, and $F(X\otimes A) = F(X)\otimes F(A)$ hold for all objects A, and objects, morphisms or 2-morphisms X. The relations for decomposing $\bigotimes_{ff',g}$ and $\bigotimes_{f,gg'}$ together with $F(\bigotimes_{f,g}) = \bigotimes_{F(f),F(g)}$ for generating 1-morphisms f, g implies $F(\bigotimes_{f,g}) = \bigotimes_{F(f),F(g)}$ for all 1-morphisms f, g, thus if F is well defined, it is a strict monoidal 2-functor. Similarly $F(X^*) = F(X)^*$ and $F(i_f) = i_{F(f)}$ for generators together with the duals relations imply that $F(X^*) = F(X)^*$ and $F(i_f) = i_{F(f)}$ for any object, morphism and 2-morphism X and morphism f.

To show that F is determined by $F(Z)$, recall that $\mathcal{T}$ is generated by Z, hence any object, morphism and 2-morphism can be written as a composition (allowing duals, tensor products, composition of morphisms, and horizontal and vertical composition of 2-morphisms) of $Z, 1_Z, i_Z, R_{Z,Z}, i_f, 1_f, R_{Z,f}, R_{f,Z}, \bigotimes_{f,g}, T_Z, W_Z$ where f, g are 1-morphisms (and therefore compositions of $i_Z, R_{Z,Z}$ and duals their duals). The restrictions on F determine the image of all such objects, morphisms and 2-morphisms, so the functor is completely determined by $F(Z) = V$. It remains to show that F is well defined; we will show this by showing that the relations (as given for the combinatorial description) are preserved by F. We begin by noting that the source and target maps are preserved.

The only relation for objects is that $\otimes$ is associative; $F(A \otimes B) = F(A) \otimes F(B)$, and $\otimes$ is also associative in $\mathcal{A}$, so F is well defined on objects.

Note that the 1-morphism generators are:

$X_{0,0} = R_{Z,Z}$

$\bar{X}_{0,0} = R^*_{Z,Z}$

$\cap_{0,0} = i_Z$

$\cup_{0,0} = e_Z$

$1_n = 1_{A_n}$

The only two relations for morphisms in $\mathcal{T}$ are that composition is associative, and that $1_A f = f 1_B = f$ for $f\colon A \to B$. These also hold in $\mathcal{A}$, so F is well defined on 1-morphisms.

We first define F on the 2-morphism generators in the combinatorial definition of 2-morphisms of $\mathcal{C}$. Recall the descriptions of the 2-morphism generators in terms of Z:

$I_{0,0} = i_{i_Z}$

$E_{0,0} = e_{i_Z}$

$W_{0,0} = W_Z$

$II_{0,0} = i_{R_{Z,Z}}$

$S_{0;0,0} = R^{-1}_{Z,R_{Z,Z}} = R_{R_{Z,Z},Z}$

$S_{1;0,0} = R^{-1}_{Z,R^*_{Z,Z}}$

$S_{2;0,0} = R^{\dagger}_{Z,R^*_{Z,Z}}$

$H_{0,0} = ((i_Z \otimes Z)(Z \otimes R_{Z,Z})(i_{R_{Z,Z}} \otimes Z)) \cdot ((R_{i_Z,Z} \otimes Z)(R^*_{Z,Z} \otimes Z))$

$T_{0,0} = T_Z$

$N_{Y_{m,n},Z_{i,j}} = \otimes_{Y_{m,n-j''},Z_{0,j}}$ where j'' is the number of points in a representative of source$(Z_{0,j})$

$\bar{W}_{0,0} = (i_Z i_{R_{Z,Z}}) \cdot (W_Z^{-1} R^*_{Z,Z})$

$\bar{\Pi}_{0,0} = i_{R^*_{Z,Z}}$

$\bar{S}_{0;0,0} = R^\dagger_{Z,R_{Z,Z}} = R^{\dagger-1}_{R_{Z,Z},Z}$

$\bar{S}_{1;0,0} = R^{\dagger-1}_{R^*_{Z,Z},Z}$

$\bar{S}_{2;0,0} = R_{R^*_{Z,Z},Z}$

$\bar{H}_{0,0} = ((i_Z \otimes Z)(Z \otimes R^*_{Z,Z})(i_{R^*_{Z,Z}} \otimes Z)) \cdot ((R^\dagger_{Z,i_Z})(R_{Z,Z} \otimes Z))$

The properties of a monoidal 2-functor that preserves braiding and duals strictly on the generator, together with the descriptions above, to define the action of F on the combinatorial generators for 2-morphisms. Since F is well defined on morphisms, the definition $F(1_f) = 1_{F}(f)$ gives the same 2-morphism in $\mathcal{A}$ for any combinatorial description of f. We also note that the $F(R^{-1}_{Z,R_{Z,Z}}) = F(R_{R_{Z,Z},Z})$ corresponds precisely to $S^+ = S^-$ in $\mathcal{A}$, hence $F(S_{0;0,0})$ and $F(\bar{S}_{0;0,0})$ are consistent with either description of $S_{0;0,0}$ and $\bar{S}_{0;0,0}$.

Next we check that the relations given for the combinatorial definition of 2-morphisms all are preserved by F. Note that if a particular relation $\alpha = \beta$ holds in the image, then $\alpha^* = \beta^*$ and $\alpha^\dagger = \beta^\dagger$ also hold, so only $\bar{\alpha} = \bar{\beta}$ must be checked separately.

Relations 1-4, 6, 7, and 39, when mapped to $\mathcal{A}$, hold because $\mathcal{A}$ is a braided monoidal 2-category with duals, and F preserves sources and targets; the $\bar{\bullet}$ versions also hold similarly.

Relations 9, 10, and 15 hold in the image, since W_V, $i_{R_{V,V}}$, and T_V are unitary. The $\bar{\bullet}$ version of 9 follows from W_V, $i_{R_{V,V}}$ are unitary, and the $\bar{\bullet}$ version of 10 corresponds to the fact that $i_{R^*_{V,V}}$ is unitary. Relation 11 and the $\bar{\bullet}$ version of it follow from the fact that $\tilde{R}_{(\cdot|\cdot,\cdot)}$, $\tilde{R}_{(\cdot,\cdot|\cdot)}$, $R_{R_{V,V},V}$, and $R_{V,R_{V,V}}$ are unitary, and compositions of unitary morphisms are unitary. Relation 17 and the $\bar{\bullet}$ version of it follow from the fact that the 2-morphisms $\tilde{R}_{\cdot|\cdot,\cdot}$, $\tilde{R}_{\cdot,\cdot|\cdot}$, $R_{i_V,V}$, R_{V,i_V}, $i_{R_{V,V}}$, and $i_{R^*_{V,V}}$ are unitary. Relation 22 and the $\bar{\bullet}$ version of it follow from the fact that $\otimes_{f,g}$ is unitary for all morphisms f, g.

Relations 5, 8, and 34 are used in defining F, so they hold trivially in the image.

Relations 30, its $\bar{\bullet}$ version, and 31 map to the triangle relation $i_f f \cdot f e_f = 1_f$ for $f = R_{V,V}, R^*_{V,V}$ and i_V respectively.

Relations 23-27 and 29 can be rewritten as relations of the form:

$$\bigotimes_{f,g} \cdot ((f \otimes \text{source}(g))(\text{target}(f) \otimes \beta)) = ((\text{source}(f) \otimes \beta)(f \otimes \text{target}(g))) \cdot \bigotimes_{f,g'},$$

and

$$\bigotimes_{f,g} \cdot ((\alpha \otimes \text{source}(g))(\text{target}(f) \otimes g) = ((\text{source}(f) \otimes g)(\alpha \otimes \text{target}(g))) \cdot \bigotimes_{f',g}$$

where $\alpha\colon f \to f'$, and $\beta\colon g \to g'$. The image of a relation of the first form is

$$\bigotimes_{F(f),F(g)} \cdot ((F(f) \otimes \text{source}(F(g)))(\text{target}(F(f)) \otimes \beta))$$

$$= ((\text{source}(F(f))\beta)(F(f) \otimes \text{target}(F(g)))) \cdot \bigotimes_{F(f),F(g')},$$

which holds because $\mathcal{A}$ is a monoidal 2-category, and similarly for relations of the second form. Note that for all of these, the versions with $\bar{\bullet}$ follow from the same argument.

Relation 23 corresponds to a relation of the first form where $g = Y''_{0,l}$, and $\alpha = N_{Y_{m,n-l'}, Y'_{i,j-l'}}$, where l' is the number of points in a representative of source$(Y_{0,l})$. For all of these cases the 2-morphisms, $\alpha\colon f \Rightarrow f'$, morphisms $Y, Y', Y'' \in \{X, \bar{X}, \cap, \cup\}$, and subscripts are chosen so that all the above 2-morphisms and compositions are defined. Relation 24 corresponds to relations where $f = Y_{m,n-k-2}$, and $\beta = S_{i;0,k}$ for the first form, and $g = Y_{0,n}$, and $\alpha = S_{i;j,k-n''}$, where n'' is the number of points in a representative of source$(Y_{0,n})$ for the second form. Relation 25 corresponds to relations where $f = Y_{m,n-k-1}$, and $\beta = T_{0,k}$ for the first form, and $g = Y_{0,n}$, and $\alpha = T_{j,k-n''}$, where n'' is the number of points in a representative of source$(Y_{0,n})$ for the second form. Relation 26 corresponds to relations where $f = Y_{m,n-k}$, and $\beta = W_{0,k}$ for the first form, and $g = Y_{0,n}$, and $\alpha = W_{j,k-n''}$, where n'' is the number of points in a representative of source$(Y_{0,n})$ for the second form. Relation 27 corresponds to relations where $f = Y_{m,n-k-1}$, and $\beta = \bar{H}^*_{0,k}$ for the first form, and $g = Y_{0,n}$, and

$\alpha = \bar{H}^*_{j,k-n''}$, where n'' is the number of points in a representative of source$(Y_{0,n})$ for the second form. Relation 29 corresponds to relations where $\beta = II_{0,j}, I_{0,j}$, or $E_{0,j}$ and $f = Y_{m,n-k''}$, where k'' is the number of points in a representative of source(β) for the first form, and $g = Y_{0,n}$, and $\alpha = II_{j,k-n''}, I_{j,k-n''}$, or $E_{j,k-n''}$, where n'' is the number of points in a representative of source$(Y_{0,n})$ for the second form.

Relations 12, 13, 14, 20, 33, 36, and 38 are consequences of the braiding relations:

$$R_{f,A} \cdot (R_{\text{source}(f),A}(A \otimes \alpha)) = ((\alpha \otimes A)R_{\text{target}(f),A}) \cdot R_{f',A}$$

and

$$R_{A,g} \cdot (R_{A,\text{source}(g)}(\beta \otimes A)) = ((A \otimes \beta)R_{A,\text{target}(g)}) \cdot R_{A,g'}.$$

Note that $\bar{R}_{f,a} = R^{\dagger}_{a,\bar{f}}$ and $\bar{R}_{a,f} = R\dagger_{\bar{f},a}$, hence the $\bar{\bullet}$ versions of these relations also follow as a consequence of the same relations in $\mathcal{A}$.

We write these relations in the above form in $\mathcal{T}$. The fact that they also hold when mapped to $\mathcal{A}$ is a consequence of the properties of F. For example, for $f, g = R_{Z,Z}, R^*_{Z,Z}, i_Z, e_Z$, we can show (by the relations for decomposing $R_{Z,fg}$ and the fact that $\tilde{R}$ is unitary) that

$$F(R_{Z,fg}) = ((V \otimes F(fg))\tilde{R}_{(V|V,V)}) \cdot R_{V,F(fg)} \cdot (\tilde{R}^{-1}_{(V|V,V)}(F(fg) \otimes V)$$

(see Fig. 28) where the 2-morphisms $\tilde{R}$ are dropped where appropriate for $f, g = i_Z, e_Z$.

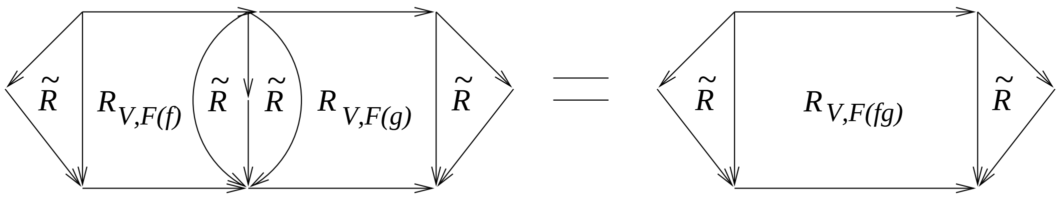

Figure 28: $F(R_{Z,fg})$ and the equivalent 2-morphism which includes $R_{V,F(fg)}$

We can also show (by the relations for decomposing $R_{Z,(f\otimes Z)(Z\otimes g)}$, the fact that $\tilde{R}$ is unitary, and the tetrahedron relations for $\tilde{R}$) that

$$F(R_{Z,(f\otimes Z)(Z\otimes g)}) = ((V \otimes F((f \otimes Z)(Z \otimes g))(\tilde{R}_{V|V\otimes V,V}) \cdot (\tilde{R}_{V|V,V}(VV \otimes R_{V,V}))))$$

$$\cdot R_{V,F((f\otimes Z)(Z\otimes g))}\cdot(((\tilde{R}_{V|V\otimes V,V}\cdot(\tilde{R}_{V|V,V}(VV\otimes R_{V,V})))^{-1})(F((f\otimes Z)(Z\otimes g))\otimes V)),$$

(see Fig. 29) where again $f, g = R_{Z,Z}, R^*_{Z,Z}, i_Z, e_Z$, and the 2-morphisms $\tilde{R}$ are dropped or replaced by $\tilde{R}_{(I,I|V)}$ or $\tilde{R}_{V|I,I}$ where appropriate for $f, g = i_Z, e_Z$. In particular the original $\tilde{R}$ 2-morphisms determined by F may be omitted, whereas the 2-morphism pairs $\tilde{R}, \tilde{R}^{-1}$ which are introduced to put it in desired form should remain.

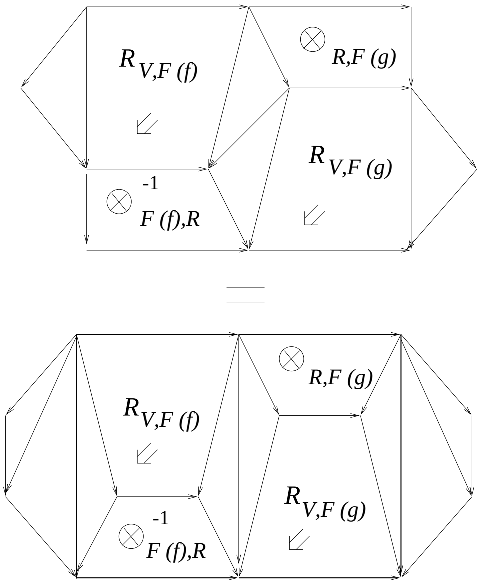

Figure 29: $F(R_{Z,(f\otimes Z)(Z\otimes g)})$ and $R_{V,F((f\otimes Z)(Z\otimes g)}$

These rules are sufficient (with the obvious extensions for longer compositions of 1-morphisms of the same types, and adaptions for 2-morphisms of the form $R_{f,Z}$) to show for the relations below, that, once they are written in the form of one of the above relations they also hold in the image as a result of those relations in $\mathcal{A}$. This follows because the 2-morphisms composed with $R_{V,F(f)}$ are determined for the image by the source and target of the 2-morphism, so they are the same for the both sides of the equations. See Fig. 30 for the 2-morphism relation and its image in $\mathcal{A}$.

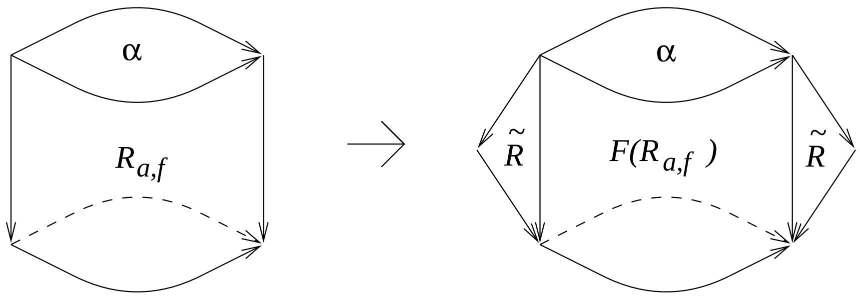

Figure 30: The relation on $\mathcal{T}$ and its image in $\mathcal{A}$

For relations 12, 13, 14, 20, 33, 36, and 38, we give equivalent relations of the form

$$R_{f,Z} \cdot (R_{\mathrm{source}(f),Z}(Z \otimes \alpha)) = ((\alpha \otimes Z)R_{\mathrm{target}(f),Z}) \cdot R_{f',Z}$$

and

$$R_{Z,g} \cdot (R_{Z,\mathrm{source}(g)}(\beta \otimes Z)) = ((Z \otimes \beta)R_{Z,\mathrm{target}(g)}) \cdot R_{Z,g'}.$$

In some cases one must use previous relations to show that the version in this form is equivalent to the original form. It then follows that the relations hold in the image of F.

Relation 12 can be written as a relation of the first form where $\beta = Z_{n;0,0}$ for $n = 0, 1, 2$. Four of the versions of relation 33 are equivalent to relations of the above type where $\alpha, \beta = i_{R_{Z,Z}}$ or $i_{\bar{R}_{Z,Z}}$. The other four versions follow from the first four together with relations 10, 29, and 39 and the duality definition of $\alpha^\dagger$ (which we previously showed is equivalent to the combinatorial definition). Relations 13 and 14 are equivalent (using relations 30, 33, and 39) to relations of the above form where $\alpha, \beta = W_Z$ or $\bar{W}_{0,0}$ (where $\bar{W}_{0,0} = (i_Z i_{R_{Z,Z}}) \cdot (W_Z^* R_{Z,Z}^*)$). Relation 20 is equivalent to relations of the above forms where $\alpha, \beta = T$ or $T^{\dagger *}$. Relation 36 is equivalent to relations of the above forms where $\alpha, \beta = i_{i_Z}$. Relation 38 is equivalent to relations of the above forms where $\alpha, \beta = e_{i_Z}$.

Relation 16 corresponds to the swallowtail relation in the definition of monoidal 2-category with duals. There are no crossings in relation 16, so the version $\bar{\bullet}$ is the same as the standard version. One version of relation 18 corresponds to the equation in the definition of unframed object, and the other versions follow from it using relations 9 and 39, and the fact that $\bar{\bullet} = \bullet^{\dagger}$. Relation 19 is a consequence of relations 9, 18, and 39.

Relations 32 and 35 follow from the categorical definition of $\alpha^{\dagger}$ and the property that $\alpha^{\dagger *} = \alpha^{*\dagger}$ for $\alpha = T_Z, W_Z$ together with previous relations. Since it is this definition we use in defining $\alpha^{\dagger}$, the relations are preserved under F.

Relations 21, and 28 hold in $\mathcal{A}$ as a consequence of the relation

$$((f \otimes B)R_{A',g}) \cdot (R_{f,B}(g \otimes A')) \cdot (R_{A,B} \bigotimes{}_{g,f}) =$$

$$(\bigotimes{}_{f,g}^{-1} R_{A',B'}) \cdot ((A \otimes g)R_{f,B'}) \cdot (R_{A,g}(B' \otimes f))$$

where $f\colon A \to A'$ and $g\colon B \to B'$ for morphisms $f, g = i_Z, R_{Z,Z}$, and i_Z, e_Z respectively. As in the previous relations involving the braiding 2-morphisms $R_{f,B}$ and $R_{A,g}$, these braiding 2-morphisms map to $\omega_{A'} \cdot R_{F(f),F(B)} \cdot \omega_A^{-1}$ and $\omega_{B'} \cdot R_{F(A),F(g)} \cdot \omega_B^{-1}$, where ω is a composition involving 2-morphisms of the type $\tilde{R}_{\cdot|\cdot,\cdot}$ and $\tilde{R}_{\cdot,\cdot|\cdot}$, which depends only on the source or target of the braiding 2-morphism, $R_{F(A),F(g)}$. The relation then maps to a similar relation in $\mathcal{A}$, but with the 2-morphisms ω inserted in a trivial way. Diagrams of the relation for $f, g = i_Z, R_{Z,Z}$ and its image are given in Fig. 31 below.

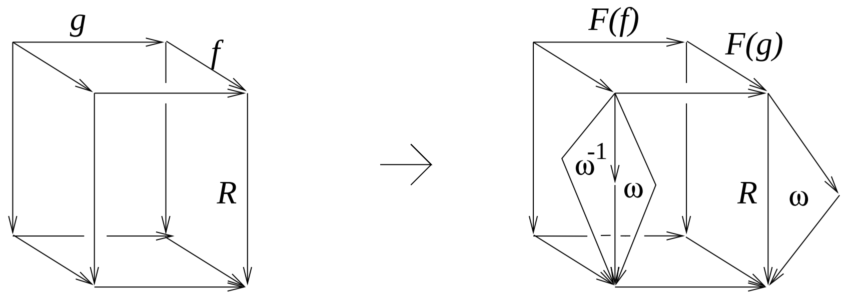

Figure 31: The relation in $\mathcal{T}$ and its image in $\mathcal{A}$

In particular, for relation 21, most of the variants are equivalent to relations of the above form with $f = i_Z$ and $g = R_{Z,Z}$ or $g = R^*_{Z,Z}$ and vice versa. The remaining variants are equivalent to relations of the form

$$R_{f,Z} \cdot (R_{\mathrm{source}(f),Z}(Z \otimes \alpha)) = ((\alpha \otimes Z)R_{\mathrm{target}(f),Z}) \cdot R_{f',Z}$$

and

$$R_{Z,g} \cdot (Z, R_{\mathrm{source}(g)}(\beta \otimes Z)) = ((Z \otimes \beta)R_{Z,\mathrm{target}(g)}) \cdot R_{Z,g'},$$

where $\alpha, \beta = R_{i_Z,Z}$, R_{Z,i_Z}, $R^{\dagger}_{e_Z,Z}$ or $R^{\dagger}_{Z,e_Z}$. These relations also hold in the image, as shown above for the similar relations 12, 13, 20, 33, and 38. Relation 28 is equivalent to relations of the above type, where $f = i_Z$, $g = e_Z$ and vice versa.

Relation 37 can be eliminated, because it follows from relations 30, 31, 35, and 39.

We note that $F(Y^*) = F(Y)^*$ for any object morphism or 2-morphism Y, and $F(\alpha^{\dagger}) = F(\alpha)^{\dagger}$ for any 2-morphism α, hence the relation $\alpha^{\dagger *} = \alpha^{* \dagger}$ hold in the image of F. By definition, $F(\bar{\alpha}^*) = F(\bar{\alpha^*})$ and $F(\bar{\alpha}^{\dagger}) = F(\bar{\alpha^{\dagger}})$. $\square$

5 Conclusion

Our main result implies that from any braided monoidal 2-category with duals containing an unframed self dual object, we can obtain an invariant of unframed, unoriented 2-manifolds in 4 dimensions. It seems likely, moreover, that we will be able to get a similar characterization of framed oriented 2-manifolds in 4 dimensions as the free braided monoidal 2-category with duals, eliminating the need to have an unframed object. It should be noted, at this point, that the definition of duality in this case is likely to be slightly different. In removing the writhe move from the list of 2-morphisms, at least one other relation must be added to the definition of duality in the braided case. Specifically, the 2-morphism which corresponds to the framed version of a Reidemeister I move exists under the current definition of a braided monoidal 2-category with duals, without using the writhe move, but is not invertible.

This brings us to the problem of finding interesting braided monoidal 2-categories with duals. Beginning at the end, we note that Baez and Neuchl give a center construction, similar to the quantum double construction one dimension lower, which produces a braided monoidal 2-category from a monoidal 2-category [4]. It seems likely that this can be extended to give a braided monoidal 2-category with duals from a monoidal 2-category with duals. This pushes the problem back to looking for a monoidal 2-category with duals. One place to look for monoidal 2-categories with duals, analogous to the case one dimension lower, is representation 2-categories of 2-groups or of Hopf categories. Crane and Frenkel give a method for constructing Hopf categories from Lusztig's canonical bases for quantum groups [5]. Another possibility for finding appropriate 2 categories is homotopy 2-types of double loop spaces, which Baez and Dolan suggest are likely to be braided monoidal 2-categories [2, 3]. Further, Baez suggests that braided monoidal 3-Hilbert spaces are likely to be braided monoidal 2-categories with duals [1]. Finally, there are several examples of braided monoidal 2-categories which have been constructed from solutions of the Zamolodchikov tetrahedron equations [11, 7], and it is possible that one of these may give a braided monoidal 2-category with duals.

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