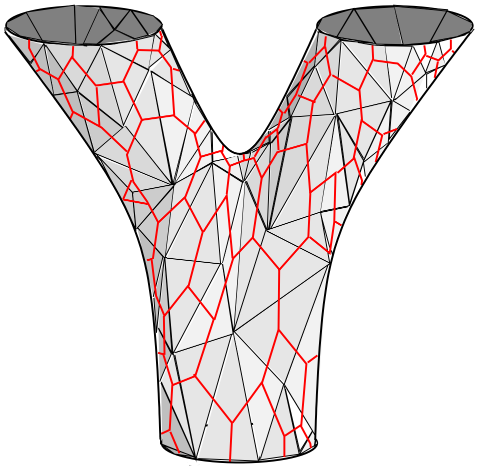
OPEN-CLOSED TOPOLOGICAL QUANTUM FIELD THEORY AND TANGLE HOMOLOGY

Aaron D. Lauda
(Hughes Hall College)

Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge, Cambridge CB3 0WB, United Kingdom
A.Lauda@dpmms.cam.ac.uk



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This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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AND TANGLE HOMOLOGY

Thesis Summary for Aaron Lauda

In this thesis we examine the relationship between 2-dimensional topological field theories and homological link invariants, or what are sometimes referred to as 'categorified' link invariants. To this end, we provide a detailed description of a special sort of 2-dimensional extended Topological Quantum Field Theories (TQFTs) which are called open-closed TQFTs. A state sum, or 'local', construction of such TQFTs is then defined and used to define an algebraic tangle homology theory based on Khovanov's link homology.

Open-closed TQFTs are defined in Chapter 2 to algebraically represent open-closed cobordisms by which we mean smooth compact oriented 2-manifolds with corners that have a particular global structure in order to model the smooth topology of open and closed string worldsheets. We show that the category of open-closed TQFTs is equivalent to the category of knowledgeable Frobenius algebras. A knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ consists of a symmetric Frobenius algebra A, a commutative Frobenius algebra C, and an algebra homomorphism $\imath\colon C \to A$ with dual $\imath^*\colon A \to C$, subject to some conditions. This result is achieved by providing a generators and relations description of the category of open-closed cobordisms. In order to prove the sufficiency of our relations, we provide a normal form for such cobordisms which is characterized by topological invariants. Starting from an arbitrary such cobordism, we construct a sequence of moves (generalized handle slides and handle cancellations) which transforms the given cobordism into the normal form. Using the generators and relations description of the category of open-closed cobordisms, we show that it is equivalent to the symmetric monoidal category freely generated by a knowledgeable Frobenius algebra. Our formalism is then generalized to the context of open-closed cobordisms with labeled free boundary components, *i.e.* to open-closed string worldsheets with D-brane labels at their free boundaries.

In Chapter 3 we present a state sum construction of two-dimensional open-closed Topolog-

ical Quantum Field Theories (TQFTs) which generalizes the state sum of Fukuma–Hosono–Kawai from conventional two-dimensional cobordisms to open-closed cobordisms. This construction reveals the topological interpretation of the associative algebra that enters the state sum construction as the vector space that the TQFT assigns to the unit interval. Extending the notion of a two-dimensional TQFT from cobordisms to suitable manifolds with corners therefore makes the relationship between the global description of the TQFT in terms of a functor into the category of vector spaces and the local description in terms of a state sum fully transparent. We also illustrate the state sum construction of an open-closed TQFT with a finite set of D-branes using the example of the groupoid algebra of a finite groupoid.

Finally, in Chapter 4 we use open-closed TQFTs in order to extend Khovanov homology from links to tangles. For every plane diagram of an oriented tangle, we construct a chain complex whose homology is invariant under Reidemeister moves. We give examples of knowledgeable Frobenius algebras for which our tangle homology theory reduces to the link homology theories of Khovanov, Lee, and Bar-Natan if it is evaluated for links.

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I should also like to thank Eugenia Cheng for teaching me about n-categories and David Kagan for 'string-y' conversations over the countless lunches in the department and dinners at Maharaja's. I am grateful to Peter May, Nick Gursky, Michael Shulman, and others at the University of Chicago. For introducing me to the subjects upon which this thesis is based, I would like to thank Nils Baas and Dror Bar-Natan. I would also like to thank Dror Bar-Natan for use of his diagrams and symbol package *dbnsymb*. For their various contributions to the papers upon which this thesis is based, I would like to thank Mikhail Khovanov, Gerd Laures, Marco Mackaay, Ingo Runkel, Alex Shannon, Ivan Smith, Simon Willerton, Jonathan Woolf, and the European Union Superstring Theory Network.

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Aaron Lauda
Cambridge
February 23, 2006

Because something is happening here

But you don't know what it is

Do you, Mister Jones?

—Bob Dylan

Contents

Chapter 1

Introduction

In this thesis we examine the relationship between 2-dimensional topological field theories and homological link invariants, or what are sometimes referred to as 'categorified' link invariants. Loosely speaking, categorification is the process of turning numbers into vector spaces and vector spaces into categories. More generally, Louis Crane coined the term categorification to refer to the idea of finding categorical notions which generalize set theoretic concepts [4]. There are many examples of categorification in the literature (for example [5–11]), but Khovanov's categorification of the Jones polynomial serves as the motivation for much of the work done in this thesis.

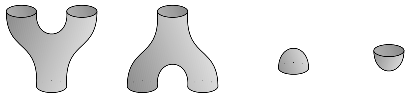
The Jones polynomial is a polynomial link invariant in $\mathbb{Z}[q, q^{-1}]$. Khovanov's novel idea was to generalize the Jones polynomial to a chain complex of graded vector spaces. For every plane diagram of an oriented link L, Khovanov's link homology theory [12] yields a chain complex $[\![L]\!]$ of graded vector spaces whose graded Euler characteristic agrees with the Jones polynomial of the link. This construction can be seen as a categorification of the suitably normalized Jones polynomial, replacing a polynomial in one indeterminate q by a chain complex of graded vector spaces where the degree corresponds to the exponent of q. The coefficients of the polynomial arise as the dimensions of the homogeneous components of the graded homology groups of the chain complex in such a way that the degree corresponds to the power of q.

If two link diagrams are related by a Reidemeister move, the corresponding chain complexes of graded vector spaces are homotopy equivalent, and so their homology groups are isomorphic as graded vector spaces. This of course implies that their graded Euler characteristics and thereby their Jones polynomials agree, but in general the homology groups contain more information about the link than just the Jones polynomial. Indeed, Bar-Natan [13, 14] has shown that there are knots and links that have the same Jones polynomial, but which can be distinguished by their Khovanov homology.

1.0.1 Two-dimensional Topological Field Theories

The construction of Khovanov's chain complex heavily relies on a 2-dimensional Topological Quantum Field Theory (TQFT). An n-dimensional Topological Quantum Field Theory (TQFT) [15] is a symmetric monoidal functor from the category **nCob** of n-dimensional cobordisms to the category $\mathbf{Vect}_k$ of vector spaces over a given field k. The objects of the category **nCob** are smooth compact oriented $(n-1)$-manifolds without boundary, and the morphisms are equivalence classes of smooth compact oriented cobordisms between these, modulo diffeomorphisms that restrict to the identity on the boundary. An n-dimensional TQFT therefore associates vector spaces with $(n-1)$-manifolds and linear maps with n-dimensional cobordisms. Disjoint unions of manifolds correspond to tensor products of vector spaces and linear maps, and gluing cobordisms along their boundaries corresponds to the composition of linear maps. Note that the empty $(n-1)$-manifold plays the role of the unit object for the tensor product and corresponds to the field k.

For $n = 2$, the category **nCob** is well understood, and so there are strong results about 2-dimensional TQFTs. For these classic results, we refer to [16–18] and to the book [19]. It is known, for example, that 2-dimensional TQFTs are characterized by commutative Frobenius algebras. The objects of **2Cob** are compact 1-manifolds without boundary, *i.e.* disjoint unions of circles S^1. For the morphisms of **2Cob**, one has a description in terms of generators and relations. The generators are these cobordisms:

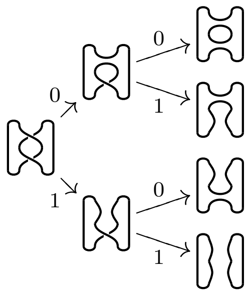


μ $\qquad$ Δ $\qquad$ η $\qquad$ ε . (1.0.1)

We have drawn them in such a way that their source is at the top and their target at the bottom of the diagram. The TQFT is a functor $Z\colon \mathbf{2Cob} \to \mathbf{Vect}_k$. If we denote by $C := Z(S^1)$ the vector space associated with the circle, the TQFT assigns linear maps $\mu\colon C \otimes C \to C$, $\Delta\colon C \to C \otimes C$, $\eta\colon k \to C$ and $\varepsilon\colon C \to k$ to the morphisms depicted in (1.0.1). The relations among the morphisms of **2Cob** then imply that $(C, \mu, \eta, \Delta, \varepsilon)$ forms a commutative Frobenius algebra. Conversely, given any commutative Frobenius algebra C, there is a functor $Z\colon \mathbf{2Cob} \to \mathbf{Vect}_k$ such that $Z(S^1) = C$.

The generators and relations description of the category **2Cob** greatly simplifies the study of 2-dimensional TQFTs. In fact, there is a categorical equivalence between the category of 2-dimensional topological field theories and the category of commutative Frobenius algebras. This makes the translation between algebra and topology a very simple matter.

1.0.2 Link homology



Given a link diagram L, both the Jones polynomial and Khovanov's categorification of it can be computed from the resolutions of the link diagram. To resolve a link diagram one locally replaces each of the crossings ($\times$) by the 0-smoothing ($\asymp$) and the 1-smoothing ()(). This produces two link diagrams built from the original by these local replacements. It is easy to see that if one resolves all of the n crossings of a given link diagram then there will be 2^n resolutions each of which consists of the disjoint union of link diagrams diffeomorphic to circles. The Khovanov complex $[\![L]\!]$ is computed from these resolutions by regarding each resolution as an object of the category **2Cob**. The differential is then computed using the morphisms of **2Cob**, that is using 2-dimensional cobordisms. Hence, applying a 2-dimensional topological quantum field theory to this complex produces an algebraic realization of the complex and provides a computable link invariant. But from the discussion above, specifying a 2-dimensional TQFT amounts to specifying a commutative Frobenius algebra.

Khovanov's original choice of Frobenius algebra $A[c]$ over the commutative ring $R = \mathbb{Z}[c]$ is such that one gets a chain complex of graded modules and a categorification of the Jones polynomial. His TQFT is actually a functor $\mathbf{2Cob} \to \mathbf{Mod}_R$. We restrict ourselves to the case $c \equiv 0$ (*c.f.* [20, 21]) and to algebras over a field k.

Definition 1.0.1. Let k be a field. Khovanov's [12] commutative Frobenius algebra $(C_{\mathrm{Kh}}, \mu, \eta, \Delta, \varepsilon)$ is the algebra $C_{\mathrm{Kh}} = k[x]/(x^2)$ with the Frobenius algebra structure given in the k-basis $\{1, x\}$ by $\mu(1 \otimes 1) = 1$, $\mu(1 \otimes x) = x$, $\mu(x \otimes 1) = x$, $\mu(x \otimes x) = 0$, $\eta(1) = 1$, $\Delta(1) = 1 \otimes x + x \otimes 1$, $\Delta(x) = x \otimes x$, $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$.

There are other choices of commutative Frobenius algebras which do not give a grading, or which give a filtration rather than a grading, and some of which are not known to categorify any interesting link invariant. The most common choices are those of Lee and Bar-Natan.

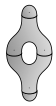
Definition 1.0.2. Let k be a field. Lee's [22] commutative Frobenius algebra $(C_{\text{Lee}}, \mu, \eta, \Delta, \varepsilon)$ is the algebra $C_{\text{Lee}} = k[x]/(x^2-1)$ with the Frobenius algebra structure given by $\mu(1\otimes 1) = 1$, $\mu(1\otimes x) = x$, $\mu(x\otimes 1) = x$, $\mu(x\otimes x) = 1$, $\eta(1) = 1$, $\Delta(1) = 1\otimes x + x\otimes 1$, $\Delta(x) = x\otimes x + 1\otimes 1$, $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$.

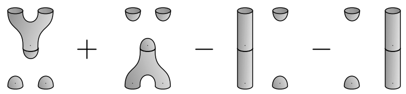
Definition 1.0.3. Let k be a field. Bar-Natan's [23] commutative Frobenius algebra $(C_{\text{BN}}, \mu, \eta, \Delta, \varepsilon)$ is the algebra $C_{\text{BN}} = k[x]/(x^2-x)$ with the Frobenius algebra structure given by $\mu(1\otimes 1) = 1$, $\mu(1\otimes x) = x$, $\mu(x\otimes 1) = x$, $\mu(x\otimes x) = x$, $\eta(1) = 1$, $\Delta(1) = 1\otimes x + x\otimes 1 - 1\otimes 1$, $\Delta(x) = x\otimes x$, $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$.

The link homology theory associated with Khovanov's Frobenius algebra is known to categorify the Jones polynomial, a *quantum invariant* of links. In some cases, the other two link homology theories are related to *classical invariants* of links: Lee's theory is related to the number of components of the link [22] whereas Bar-Natan's theory in characteristic 2 categorifies a combinatorial expression involving linking numbers [24].

Bar-Natan [23] has presented sufficient conditions for Frobenius algebras to yield link homology theories. Topologically, these conditions can be visualized as follows:

 $= 0,$ (1.0.2)

 $= 2,$ (1.0.3)

 $= 0.$ (1.0.4)

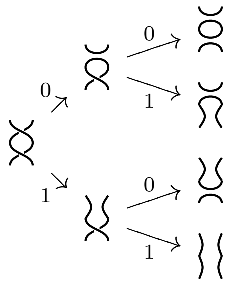
The Frobenius algebras of Definition 1.0.1, 1.0.2 and 1.0.3 satisfy these three conditions. Khovanov [25] has classified the Frobenius algebras that give rise to link homology theories. This classification includes examples, for instance his $A[c]$ without evaluation at $c = 0$, that do not satisfy Bar-Natan's conditions.

Over a field k, the most general commutative Frobenius algebra (up to base change) satisfying Bar-Natan's conditions is given by:

Definition 1.0.4 (see [25])**.** Let k be a field and $h, t \in k$. $C_{h,t}$ denotes the algebra $C_{h,t} = k[x]/(x^2 - hx - t)$ equipped with the structure of a commutative Frobenius algebra $(C_{h,t}, \mu, \eta, \Delta, \varepsilon)$ which is given in the basis $\{1, x\}$ by $\mu(1\otimes 1) = 1$, $\mu(1\otimes x) = x$, $\mu(x\otimes 1) = x$, $\mu(x\otimes x) = hx + t$, $\eta(1) = 1$, $\Delta(1) = 1\otimes x + x\otimes 1 - h\cdot 1\otimes 1$, $\Delta(x) = x\otimes x + t\cdot 1\otimes 1$, $\varepsilon(1) = 0$, and $\varepsilon(x) = 1$.

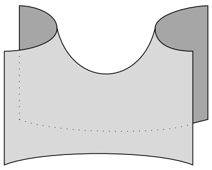
While Khovanov originally studied the case $h = 0$, $t = 0$, Lee [22] considered $h = 0$, $t = 1$, and Bar-Natan [23] studied $h = 1$, $t = 0$. Below, we refer to $C_{0,0} = C_{\mathrm{Kh}}$ as *Khovanov's*, to $C_{0,1} = C_{\mathrm{Lee}}$ as *Lee's*, and to $C_{1,0} = C_{\mathrm{BN}}$ as *Bar-Natan's* Frobenius algebra.

It is natural to ask whether one can extend Khovanov's link homology from links to tangles. For tangles, the smoothings would consist not only of circles, but rather of circles and arcs. Khovanov has already remarked in [26] that one would need an extended 2-dimensional TQFT in which the cobordisms are generalized to suitable manifolds with corners.



Even without such an extended TQFT, there are two workarounds. Khovanov [26] considers tangles with an even number of points both for the source and the target of the tangle, and presents a definition in which the tangle homology is reduced to his link homology. He therefore closes the open ends of the tangles in all possible ways and takes a formal sum over the resulting expressions for the links. He thus obtains a tangle homology theory only for even tangles, *i.e.* those with an even number of points both for the source and the target, but he is still able to say which expression this construction categorifies.

Bar-Natan [23] works with formal linear combinations of manifolds with corners and with chain complexes of these surfaces

 .

As long as one stays in this geometric 'picture world', one has good composition laws for tangles, but still one can translate the picture world into algebra only for links, *i.e.* after all open ends have been closed.

The following questions remain to be answered and form the subject of this thesis:

1. What kind of 2-dimensional topological quantum field theories naturally extend 2-dimensional topological quantum field theories to manifolds with corners and what algebraic structures characterize such extended TQFTs?

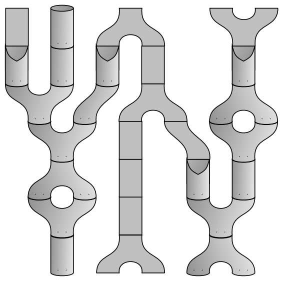
2. How does one define an algebraic homology theory for arbitrary oriented tangles?

3. Which algebraic operation corresponds to the composition of tangles and do these algebraic operations correspond naturally to operations in some version of extended 2-dimensional TQFT?

One answer to the first question is given by the notion of an open-closed 2-dimensional topological quantum field theory.

1.0.3 Open-closed TQFT

Open-closed topological field theories were originally studied in the context of open string theory and boundary conformal field theory [27,28]. By open-closed cobordisms we mean the morphisms of a category $\mathbf{2Cob}^{\mathrm{ext}}$ whose objects are compact oriented smooth 1-manifolds, *i.e.* free unions of circles S^1 and unit intervals $I = [0,1]$. The morphisms are certain compact oriented smooth 2-manifolds with corners. The corners of such a manifold f are required to coincide with the boundary points ∂I of the intervals. The boundary of f viewed as a topological manifold, minus the corners, consists of components that are either 'black' or 'coloured'. Each corner is required to separate a black component from a coloured one. The black part of the boundary coincides with the union of the source and the target objects. Two such manifolds with corners are considered equivalent if they are related by an orientation preserving diffeomorphism which restricts to the identity on the black part of the boundary. An example of such an open-closed cobordism is depicted here[1],



(1.0.5)

where the boundaries at the top and at the bottom of the diagram are the black ones. In Section 2.5, we present a formal definition which includes some additional technical properties. Gluing such cobordisms along their black boundaries, *i.e.* putting the building blocks of (1.0.5) on top of each other, is the composition of morphisms. The free union of manifolds, *i.e.* putting the building blocks of (1.0.5) next to each other, provides $\mathbf{2Cob}^{\mathrm{ext}}$ with the structure of a symmetric monoidal category.

Open-closed cobordisms can be seen as a generalization of the conventional 2-dimensional cobordism category $\mathbf{2Cob}$. The category $\mathbf{2Cob}$ is a subcategory of $\mathbf{2Cob}^{\mathrm{ext}}$ whose objects of are compact oriented smooth 1-manifolds *without boundary*; the morphisms are compact

[1]In order to get a feeling for these diagrams, the reader might wish to verify that this cobordism is diffeomorphic to the one depicted in Figure 1 of [29].

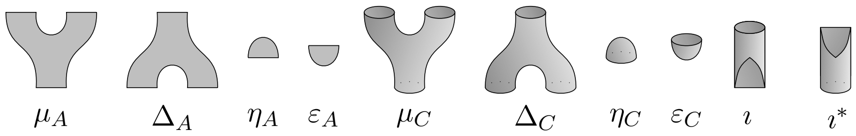
oriented smooth cobordisms between them, modulo orientation-preserving diffeomorphisms that restrict to the identity on the boundary.

The study of open-closed cobordisms plays an important role in conformal field theory if one is interested in boundary conditions, and open-closed cobordisms have a natural string theoretic interpretation. The intervals in the black boundaries are interpreted as open strings, the circles as closed strings, and the open-closed cobordisms as string worldsheets. Here we consider only the underlying smooth manifolds, but not any additional conformal or complex structure. Additional labels at the coloured boundaries are interpreted as D-branes or boundary conditions on the open strings.

An open-closed Topological Quantum Field Theory (TQFT), which we formally define in Section 2.7 below, is a symmetric monoidal functor $\mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$ into a symmetric monoidal category $\mathcal{C}$. If $\mathcal{C}$ is the category of vector spaces over a fixed field k, then the open-closed TQFT assigns vector spaces to the 1-manifolds I and S^1, it assigns tensor products to free unions of these manifolds, and k-linear maps to open-closed cobordisms.

Such an open-closed TQFT can be seen as an extension of the notion of a 2-dimensional TQFT [15] which is a symmetric monoidal functor $\mathbf{2Cob} \to \mathcal{C}$. We refer to this conventional notion of 2-dimensional TQFT as a *closed TQFT* and to the morphisms of $\mathbf{2Cob}$ as *closed cobordisms*.

In order to describe open-closed cobordisms using generators and relations, one would need a generalization of Morse theory for manifolds with corners. Such a generalization of Morse theory can be used in order to find the generators of $\mathbf{2Cob}^{\mathrm{ext}}$,



(1.0.6)

and brute force can be used to establish the necessity of certain relations. However, we are not aware of any abstract theorem that would guarantee the sufficiency of these relations.

The main result of Chapter 2 is a normal form for open-closed cobordisms with an inductive proof that a specified set of relations suffice in order to transform any handle decomposition into the normal form. As a consequence, for any two diffeomorphic open-closed cobordisms whose handle decompositions are given, we explicitly construct a diffeomorphism relating the two by constructing the corresponding sequence of moves.

The description of $\mathbf{2Cob}^{\mathrm{ext}}$ in terms of generators and relations has emerged over the last couple of years from consistency conditions in boundary conformal field theory, going back to the work of Cardy and Lewellen [27, 28], Lazaroiu [30], Alexeevski and Natanzon [31], and

Moore and Segal, see, for example [32, 33], and these results have been known to the experts for some time. One aspect of our approach that is different from the above is the normal form and our inductive proof that the relations are sufficient.

This, in turn, implies the following result in Morse theory for our sort of compact 2-manifolds with corners which has so far not been available by other means: The handle decompositions associated with any two Morse functions on the same manifold are related by a finite sequence of handle slides and handle cancellations.

Once a description of $\mathbf{2Cob}^{\mathrm{ext}}$ in terms of generators and relations is available, it is possible to find an algebraic characterization for the symmetric monoidal category of open-closed TQFTs. Whereas the category of closed TQFTs is equivalent as a symmetric monoidal category to the category of commutative Frobenius algebras [17], we prove that the category of open-closed TQFTs is equivalent as a symmetric monoidal category to the category of *knowledgeable Frobenius algebras*. We define knowledgeable Frobenius algebras in Section 2.4 precisely for this purpose. A knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ consists of a symmetric Frobenius algebra A, a commutative Frobenius algebra C, and an algebra homomorphism $\imath\colon C \to A$ with dual $\imath^*\colon A \to C$, subject to some conditions. The structure that emerges is consistent with the algebraic characterization supplied by the work of Moore and Segal [32]. The name knowledgeable Frobenius algebra was not used by Moore and Segal and our reason for this terminology will be explained in the next chapter.

The algebraic structures relevant to boundary conformal field theory have been studied by Fuchs and Schweigert [34]. In a series of papers, for example [35], Fuchs, Runkel, and Schweigert study Frobenius algebra objects in ribbon categories. Topologically, this corresponds to a situation in which the surfaces are embedded in some 3-manifold and studied up to ambient isotopy. In contrast, we consider Frobenius algebra objects in a symmetric monoidal category, and our 2-manifolds are considered equivalent as soon as they are diffeomorphic (as abstract manifolds) relative to the boundary.

Various extensions of open-closed topological field theories have also been studied. Baas, Cohen, and Ramírez have extended the symmetric monoidal category of open-closed cobordisms to a symmetric monoidal 2-category whose 2-morphisms are certain diffeomorphisms of the open-closed cobordisms [29]. This work extends the work of Tillmann who defined a symmetric monoidal 2-category extending the closed cobordism category [36]. She used this 2-category to introduce an infinite loop space structure on the plus construction of the stable mapping class group of closed cobordisms [37]. Using a similar construction to Tillmann's, Baas, Cohen, and Ramírez have defined an infinite loop space structure on the plus construction of the stable mapping class group of open-closed cobordisms, showing that infinite loop space structures are a valuable tool in studying the mapping class group.

Another extension of open-closed TQFT comes from open-closed Topological *Conformal* Field Theory (TCFT). It was shown by Costello [38] that the category of open Topological Conformal Field Theories is homotopy equivalent to the category of certain A_∞ categories with extra structure. Ignoring the conformal structure, or equivalently taking H_0 of the Hom spaces in the corresponding category, reduces this to the case of Topological Quantum Field Theory. Costello associates to a given open TCFT an open-closed TCFT where the homology of the closed states is the Hochschild homology of the A_∞ category describing the open states. This work is also useful for providing generators and relations for the category of open Riemann surfaces and, when truncated, this result also agrees with the characterization of open cobordisms and their diffeomorphisms up to isotopy given in [39] where a smaller list of generators and relations is given. In this thesis, we aim directly for an explicit description of the category of open-closed cobordisms.

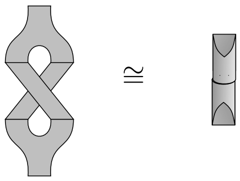
1.0.4 Tangle homology

As we have already mentioned, rather than boundary conformal field theory or topological conformal field theory, our interest in open-closed TQFTs stems from their relevance to constructing tangle homology theories. In Chapter 4, we show that open-closed TQFTs play a role in tangle homology theories analogous to the role played by closed TQFTs in Khovanov's link homology theory.

Using the detailed description and properties of the category $\mathbf{2Cob}^{\mathrm{ext}}$ developed in Chapter 2, we can apply essentially Morse theoretic techniques to the 'picture world' used by Bar-Natan to construct tangle homology theories. The only impedance to naively applying an open-closed TQFT to Bar-Natan's picture world is that additional data (orientations and labeling) must be assigned to the picture world so that the geometric objects can be regarded as objects and morphisms of the category $\mathbf{2Cob}^{\mathrm{ext}}$. This will be explained in greater detail in Chapter 4.

From Bar-Natan's work [23] and from his topological way of proving the invariance of his tangle homology under Reidemeister moves, it will become obvious that an open-closed TQFT is just the right tool for turning Bar-Natan's picture world into an algebraic tangle homology theory. What remains to be done is to find knowledgeable Frobenius algebras and thereby open-closed TQFTs that satisfy the conditions (1.0.2) to (1.0.4). In Chapter 4, we present examples of such knowledgeable Frobenius algebras for which the resulting tangle homology theories reduce to the link homology theories of Khovanov, Lee, and Bar-Natan (Definitions 1.0.1 to 1.0.3) when they are evaluated for links. This resolves the second question posed at the end of Section 1.0.2, namely to find algebraic homology theories for arbitrary oriented tangles.

In this thesis we will only provide examples that extend the traditional link homologies (those of Khovanov, Lee and Bar-Natan) for fields of finite characteristic. Hence, we do not strictly extend these traditional link homologies. In [3] it is shown that *in the symmetric monoidal category of vector spaces* $\mathbf{Vect}_k$ (with the usual tensor product) the only possible knowledgeable Frobenius algebras $(A, C, \imath, \imath^*)$ whose centre C satisfies Bar-Natan's conditions (1.0.2)-(1.0.4) (or equivalently having C given by Definition (1.0.4)) and which possess a grading or filtration compatible with a traditional link homology exist in finite characteristic[2]. This is due to the Cardy condition, an axiom in the definition of a knowledgeable Frobenius algebra which is a consequence of the following diffeomorphism,



. (1.0.7)

Here we will present a simplified form of tangle homology in which gradings and filtrations are ignored. The gradings and filtrations can easily be accounted for, providing a much richer theory, see [3], but we neglect them here to simplify our discussion and highlight the new features of our theory. For example, although we present a simplified theory, the tangle homology described in this thesis possesses the desirable property of being *monoidal*. This

[2]The Cardy condition together with the grading requirements of Khovanov's theory lead to the requirement that the algebra A have quantum trace equal to zero [3]. In the category $\mathbf{Vect}_k$ this implies that the dimension of A is zero in the ground field k, hence in order to have a nontrivial theory one must restrict to finite characteristic. The question as to whether or not the requirement of finite characteristic can be relaxed using a different category, such as the category of super vector spaces $\mathbf{SVect}_k$ with its graded tensor product, is still open at this time. However, one should note that any naive attempt at utilizing the category $\mathbf{SVect}_k$ will surely fail, since a *symmetric monoidal* structure only exists on the category of super vector spaces with morphisms the *even* maps and Khovanov's graded theory requires the counit to be of odd degree. There does not exists a symmetric monoidal category whose objects are super vector spaces over k and whose morphisms *all* linear maps between them. Hence, a knowledgeable Frobenius algebra with the appropriate gradings can not even be defined in the symmetric monoidal category of super vector spaces.

means that monoidal structure of the category of tangles, given by the disjoint union of tangles, is mapped to the monoidal structure in the category of complexes given by the tensor product. Hence, the complex associated to the disjoint union of two tangles is the tensor product of their associated complexes. This is a new feature of the tangle homology presented here that is not present in other algebraic approaches to tangle homology.

1.0.5 Open-closed TQFTs from state sum constructions

In light of the discussion above, the final question posed at the end of Section 1.0.2 can be reformulated as the problem of finding which open-closed TQFTs carry the relevant additional structure to facilitate a natural notion of gluing tangles. In Chapter 3, we define such TQFTs using a state sum construction. This construction generalizes the state sum construction of 2-dimensional (closed) TQFTs introduced by Fukuma–Hosono–Kawai [40].

The state sum of Fukuma–Hosono–Kawai forms a different and *a priori* independent way of defining a 2-dimensional TQFT. This construction starts with a finite-dimensional semisimple algebra A over a field k of characteristic zero. For every 2-dimensional cobordism $M\colon \Sigma_1 \to \Sigma_2$, one considers a triangulation of M, and from the data A, μ, η, and from the triangulation, one computes the linear map $Z(M)\colon Z(\Sigma_1) \to Z(\Sigma_2)$ as a so-called *state sum*. In a state sum, roughly speaking, one colours the simplices of the triangulated manifold M with algebraic data such as the vector space underlying A or the linear maps μ, η, and then one 'sums over all colourings' following certain rules. We present this construction in detail in Section 3.4 . In particular, one can compute the vector space associated with the circle, and it turns out that this is the centre

$$Z(S^1) = Z(A) \tag{1.0.8}$$

of the algebra one has started with. The first 'Z' in (1.0.8) refers to the functor $Z\colon \mathbf{2Cob} \to \mathbf{Vect}_k$ whereas the second 'Z' means centre. The structure of $Z(S^1)$ as a commutative Frobenius algebra can be computed from the algebra A, too.

In the functorial definition of 2-dimensional TQFTs we say that the commutative Frobenius algebra provides a *global description* of the 2-dimensional TQFT. The relevant algebraic structure, namely the commutative Frobenius algebra $(C, \mu, \eta, \Delta, \varepsilon)$, has an immediate topological interpretation in terms of the vector space C associated with the circle, the linear maps μ, η, Δ, and ε associated with the generators (1.0.1), and in terms of the relations among the morphisms of $\mathbf{2Cob}$.

Hence, the centre $Z(A)$ of the semisimple algebra A has a clear topological interpretation, whilst the algebra A is so far just part of a 'recipe' (the state sum construction), but it is far from obvious whether A itself plays any role in the topology of 2-manifolds.

Given a 2-dimensional TQFT $Z\colon \mathbf{2Cob} \to \mathbf{Vect}_k$ where k is a field of characteristic zero, one can ask the converse question, namely, whether there is a finite-dimensional semisimple algebra A over k such that one can obtain the given TQFT from the state sum of Fukuma–Hosono–Kawai. Of course, the algebra structure of A needs to be such that $Z(A) = Z(S^1)$, but one also has to understand which Frobenius algebra structure to choose for A in order to recover the appropriate one for $Z(A)$. In order to answer this question, a topological interpretation of the algebra A is clearly desirable.

In Chapter 3, we show (see Theorem 3.4.8) that for every strongly separable[3] algebra A over any field k and for every choice of a symmetric Frobenius algebra structure for A, there is a knowledgeable Frobenius algebra $(A, Z(A), \imath, \imath^*)$ and a generalization to $\mathbf{2Cob}^{\mathrm{ext}}$ of the state sum of Fukuma–Hosono–Kawai that yields the open-closed TQFT characterized by $(A, Z(A), \imath, \imath^*)$. Extending the notion of a 2-dimensional TQFT to suitable manifolds with corners therefore reveals which topological role is played by the algebra A that enters the state sum construction.

Why is it important to better understand the role of the algebra A? After all, 2-dimensional TQFTs are well understood, and the state sum of Fukuma–Hosono–Kawai is just one of several ways of finding examples. The primary reason for understanding the role of the algebra A will be established in Chapter 4, where we associate A to a trivial tangle on a single strand and interpret the composition of tangles as triangulated 1-manifolds. However, this question has various other answers depending on the view point taken.

Going back to the string theoretic interpretation of open-closed cobordisms, state sum constructions of topological field theory tend to be the most relevant to path integral constructions and hence to the standard approach to modern physics. In the state sum, the algebra A turns out to be the algebra associated to the open string and the state sum itself is interpreted as a discrete path integral. This suggests that the state sum construction may be useful for studying topological string theory. The state sum of Fukuma–Hosono–Kawai is also relevant to recent work on boundary conformal field theory, see, for example [35, 41]

[3]It turns out that for a field of arbitrary characteristic, the appropriate class of algebras is that of the strongly separable ones. Strongly separable algebras are defined in Section 3.2. They are characterized by the nondegeneracy of a certain canonically associated bilinear form.

where the algebra A already appears in connection with the boundary conditions, and so this thesis is immediately relevant in this context.

Another reason for better understanding the topological significance of the algebra A is given by attempts to generalize the framework to higher dimensions. For $n \geq 3$, the cobordism category **nCob** is not fully understood, *i.e.* n-dimensional cobordisms have not been (or even cannot be) classified, and in particular one does not have any description of **nCob** in terms of generators and relations. This makes a full understanding of n-dimensional TQFTs much harder if not impossible.

On the other hand, there are some generalizations of the state sum construction of Fukuma–Hosono–Kawai to higher dimensions, notably the 3-dimensional TQFT of Turaev and Viro [42], extended by Barrett and Westbury [43], which produces a 3-dimensional TQFT for any given modular category or, more generally, for suitable spherical categories [44]. The step from dimension 2 to 3, *i.e.* from the state sum of Fukuma–Hosono–Kawai to that of Turaev–Viro, can be understood as an example of *categorification*, see for example [45]. The dimensional ladder of Crane and Frenkel [46] sketches which sort of algebraic structures one would need in order to construct n-dimensional TQFTs from state sums:

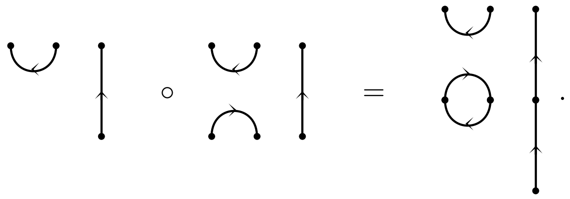
$$
\begin{tikzcd}
n=4 & \text{trialgebras} \arrow[dr, dash] & & \text{Hopf categories} \arrow[dl, dash] \arrow[dr, dash] & & \text{monoidal 2-categories} \arrow[dl, dash] \\
n=3 & & \text{Hopf algebras} \arrow[dr, dash] & & \text{monoidal categories} \arrow[dl, dash] & \\
n=2 & & & \text{associative algebras} & &
\end{tikzcd}
\tag{1.0.9}
$$

In this diagram, the entry 'associative algebras' refers to the state sum of Fukuma–Hosono–Kawai whereas 'monoidal categories' refers to the Turaev–Viro state sum. For $n = 2$, it should actually read '*strongly separable* associative algebras'. The appropriate choice of adjectives for the other cases is in fact not systematically understood. In order to settle this question and in order to extend the diagram upwards to higher dimensions, one can ask whether it is possible to classify the algebraic structures from which one can construct n-dimensional state sum TQFTs for generic n.

Whereas the algebraic structures of (1.0.9) that are relevant to the state sum construction are closely related to Pachner moves [47] and to the coherence conditions in higher categories, they have no obvious relationship to the global description of the TQFT as a functor $Z\colon \mathbf{nCob} \to \mathbf{Vect}_k$.

By showing that the associative algebra A of the Fukuma–Hosono–Kawai state sum is precisely the vector space $A = Z(I)$ associated with the unit interval in an appropriately extended notion of 2-dimensional TQFT, we have revealed such a relationship for the simplest case $n = 2$ of the dimensional ladder (1.0.9). This raises the question of whether one can find topological interpretations for the other algebraic structures featured in (1.0.9), presumably by extending the notion of TQFT from conventional cobordisms to manifolds with corners of higher and higher codimension. Further evidence for such a relationship is provided by the Hopf algebra object in 3-dimensional extended TQFTs [48–50] in connection with Kuperberg's 3-manifold invariant which is based on certain Hopf algebras [51].

Putting these other interests aside, the state sum construction allows one to compute an open-closed TQFT from the input data of a strongly separable algebra. The result of composing the resolutions of two tangles along their boundaries can then be interpreted as a triangulated 1-manifold where the boundaries that we glued along appear as vertices in the triangulation. For example,



(1.0.10)

Evaluating the result with a state sum TQFT then translates this composite into algebra. We will see in Chapter 4 that this idea extends to define the algebraic operations needed in order to algebraically compose the complexes constructed from the composite of two tangles.

1.0.6 Summary

The essential goal of this thesis can be seen as developing the theory of open-closed topological field theories to a level comparable with the theory of closed 2-dimensional TQFTs. In this vein, we prove an equivalence of categories between open-closed topological quantum field theories and the algebraic category of knowledgeable Frobenius algebras. This result generalizes Abrams work [17] providing an equivalence of categories between closed TQFTs and commutative Frobenius algebras, a result which is the foundation of many applications of closed TQFTs.

The state sum construction of Fukuma, Hosono, and Kawai provides another way to view closed 2-dimensional TQFTs. This constructions relates axiomatic TQFTs to physics in a natural way. By defining a state sum construction for open-closed TQFTs we also extend this important construction from the context of closed 2-dimensional cobordisms.

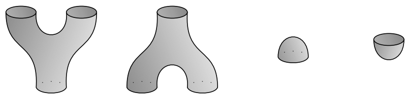
Finally, we extend one of the most important applications of closed TQFTs, namely Khovanov link homology. This extension provides an important example of the relevance of open-closed cobordisms to solving concrete mathematical problems. Using both the state sum construction, as well as the basic machinery developed for open-closed TQFTs, we define algebraic tangle homology theories.

Chapter 2

Open-Closed Topological Field Theories

2.1 Introduction

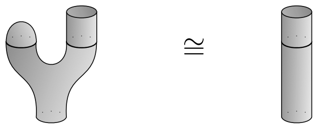
The most powerful results on closed TQFTs crucially depend on results from Morse theory. Morse theory provides a generators and relations description of the category **2Cob**. First, any compact cobordism Σ can be obtained by gluing a finite number of elementary cobordisms along their boundaries. In order to see this, one chooses a Morse function $f\colon \Sigma \to \mathbb{R}$ such that all critical points have distinct critical values and considers the pre-images $f^{-1}([x_0-\varepsilon, x_0+\varepsilon]) \subseteq \Sigma$ of intervals that contain precisely one critical value $x_0 \in \mathbb{R}$. Each such pre-image is the free union of one of the elementary cobordisms,



(2.1.1)

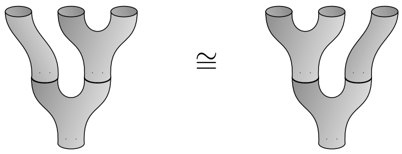
with zero or more cylinders over S^1. The different elementary cobordisms (2.1.1) are precisely the Morse data that characterize the critical points, and the way they are glued corresponds to the handle decomposition associated with f. The Morse data of (2.1.1) provide the *generators* for the morphisms of **2Cob**. Our diagrams of open-closed cobordisms, for example (1.0.5), are organized in such a way that the vertical axis of the drawing plane serves as a Morse function, and the cobordisms are composed of building blocks that contain at most one critical point.

Second, given two Morse functions $f_1, f_2\colon \Sigma \to \mathbb{R}$, the handle decompositions associated with f_1 and f_2 are related by a finite sequence of *moves*, *i.e.* handle slides and handle cancellations. This means that there are diffeomorphisms such as,

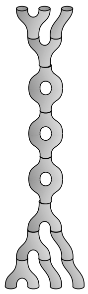


(2.1.2)

which provide us with the *relations* of **2Cob**. When we explicitly construct the diffeomorphism that relates two handle decompositions of some manifold, we call these diffeomorphisms *moves*. The example (2.1.2) corresponds to a cancelation of a 1-handle and a 2-handle. Below is an example of sliding a 1-handle past another 1-handle.

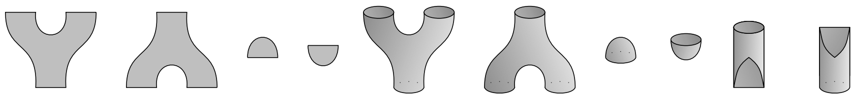


(2.1.3)



Whereas it is not too difficult to construct by brute-force a set of diffeomorphisms between manifolds such as those in (2.1.2) and (2.1.3), *i.e.* to show that a set of relations is necessary, it is much harder to show that they are also sufficient, *i.e.* that any two handle decompositions are related by a finite sequence of moves such as (2.1.2) and (2.1.3). In order to establish this result, one strategy is to prove that there exists a *normal form* for the morphisms of **2Cob** which is characterized by topological invariants, and then to show that the relations suffice in order to transform an arbitrary handle decomposition into this normal form. The normal form for closed cobordisms is determined by the number of incoming and outgoing boundary components together with the genus. The example to the right shows the normal form of a closed cobordism with three incoming boundary components, four outgoing boundary components, and genus three. For closed cobordisms, the normal form and proof of the sufficiency of the relations is done in detail in [17, 19, 52].

As was mentioned in the introduction, the purpose of this chapter is to develop a similar story for open-closed cobordisms. We will define the relevant Morse theory that decomposes an open-closed cobordism into the elementary generators



μ_A Δ_A η_A ε_A μ_C Δ_C η_C ε_C $\imath$ $\imath^*$. (2.1.4)

The relevant topological invariants are also defined. We then go on to define the normal form of an open-closed cobordism and provide a combinatorial proof that a well known set of relations is sufficient. As a result, we can provide a complete algebraic description of open-closed TQFTs in terms of knowledgeable Frobenius algebras (in fact a categorical equivalence). We should comment here that since the appearance of [1] alternative non Morse-theoretic proofs of the generators and relations description of $\mathbf{2Cob}^{\mathrm{ext}}$ have appeared in [53].

This chapter is structured as follows: in Section 2.2 we introduce some convenient diagrams and use them in Section 2.3 to express the definition of a Frobenius algebra. In Section 2.4, we define the notion of a knowledgeable Frobenius algebra and introduce the symmetric monoidal category $\mathbf{K\text{-}Frob}(\mathcal{C})$ of knowledgeable Frobenius algebras in a symmetric monoidal category $\mathcal{C}$. We provide an abstract description in terms of generators and relations of this category by defining a category $\mathbf{Th}(\mathbf{K\text{-}Frob})$, called the *theory of knowledgeable Frobenius algebras*, and by showing that the category of symmetric monoidal functors and monoidal natural transformations $\mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathcal{C}$ is equivalent as a symmetric monoidal category to $\mathbf{K\text{-}Frob}(\mathcal{C})$. In Section 2.5, we introduce the category $\mathbf{2Cob}^{\mathrm{ext}}$ of open-closed cobordisms. We present a normal form for such cobordisms and characterize the category in terms of generators and relations. In Section 2.7, we define open-closed TQFTs as symmetric monoidal functors $\mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$ into some symmetric monoidal category $\mathcal{C}$. We show that the category $\mathbf{2Cob}^{\mathrm{ext}}$ is equivalent as a symmetric monoidal category to $\mathbf{Th}(\mathbf{K\text{-}Frob})$ which in turn implies that the category of open-closed TQFTs in $\mathcal{C}$ is equivalent as a symmetric monoidal category to the category of knowledgeable Frobenius algebras $\mathbf{K\text{-}Frob}(\mathcal{C})$. In Section 2.8, we generalize our results to the case of labeled free boundaries. Section 2.9 contains a summary and an outlook on open problems related to open-closed TQFTs.

2.2 Symmetric monoidal categories and string diagrams

In this section, we review the basics of string diagrams in a symmetric monoidal category. The definitions and required facts about symmetric monoidal categories are collected for convenience in Appendix A. We denote the class of objects of a category $\mathcal{C}$ by $|\mathcal{C}|$ and for each object $X \in |\mathcal{C}|$, the identity morphism by $\mathrm{id}_X \colon X \to X$.

Definition 2.2.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category with tensor product $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, unit object $\mathbb{1} \in |\mathcal{C}|$, associativity constraint $\alpha_{X,Y,Z} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, left- and right-unit constraints $\lambda_X \colon \mathbb{1} \otimes X \to X$ and $\rho_X \colon X \otimes \mathbb{1} \to X$, and the symmetric braiding $\tau_{X,Y} \colon X \otimes Y \to Y \otimes X$, for objects X, Y, Z of $\mathcal{C}$ (in symbols $X, Y, Z \in |\mathcal{C}|$).

1. An object X of $\mathcal{C}$ is called *rigid* if it has a *left-dual* $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$. This is an object

X^* of $\mathcal{C}$ with morphisms $\mathrm{ev}_X\colon X^* \otimes X \to \mathbb{1}$ (*evaluation*) and $\mathrm{coev}_X\colon \mathbb{1} \to X \otimes X^*$ (*coevaluation*) which satisfy the *zig-zag identities*,

$$\rho_X \circ (\mathrm{id}_X \otimes \mathrm{ev}_X) \circ \alpha_{X,X^*,X} \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \circ \lambda_X^{-1} = \mathrm{id}_X, \quad \text{and} \tag{2.2.1}$$

$$\lambda_{X^*} \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^*}) \circ \alpha^{-1}_{X^*,X,X^*} \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X) \circ \rho^{-1}_{X^*} = \mathrm{id}_{X^*}\,. \tag{2.2.2}$$

2. Let X be a rigid object of $\mathcal{C}$ and $f \in \mathrm{Hom}(X,X)$. The *categorical trace* $\mathrm{tr}_X(f)$ is defined by,

$$\mathrm{tr}_X(f) := \mathrm{ev}_X \circ \tau_{X,X^*} \circ (f \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X \in \mathrm{Hom}(\mathbb{1},\mathbb{1}). \tag{2.2.3}$$

3. The *categorical dimension* $\dim X$ of a rigid object X of $\mathcal{C}$ is defined by,

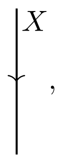
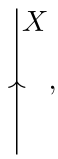
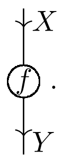
$$\dim X := \mathrm{tr}_X(\mathrm{id}_X) \in \mathrm{Hom}(\mathbb{1},\mathbb{1}). \tag{2.2.4}$$

4. For rigid objects X and Y of $\mathcal{C}$ and $f \in \mathrm{Hom}(X,Y)$, the morphism,

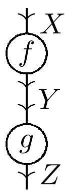
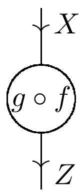
$$f^* := \lambda_{X^*} \circ (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ ((\mathrm{id}_{Y^*} \otimes f) \otimes \mathrm{id}_{X^*}) \circ \alpha^{-1}_{Y^*,X,X^*} \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X) \circ \rho^{-1}_{Y^*}\colon Y^* \to X^*, \tag{2.2.5}$$

is called the *dual* of f.

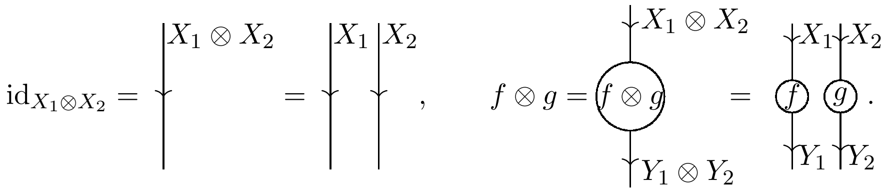
In the following, we use *string diagrams* [54,55] to visualize morphisms of a given symmetric monoidal category $\mathcal{C}$ and the identities between them. The diagrams are read from top to bottom. For each object $X \in |\mathcal{C}|$, the identity morphism id_X is denoted by a line labeled 'X' with an arrow pointing down. The identity morphism id_{X^*} of the dual object has the arrow pointing up. For a morphism $f\colon X \to Y$, we write a disc labeled 'f', called a *coupon*. This disc has a white side which always faces the reader and a black side which never does so,

$\mathrm{id}_X =$  , $\quad \mathrm{id}_{X^*} =$  , $\quad f =$  . (2.2.6)

Composition of morphisms is depicted by vertically concatenating the corresponding diagrams; for example, for morphisms $f\colon X \to Y$ and $g\colon Y \to Z$,

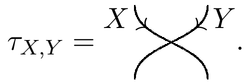
$g \circ f =$  $=$  . (2.2.7)

The tensor product of morphisms is visualized by putting diagrams next to each other; for example, for $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$,



(2.2.8)

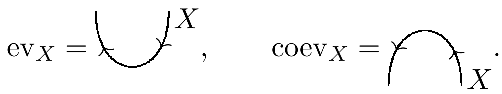
The symmetric braiding is denoted by,



(2.2.9)

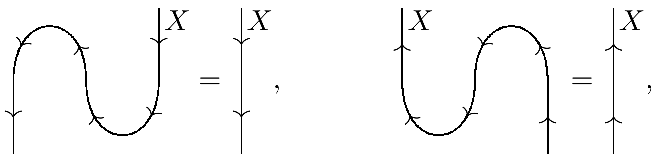
Mac Lane's coherence theorem for monoidal categories [56] then ensures that once a choice of parenthesis for the source and target of the string diagram has been chosen, one can unambiguously translate any such string diagram into a morphism of $\mathcal{C}$. One therefore chooses parentheses for all tensor products that occur in the source and target objects of the morphism and inserts the structure isomorphisms α, λ, ρ appropriately. The coherence theorem implies that all possible ways of inserting the structure isomorphisms yield equal morphisms, *i.e.* that there is a well-defined morphism of $\mathcal{C}$ specified by the diagram. In addition, the coherence theorem allows us to suppress the lines associated with the unit object $\mathbb{1}$.

For a rigid object $X \in \mathcal{C}$, evaluation and coevaluation are represented by these diagrams:



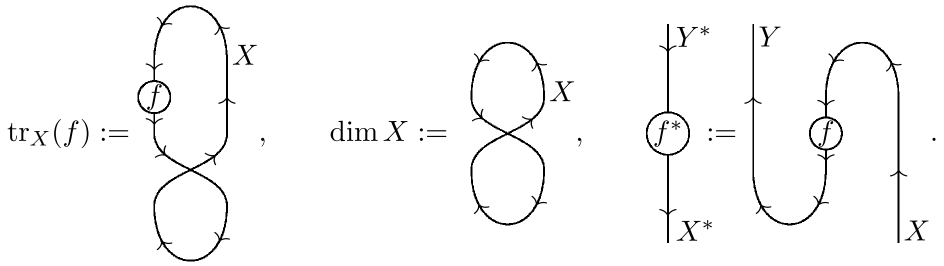
(2.2.10)

The zig-zag identities of (2.2.1) are represented in string diagrams as:



(2.2.11)

and the definitions of trace (2.2.3), dimension (2.2.4) and dual morphism (2.2.5) are:



(2.2.12)

If $\mathcal{C}$ is locally small (meaning that the morphisms $\mathrm{Hom}(X,Y)$ between every pair of objects X and Y form a set), the set $\mathrm{Hom}(\mathbb{1},\mathbb{1})$ forms a commutative monoid with multiplication

$\xi_1 \cdot \xi_2 := \lambda_{\mathbb{1}} \circ (\xi_1 \otimes \xi_2) \circ \lambda_{\mathbb{1}}^{-1}$ for $\xi_1, \xi_2 \in \mathrm{Hom}(\mathbb{1}, \mathbb{1})$ and unit $\mathrm{id}_{\mathbb{1}}$. The monoid $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ acts on $\mathrm{Hom}(X, Y)$ for all $X, Y \in |\mathcal{C}|$ by $\xi \cdot f := \lambda_Y \circ (\xi \otimes f) \circ \lambda_X^{-1}$ where $f \in \mathrm{Hom}(X, Y)$ and $\xi \in \mathrm{Hom}(\mathbb{1}, \mathbb{1})$.

The coherence theorem now allows us to view the elements of $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ as scalars by which the entire diagram is multiplied.

2.3 Frobenius algebras

In this section, we define the notion of a Frobenius algebra. We consider these Frobenius algebras not only in the symmetric monoidal category $\mathbf{Vect}_k$ of vector spaces over some fixed field k, but in an arbitrary symmetric monoidal category. Other examples include the symmetric monoidal categories of Abelian groups, graded-vector spaces, and chain complexes.

2.3.1 Definitions

Definition 2.3.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category.

1. An *algebra object* (A, μ, η) in $\mathcal{C}$ consists of an object A and morphisms $\mu\colon A \otimes A \to A$ and $\eta\colon \mathbb{1} \to A$ of $\mathcal{C}$ such that:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\alpha_{A,A,A}"] \arrow[d, "\mu \otimes \mathrm{id}_A"'] & & A \otimes (A \otimes A) \arrow[d, "\mathrm{id}_A \otimes \mu"] \\
A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd} \tag{2.3.1}$$

and

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "\eta \otimes \mathrm{id}_A"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{1} \arrow[l, "\mathrm{id}_A \otimes \eta"'] \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd} \tag{2.3.2}$$

commute.

2. A *coalgebra object* (A, Δ, ε) in $\mathcal{C}$ consists of an object A and morphisms $\Delta\colon A \to A \otimes A$

and $\varepsilon\colon A \to \mathbb{1}$ of $\mathcal{C}$ such that:

$$\begin{tikzcd}
& A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[d, "\Delta\otimes\mathrm{id}_A"'] & & A \otimes A \arrow[d, "\mathrm{id}_A\otimes\Delta"] \\
(A \otimes A) \otimes A \arrow[rr, "\alpha_{A,A,A}"'] & & A \otimes (A \otimes A)
\end{tikzcd} \tag{2.3.3}$$

and

$$\begin{tikzcd}
& A \arrow[dl, "\lambda_A^{-1}"'] \arrow[d, "\Delta"'] \arrow[dr, "\rho_A^{-1}"] & \\
\mathbb{1} \otimes A & A \otimes A \arrow[l, "\varepsilon\otimes\mathrm{id}_A"] \arrow[r, "\mathrm{id}_A\otimes\varepsilon"'] & A \otimes 1
\end{tikzcd} \tag{2.3.4}$$

commute.

3. A *homomorphism of algebras* $f\colon A \to A'$ between two algebra objects (A, μ, η) and (A', μ', η') in $\mathcal{C}$ is a morphism f of $\mathcal{C}$ such that:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"] \\
A' \otimes A' \arrow[r, "\mu'"'] & A'
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathbb{1} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & A \arrow[d, "f"] \\
& A'
\end{tikzcd} \tag{2.3.5}$$

commute.

4. A *homomorphism of coalgebras* $f\colon A \to A'$ between two coalgebra objects (A, Δ, ε) and $(A', \Delta', \varepsilon')$ in $\mathcal{C}$ is a morphism f of $\mathcal{C}$ such that:

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "f"'] & A \otimes A \arrow[d, "f\otimes f"] \\
A' \arrow[r, "\Delta'"'] & A' \otimes A'
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "\varepsilon"] & \\
A' \arrow[r, "\varepsilon'"'] & \mathbb{1}
\end{tikzcd} \tag{2.3.6}$$

commute.

Definition 2.3.2. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category.

1. A *Frobenius algebra object* $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ consists of an object A and of morphisms μ, η, Δ, ε of $\mathcal{C}$ such that:

(a) (A, μ, η) is an algebra object in $\mathcal{C}$,

(b) (A, Δ, ε) is a coalgebra object in $\mathcal{C}$, and

(c) the following compatibility condition, called the *Frobenius relation*, holds,

$$\begin{tikzcd}
 & A \otimes A \arrow[dl, "\Delta\otimes\mathrm{id}_A"'] \arrow[dd, "\mu"] \arrow[dr, "\mathrm{id}_A\otimes\Delta"] & \\
(A \otimes A) \otimes A \arrow[dd, "\alpha_{A,A,A}"'] & & A \otimes (A \otimes A) \arrow[dd, "\alpha^{-1}_{A,A,A}"] \\
 & A \arrow[dd, "\Delta"] & \\
A \otimes (A \otimes A) \arrow[dr, "\mathrm{id}_A\otimes\mu"'] & & (A \otimes A) \otimes A \arrow[dl, "\mu\otimes\mathrm{id}_A"] \\
 & A \otimes A &
\end{tikzcd} \tag{2.3.7}$$

2. A Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is called *symmetric* if:

$$\varepsilon \circ \mu = \varepsilon \circ \mu \circ \tau. \tag{2.3.8}$$

It is called *commutative* if:

$$\mu = \mu \circ \tau. \tag{2.3.9}$$

3. Let $\mathcal{C}$ be locally small. A Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is called *special* (as defined in [34]) if

$$\varepsilon \circ \eta = \xi_{\mathbb{1}} \cdot \mathrm{id}_{\mathbb{1}} \qquad \text{and} \qquad \mu \circ \Delta = \xi_A \cdot \mathrm{id}_A \tag{2.3.10}$$

for some $\xi_{\mathbb{1}}, \xi_A \in \mathrm{Hom}(\mathbb{1}, \mathbb{1})$ that are invertible in the monoid $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$.

4. Let $(A, \mu, \eta, \Delta, \varepsilon)$ and $(A', \mu', \eta', \Delta', \varepsilon')$ be Frobenius algebra objects in $\mathcal{C}$. A *homomorphism of Frobenius algebras* $f\colon A \to A'$ is a morphism f of $\mathcal{C}$ which is both a homomorphism of algebra objects and a homomorphism of coalgebra objects.

Notice that for any Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$, the object A is always a rigid object of $\mathcal{C}$. In $\mathbf{Vect}_k$ the rigid objects are the finite-dimensional vector spaces; hence every Frobenius algebra object in $\mathbf{Vect}_k$ is finite-dimensional.

The unit object $\mathbb{1} \in |\mathcal{C}|$ forms an algebra object $(\mathbb{1}, \lambda_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$ in $\mathcal{C}$ with multiplication $\lambda_{\mathbb{1}}\colon \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ and unit $\mathrm{id}_{\mathbb{1}}\colon \mathbb{1} \to \mathbb{1}$ as well as a coalgebra object $(\mathbb{1}, \lambda_{\mathbb{1}}^{-1}, \mathrm{id}_{\mathbb{1}})$ defining a commutative Frobenius algebra object in $\mathcal{C}$. Given two algebra objects (A, μ_A, η_A) and (B, μ_B, η_B) in $\mathcal{C}$, the tensor product $(A \otimes B, \mu_{A\otimes B}, \eta_{A\otimes B})$ forms an algebra object in $\mathcal{C}$ with,

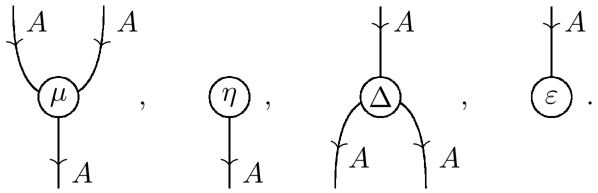
$$\begin{aligned} \mu_{A\otimes B} &= (\mu_A \otimes \mu_B) \circ \alpha^{-1}_{A,A,B\otimes B} \circ (\mathrm{id}_A \otimes \alpha_{A,B,B}) \circ (\mathrm{id}_A \otimes (\tau_{B,A} \otimes \mathrm{id}_B)) \\ &\quad \circ (\mathrm{id}_A \otimes \alpha^{-1}_{B,A,B}) \circ \alpha_{A,B,A\otimes B}, \qquad (2.3.11) \\ \eta_{A\otimes B} &= (\eta_A \otimes \eta_B) \circ \lambda^{-1}_{\mathbb{1}}. \qquad (2.3.12) \end{aligned}$$

A similar result holds for coalgebra objects and for Frobenius algebra objects in $\mathcal{C}$. Given two homomorphisms of algebra objects $f\colon (A, \mu_A, \eta_A) \to (A', \mu_{A'}, \eta_{A'})$ and $g\colon (B, \mu_B, \eta_B) \to (B', \mu_{B'}, \eta_{B'})$, their tensor product $f \otimes g\colon (A \otimes B, \mu_{A\otimes B}, \eta_{A\otimes B}) \to (A' \otimes B', \mu_{A'\otimes B'}, \eta_{A'\otimes B'})$ forms a homomorphism of algebra objects. A similar result holds for homomorphisms of coalgebra and for homomorphisms of Frobenius algebra objects.

Definitions (1.0.1)–(1.0.3) provide examples of commutative Frobenius algebra objects in the symmetric monoidal category $\mathbf{Vect}_k$ of vector spaces over the field k. We will sometimes omit the 'object' in 'Frobenius algebra object' for simplicity.

2.3.2 String diagrams for Frobenius algebras

Using the string diagram notation from Section 2.2, the string diagrams for the operations of a Frobenius algebra $(A, \mu, \eta, \Delta, \varepsilon)$ are as follows:



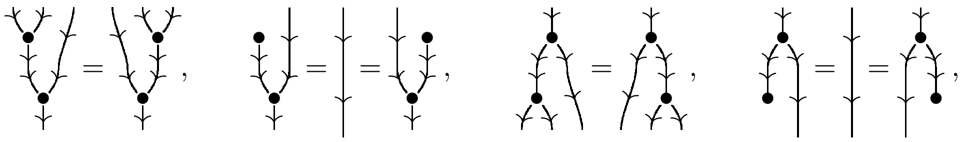
(2.3.13)

In order to keep the diagrams small, from now on we replace the coupons by vertices and also drop the label 'A' wherever it is clear from the context:

(2.3.14)

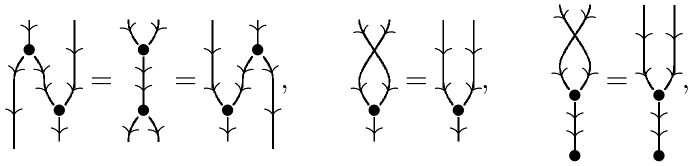
It is understood that the vertices have to be replaced by discs in the paper plane with their white side facing the reader. Furthermore, we drop all labels μ, η, Δ and ε where these are evident from the context. For example, we distinguish the operation Δ from μ by the number of incoming and outgoing lines.

The axioms of an algebra and those of a coalgebra then read:



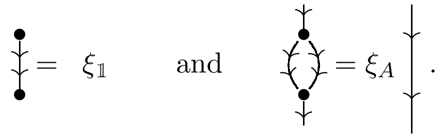
(2.3.15)

and the Frobenius relation, commutativity and symmetry are depicted as follows:



(2.3.16)

The conditions for a the Frobenius algebra to be special are these:



(2.3.17)

The string diagram presentation of the Frobenius structure will be important for the state sum construction in Chapter 3.

2.4 Knowledgeable Frobenius algebras

The following definition plays a central role in the structure of open-closed TQFTs.

Definition 2.4.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category. A *knowledgeable Frobenius algebra* $\mathbb{A} = (A, C, \imath, \imath^*)$ in $\mathcal{C}$ consists of,

- a symmetric Frobenius algebra $A = (A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$,
- a commutative Frobenius algebra $C = (C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$,
- morphisms $\imath \colon C \to A$ and $\imath^* \colon A \to C$ of $\mathcal{C}$,

such that $\imath \colon C \to A$ is a homomorphism of algebra objects in $\mathcal{C}$ and,

$$\mu_A \circ (\imath \otimes \mathrm{id}_A) = \mu_A \circ \tau_{A,A} \circ (\imath \otimes \mathrm{id}_A) \qquad \text{(knowledge)}, \tag{2.4.1}$$

$$\varepsilon_C \circ \mu_C \circ (\mathrm{id}_C \otimes \imath^*) = \varepsilon_A \circ \mu_A \circ (\imath \otimes \mathrm{id}_A) \qquad \text{(duality)}, \tag{2.4.2}$$

$$\mu_A \circ \tau_{A,A} \circ \Delta_A = \imath \circ \imath^* \qquad \text{(Cardy condition)}. \tag{2.4.3}$$

Condition (2.4.2) says that $\imath^*$ is the morphism dual to $\imath$. Together with the fact that $\imath$ is an algebra homomorphism, this implies that $\imath^* \colon A \to C$ is a homomorphism of coalgebras in $\mathcal{C}$. In the category of (super) vector spaces the structure of a knowledgeable Frobenius algebra first appeared in the work of Moore and Segal [32], and later in the work of Lazaroiu [30].

If $\mathcal{C} = \mathbf{Vect}_k$, the condition (2.4.1) states that the image of C under $\imath$ is contained in the centre of A, $\imath(C) \subseteq Z(A)$. The name *knowledgeable Frobenius algebra* is meant to indicate that the symmetric Frobenius algebra A is equipped with knowledge about its centre. This is specified precisely by C, $\imath$ and $\imath^*$. Notice that the centre $Z(A)$ itself cannot be characterized[1]

[1] We thank James Dolan and John Baez for pointing this out.

by requiring the commutativity of diagrams labeled by objects and morphisms of $\mathcal{C}$. Within the language of category theory one can not express using diagrams the assertion that a given object is the centre of another object. While the notion of subobject makes sense, and diagrams can be used to assert that a subobject commutes with a given object, no string diagram, or equivalently commutative diagram, can be drawn that asserts that a given subobject is *all* of the centre, even when the notion of centre makes sense within the category $\mathcal{C}$.

In Chapter 3 we will show that every strongly separable algebra A can be equipped with the structure of a Frobenius algebra such that $(A, Z(A), \imath, \imath^*)$ forms a knowledgeable Frobenius algebra with the inclusion $\imath\colon Z(A) \to A$ and an appropriately chosen Frobenius algebra structure on $Z(A)$.

Sometimes the folk theorem on the characterization of open-closed TQFTs is stated in such a way that it includes the condition $C = Z(A)$. In Example 3.2.15 we provide an example of a knowledgeable Frobenius algebra, and thereby open-closed TQFT, in which this condition does not hold. While it is obvious that the this condition need not hold — one can modify the centre in some trivial way — in Examples (4.5.6) and (4.5.7) nontrivial examples with this property are also supplied. Several other examples of knowledgeable Frobenius algebras are presented in Section 4.5.1.

2.4.1 The category K-Frob($\mathcal{C}$)

Definition 2.4.2. A *homomorphism of knowledgeable Frobenius algebras*

$$\varphi\colon (A, C, \imath, \imath^*) \to (A', C', \imath', \imath'^*) \tag{2.4.4}$$

in the symmetric monoidal category $\mathcal{C}$ is a pair $\varphi = (\varphi_1, \varphi_2)$ of Frobenius algebra homomorphisms $\varphi_1\colon A \to A'$ and $\varphi_2\colon C \to C'$ such that

$$\begin{array}{ccccc} C & \xrightarrow{\varphi_2} & C' \\ {\scriptstyle \imath}\downarrow & & \downarrow{\scriptstyle \imath'} \\ A & \xrightarrow[\varphi_1]{} & A' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\varphi_1} & A' \\ {\scriptstyle \imath^*}\downarrow & & \downarrow{\scriptstyle \imath'^*} \\ C & \xrightarrow[\varphi_2]{} & C' \end{array} \tag{2.4.5}$$

commute.

Definition 2.4.3. Let $\mathcal{C}$ be a symmetric monoidal category. By **K-Frob**($\mathcal{C}$) we denote the category of knowledgeable Frobenius algebras in $\mathcal{C}$ and their homomorphisms.

Proposition 2.4.4. Let $\mathcal{C}$ be a symmetric monoidal category. The category **K-Frob**$(\mathcal{C})$ forms a symmetric monoidal category as follows. The tensor product of two knowledgeable Frobenius algebra objects $\mathbb{A} = (A, C, \imath, \imath^*)$ and $\mathbb{A}' = (A', C', \imath', \imath'^*)$ is defined as $\mathbb{A} \otimes \mathbb{A}' := (A \otimes A', C \otimes C', \imath \otimes \imath', \imath^* \otimes \imath'^*)$. The unit object is given by $\underline{\mathbb{1}} := (\mathbb{1}, \mathbb{1}, \mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$, and the associativity and unit constraints and the symmetric braiding are induced by those of $\mathcal{C}$. Given two homomorphisms $\varphi = (\varphi_1, \varphi_2)$ and $\psi = (\psi_1, \psi_2)$ of knowledgeable Frobenius algebras, their tensor product is defined as $\varphi \otimes \psi := (\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2)$.

2.4.2 The category Th(K-Frob)

In this section, we define the category **Th**(**K-Frob**), called the *theory of knowledgeable Frobenius algebras*. The description that follows is designed to make **Th**(**K-Frob**) the symmetric monoidal category freely generated by a knowledgeable Frobenius algebra, and the terminology 'theory of ...' indicates that knowledgeable Frobenius algebras in any symmetric monoidal category $\mathcal{C}$ arise precisely as the symmetric monoidal functors **Th**(**K-Frob**) $\to \mathcal{C}$. This is in analogy to the theory of algebraic theories in which one uses 'with finite products' rather than 'symmetric monoidal'. The category **Th**(**K-Frob**) can also be described as the 'walking knowledgeable Frobenius algebra' in the terminology of [39, 57]. Readers who are interested in the topology of open-closed cobordisms rather than in the abstract description of knowledgeable Frobenius algebras may wish to look briefly at Proposition 2.4.6 and then directly proceed to Section 2.5.

The subsequent definition follows the construction of the 'free category with group structure' given by Laplaza [58]. It forms an example of a symmetric monoidal sketch, a structure slightly more general than an operad or a PROP, see for example [59] for the definition of symmetric monoidal sketches and a discussion of their freeness properties.

Definition 2.4.5. The category **Th**(**K-Frob**) is defined as follows: its objects are the elements of the free $\{\mathbb{1}, \otimes\}$-algebra over the two element set $\{A, C\}$. These are words of a formal language that are defined by the following requirements,

- The symbols $\mathbb{1}$, A and C are objects of **Th**(**K-Frob**).
- If X and Y are objects of **Th**(**K-Frob**), then $(X \otimes Y)$ is an object of **Th**(**K-Frob**).

We now describe the edges of a graph $\mathcal{G}$ whose vertices are the objects of **Th**(**K-Frob**). There are edges,

$$
\begin{gathered}
\mu_A\colon A\otimes A\to A,\quad \eta_A\colon \mathbb{1}\to A,\quad \Delta_A\colon A\to A\otimes A,\quad \varepsilon_A\colon \mathbb{1}\to A,\\
\mu_C\colon C\otimes C\to C,\quad \eta_C\colon \mathbb{1}\to C,\quad \Delta_C\colon C\to C\otimes C,\quad \varepsilon_C\colon \mathbb{1}\to C,\\
\imath\colon C\to A,\qquad \imath^*\colon A\to C,
\end{gathered}
\tag{2.4.6}
$$

and for all objects X,Y,Z there are to be edges

$$
\begin{gathered}
\alpha_{X,Y,Z}\colon (X\otimes Y)\otimes Z\to X\otimes(Y\otimes Z),\quad \tau_{X,Y}\colon X\otimes Y\to Y\otimes X,\\
\lambda_X\colon \mathbb{1}\otimes X\to X,\quad \rho_X\colon X\otimes\mathbb{1}\to X,
\end{gathered}
\tag{2.4.7}
$$

$$
\begin{gathered}
\bar\alpha_{X,Y,Z}\colon X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z,\quad \bar\tau_{X,Y}\colon Y\otimes X\to X\otimes Y,\\
\bar\lambda_X\colon X\to\mathbb{1}\otimes X,\quad \bar\rho_X\colon X\to X\otimes\mathbb{1}.
\end{gathered}
\tag{2.4.8}
$$

For every edge $f\colon X\to Y$ and for every object Z, there are to be edges $Z\otimes f\colon Z\otimes X\to Z\otimes Y$, $f\otimes Z\colon X\otimes Z\to Y\otimes Z$. These edges are to be interpreted as words in a formal language and are considered distinct if they have distinct names.

Let $\mathcal{H}$ be the category freely generated by the graph $\mathcal{G}$. We now describe a congruence on the category $\mathcal{H}$. We define a relation $\sim$ as follows. We require the relations making $(A,\mu_A,\eta_A,\Delta_A,\varepsilon_A)$ a symmetric Frobenius algebra object, those making $(C,\mu_C,\eta_C,\Delta_C,\varepsilon_C)$ a commutative Frobenius algebra object, those making $\imath\colon C\to A$ an algebra homomorphism as well as (2.4.1), (2.4.2), and (2.4.3). The relations making $\alpha_{X,Y,Z}$, λ_X, and ρ_X satisfy the pentagon and triangle axioms of a monoidal category as well as those making $\tau_{X,Y}$ a symmetric braiding, are required for all objects X,Y,Z. We also require the following relations for all objects X,Y and morphisms p,q,t,s of $\mathcal{H}$,

$$
\begin{gathered}
(X\otimes p)(X\otimes q)\sim X\otimes(pq),\qquad (p\otimes X)(q\otimes X)\sim(pq)\otimes X,\\
(t\otimes Y)(X\otimes s)\sim(X\otimes s)(t\otimes Y),\quad \mathrm{id}_{X\otimes Y}\sim X\otimes\mathrm{id}_Y\sim\mathrm{id}_X\otimes Y,
\end{gathered}
\tag{2.4.9}
$$

that make $\otimes$ a functor. Then we require the relations that assert the naturality of $\alpha,\lambda,\rho,\tau,\bar\alpha,\bar\lambda,\bar\rho,\bar\tau$ and that each pair e and $\bar e$ of edges of the graph form the inverses of each other. Finally, we have all expansions by $\otimes$, *i.e.* for each relation $a\sim b$, we include the relations $a\otimes X\sim b\otimes X$ and $X\otimes a\sim X\otimes b$ for all objects X, and all those relations obtained from these by a finite number of applications of this process. The category **Th**(**K-Frob**) is the category $\mathcal{H}$ modulo the category congruence generated by $\sim$.

It is clear from the description above that **Th**(**K-Frob**) contains a knowledgeable Frobenius algebra object $(A,C,\imath,\imath^*)$ which we call the knowledgeable Frobenius algebra object *generating* **Th**(**K-Frob**). Indeed, **Th**(**K-Frob**) is the symmetric monoidal category freely generated by a knowledgeable Frobenius algebra. Its basic property is that for any knowledgeable Frobenius algebra $\mathbb{A}' = (A', C', \imath', \imath'^*)$ in $\mathcal{C}$, there is exactly one *strict* symmetric monoidal functor $F_{\mathbb{A}'}\colon \mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathcal{C}$ which maps $(A, C, \imath, \imath^*)$ to $(A', C', \imath', \imath'^*)$ and $\mathbb{1} \in \mathbf{Th}(\mathbf{K\text{-}Frob})$ to $\mathbb{1} \in \mathcal{C}$.

An interesting question to ask is whether or not homomorphisms of knowledgeable Frobenius algebras are induced in some way by **Th**(**K-Frob**). This question is answered by the following proposition.

Proposition 2.4.6. Let $\mathcal{C}$ be a symmetric monoidal category. The category

$$\mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob}), \mathcal{C}) \tag{2.4.10}$$

of symmetric monoidal functors $\mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathcal{C}$ and their monoidal natural transformations is equivalent as a symmetric monoidal category to the category $\mathbf{K\text{-}Frob}(\mathcal{C})$.

This proposition is the reason for calling **Th**(**K-Frob**) the *theory of knowledgeable Frobenius algebras*. For easier reference, we collect the definitions of symmetric monoidal functors, monoidal natural transformations and of **Symm-Mon** in Appendix A.

Proof. Let $(A, C, \imath, \imath^*)$ be the knowledgeable Frobenius algebra generating **Th**(**K-Frob**), and let
$\psi\colon \mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathcal{C}$ be a symmetric monoidal functor. It is clear that the image of $(A, C, \imath, \imath^*)$ under ψ, together with the coherence isomorphisms ψ_0 and ψ_2 of $\psi = (\psi, \psi_2, \psi_0)$, defines a knowledgeable Frobenius algebra $(\psi(A), \psi(C), \psi(\imath), \psi(\imath^*))$ in $\mathcal{C}$. The symmetric Frobenius algebra structure on $\psi(A)$ is given by

$$\psi(A) = \big(\psi(A), \psi(\mu_A) \circ \psi_2, \psi(\eta_A) \circ \psi_0, \psi_2^{-1} \circ \psi(\Delta_A), \psi_0^{-1} \circ \psi(\varepsilon_A)\big). \tag{2.4.11}$$

The commutative Frobenius algebra structure on $\psi(C)$ is given by

$$\psi(C) = \big(\psi(C), \psi(\mu_C) \circ \psi_2, \psi(\eta_C) \circ \psi_0, \psi_2^{-1} \circ \psi(\Delta_C), \psi_0^{-1} \circ \psi(\varepsilon_C)\big). \tag{2.4.12}$$

This defines a mapping on objects

$$\begin{aligned} \Gamma\colon \mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob}), \mathcal{C}) &\to \mathbf{K\text{-}Frob}(\mathcal{C}) \\ \psi &\mapsto (\psi(A), \psi(C), \psi(\imath), \psi(\imath^*)). \end{aligned} \tag{2.4.13}$$

We now extend Γ to a functor by defining it on morphisms.

If $\varphi\colon \psi \Rightarrow \psi'$ is a monoidal natural transformation, then φ assigns to each object X in $\mathbf{Th}(\mathbf{K\text{-}Frob})$ a map $\varphi_X\colon \psi(X) \to \psi'(X)$ in $\mathcal{C}$. However, since every object in $\mathbf{Th}(\mathbf{K\text{-}Frob})$ is the tensor product of A's and C's and $\mathbb{1}$'s, the fact that φ is a *monoidal* natural transformation means that the φ_X are completely determined by two maps $\varphi_1\colon \psi(A) \to \psi'(A)$ and $\varphi_2\colon \psi(C) \to \psi'(C)$. The *naturality* of φ means that the φ_i are compatible with the images of all the morphisms in $\mathbf{Th}(\mathbf{K\text{-}Frob})$. Since all of the morphisms in $\mathbf{Th}(\mathbf{K\text{-}Frob})$ are built up from the generators:

$$\begin{gathered}
\mu_A\colon A\otimes A \to A, \quad \eta_A\colon \mathbb{1} \to A, \quad \Delta_A\colon A \to A\otimes A, \quad \varepsilon_A\colon \mathbb{1} \to A,\\
\mu_C\colon C\otimes C \to C, \quad \eta_C\colon \mathbb{1} \to C, \quad \Delta_C\colon C \to C\otimes C, \quad \varepsilon_C\colon \mathbb{1} \to C,\\
\imath\colon C \to A, \qquad \imath^*\colon A \to C,
\end{gathered} \tag{2.4.14}$$

(and the structure maps α, ρ, λ, τ), naturality can be expressed by the commutativity of 10 diagrams involving the 10 generating morphisms of $\mathbf{Th}(\mathbf{K\text{-}Frob})$. For example, corresponding to $\mu_A\colon A\otimes A \to A$ and $\eta_A\colon \mathbb{1} \to A$, we have the two diagrams:

$$\begin{tikzcd}
\psi(A\otimes A) \arrow[r, "\psi_2^{-1}"] \arrow[d, "\psi(\mu_A)"'] & \psi(A)\otimes\psi(A) \arrow[r, "\varphi_1\otimes\varphi_1"] & \psi'(A)\otimes\psi'(A) \arrow[r, "\psi_2'^{-1}"] & \psi'(A\otimes A) \arrow[d, "\psi'(\mu_A)"] \\
\psi(A) \arrow[rrr, "\varphi_1"'] & & & \psi'(A)
\end{tikzcd} \tag{2.4.15}$$

$$\begin{tikzcd}
\psi(\mathbb{1}) \arrow[r, "\psi_0^{-1}"] \arrow[d, "\psi(\eta_A)"'] & \mathbb{1} \arrow[r, "\psi_0'"] & \psi'(\mathbb{1}) \arrow[d, "\psi'(\eta_A)"] \\
\psi(A) \arrow[rr, "\varphi_1"'] & & \psi'(A)
\end{tikzcd} \tag{2.4.16}$$

which amount to saying that φ_1 is an algebra homomorphism $\psi(A) \to \psi'(A)$. Together with the conditions for the generators $\Delta_A\colon A \to A\otimes A$ and $\varepsilon_A\colon A \to \mathbb{1}$, we have that φ_1 is a Frobenius algebra homomorphism from $\psi(A)$ to $\psi'(A)$. Similarly, the diagrams corresponding to the generators with a C subscript imply that φ_2 is a Frobenius algebra homomorphism $\psi(C) \to \psi'(C)$. The conditions on the images of the generators $\imath\colon C \to A$ and $\imath^*\colon A \to C$ produce the requirement that the two diagrams:

$$\begin{tikzcd}
\psi(C) \arrow[r, "\varphi_2"] \arrow[d, "\psi(\imath)"'] & \psi'(C) \arrow[d, "\psi'(\imath)"] \\
\psi(A) \arrow[r, "\varphi_1"'] & \psi'(A)
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\psi(A) \arrow[r, "\varphi_1"] \arrow[d, "\psi(\imath^*)"'] & \psi'(A) \arrow[d, "\psi'(\imath^*)"] \\
\psi(C) \arrow[r, "\varphi_2"'] & \psi'(C)
\end{tikzcd} \tag{2.4.17}$$

commute. Hence, the monoidal natural transformation φ defines a morphism of knowledgeable Frobenius algebras in $\mathcal{C}$. This assignment clearly preserves the monoidal structure and symmetry up to isomorphism. Thus, it is clear that one can define a symmetric monoidal functor

$$\Gamma = (\Gamma, \Gamma_2, \Gamma_0)\colon \mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob})) \to \mathbf{K\text{-}Frob}(\mathcal{C}).$$

Conversely, given any knowledgeable Frobenius algebra $\mathbb{A}' = (A', C', \imath', \imath'^*)$ in $\mathcal{C}$, then by the remarks preceding this proposition, there is an assignment

$$\begin{aligned} \overline{\Gamma}\colon \mathbf{K\text{-}Frob}(\mathcal{C}) &\to \mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob}), \mathcal{C}) && (2.4.18)\\ (A', C', \imath', \imath'^*) &\mapsto F_{\mathbb{A}'}, && (2.4.19) \end{aligned}$$

where $F_{\mathbb{A}'}$ is the strict symmetric monoidal functor sending the knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ generating $\mathbf{Th}(\mathbf{K\text{-}Frob})$ to the knowledgeable Frobenius algebra $(A', C', \imath', \imath'^*)$ in $\mathcal{C}$. Furthermore, it is clear from the discussion above that a homomorphism of knowledgeable Frobenius algebras $\varphi\colon \mathbb{A}_1 \to \mathbb{A}_2$ defines a monoidal natural transformation $\varphi\colon F_{\mathbb{A}_1} \to F_{\mathbb{A}_2}$. Thus, it is clear that $\overline{\Gamma}$ extends to a symmetric monoidal functor $\overline{\Gamma} = (\overline{\Gamma}, \overline{\Gamma}_2, \overline{\Gamma}_0)\colon \mathbf{K\text{-}Frob}(\mathcal{C}) \to \mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob}), \mathcal{C})$.

To see that Γ and $\overline{\Gamma}$ define an equivalence of categories, let $\mathbb{A}' = (A', C', \imath', \imath'^*)$ be a knowledgeable Frobenius algebra in $\mathcal{C}$. The composite $\Gamma\overline{\Gamma}(\mathbb{A}') = \Gamma(F_{\mathbb{A}'}) = \mathbb{A}'$ since $F_{\mathbb{A}'}$ is a strict symmetric monoidal functor. Hence, $\Gamma\overline{\Gamma} = \mathrm{id}_{\mathbf{K\text{-}Frob}(\mathcal{C})}$. Now let $\psi\colon \mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathcal{C}$ be a symmetric monoidal functor and consider the composite $\overline{\Gamma}\Gamma(\psi)$. Let $\widetilde{\mathbb{A}} = \big(\psi(A), \psi(\mu_A) \circ \psi_2, \psi(\eta_A) \circ \psi_0, \psi_2^{-1} \circ \psi(\Delta_A), \psi_0^{-1} \circ \psi(\varepsilon_A)\big)$ so that $\overline{\Gamma}\Gamma(\psi) = F_{\widetilde{\mathbb{A}}}$. We define a monoidal natural isomorphism $\vartheta\colon \psi \Rightarrow F_{\widetilde{\mathbb{A}}}$ on the generators as follows:

$$\begin{aligned} \vartheta_{\mathbb{1}}\colon \psi(\mathbb{1}) \to F_{\widetilde{\mathbb{A}}}(\mathbb{1}) = \mathbb{1} &:= \psi_0^{-1}, \\ \vartheta_A\colon \psi(A) \to F_{\widetilde{\mathbb{A}}}(A) = \psi(A) &:= 1_A, && (2.4.20)\\ \vartheta_C\colon \psi(C) \to F_{\widetilde{\mathbb{A}}}(C) = \psi(C) &:= 1_C. \end{aligned}$$

The condition that ϑ be monoidal implies that $\vartheta_{A\otimes A} = (\psi_2^{-1})_{A\otimes A}$, $\vartheta_{A\otimes C} = (\psi_2^{-1})_{A\otimes C}$, $\vartheta_{C\otimes A} = (\psi_2^{-1})_{C\otimes A}$, and $\vartheta_{C\otimes C} = (\psi_2^{-1})_{C\otimes C}$. Since $\mathbf{Th}(\mathbf{K\text{-}Frob})$ is generated by $\mathbb{1}$,A, and C, this assignment uniquely defines a monoidal natural isomorphism. Hence, $\overline{\Gamma}\Gamma(\psi) \cong \psi$ so that $\overline{\Gamma}$ and Γ define a monoidal equivalence of categories. □

2.5 The category of open-closed cobordisms

In this section, we define and study the category $\mathbf{2Cob}^{\mathrm{ext}}$ of open-closed cobordisms. Open-closed cobordisms form a special sort of compact smooth 2-manifolds with corners that have a particular global structure. If one decomposes their boundary minus the corners into connected components, these components are either *black* or *coloured* with elements of some given set S. Every corner is required to separate a black boundary component from a coloured one[2].

These 2-manifolds with corners are viewed as cobordisms between their black boundaries, and they can be composed by gluing them along their black boundaries subject to a matching condition for the colours of the other boundary components. In the conformal field theory literature, the coloured boundary components are referred to as *free boundaries* and the colours as *boundary conditions*.

2-manifolds with corners with this sort of global structure form a special case of $\langle 2\rangle$-manifolds according to Jänich [60]. For an overview and a very convenient notation, we refer to the introduction of the article [61] by Laures.

In the following two subsections, we present all definitions for a generic set of colours S. Starting in Subsection 2.5.4, the generators and relations description of $\mathbf{2Cob}^{\mathrm{ext}}$ is developed only for the case of a single colour, $S = \{*\}$. We finally return to the case of a generic set of colours S in Section 2.8.

2.5.1 $\langle 2\rangle$-manifolds

Manifolds with corners

A k-dimensional *manifold with corners* M is a topological manifold with boundary that is equipped with a smooth structure with corners. A smooth structure with corners is defined as follows. A *smooth atlas with corners* is a family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ of coordinate systems such that the $U_\alpha \subseteq M$ are open subsets which cover M, and the

$$\varphi_\alpha\colon U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^k_+ \tag{2.5.1}$$

are homeomorphisms onto open subsets of $\mathbb{R}^k_+ := [0,\infty)^k$. The transition functions

$$\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta}) \tag{2.5.2}$$

for $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ are required to be the restrictions to $\mathbb{R}^k_+$ of diffeomorphisms between open subsets of $\mathbb{R}^k$. Two such atlases are considered equivalent if their union is a smooth atlas with corners, and a *smooth structure with corners* is an equivalence class of such atlases.

A *smooth map* $f\colon M \to N$ between manifolds with corners M and N is a continuous map for which the following condition holds. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in J}$ be atlases that

[2] In this terminology, black is not considered a colour.

represent the smooth structures with corners of M and N, respectively. For every $p \in M$ and for every $\alpha \in I$, $\beta \in J$ with $p \in U_\alpha$ and $f(p) \in V_\beta$, we require that the map

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta))} \colon \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(f(U_\alpha) \cap V_\beta) \tag{2.5.3}$$

is the restriction to $\mathbb{R}_+^m$ of a smooth map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^p$ for m and p the dimensions of M and N, respectively.

Manifolds with faces

For each $p \in M$, we define $c(p) \in \mathbb{N}_0$ to be the number of zero coefficients of $\varphi_\alpha(p) \in \mathbb{R}^k$ for some $\alpha \in I$ for which $p \in U_\alpha$. A *connected face* of M is the closure of a component of $\{\, p \in M \colon\ c(p) = 1 \,\}$. A *face* is a free union of pairwise disjoint connected faces. This includes the possibility that a face can be empty.

A k-dimensional *manifold with faces* M is a k-dimensional manifold with corners such that each $p \in M$ is contained in $c(p)$ different connected faces. Notice that every face of M is itself a manifold with faces.

$\langle n \rangle$-manifolds

A k-dimensional $\langle n \rangle$-manifold M is a k-dimensional manifold with faces with a specified tuple $(\partial_0 M, \ldots, \partial_{n-1} M)$ of faces of M such that the following two conditions hold.

1. $\partial_0 M \cup \cdots \cup \partial_{n-1} M = \partial M$. Here ∂M denotes the boundary of M as a topological manifold.

2. $\partial_j M \cap \partial_\ell M$ is a face of both $\partial_j M$ and $\partial_\ell M$ for all $j \neq \ell$.

Notice that a $\langle 0 \rangle$-manifold is just a manifold without boundary while a $\langle 1 \rangle$-manifold is a manifold with boundary. A diffeomorphism $f \colon M \to N$ between two $\langle n \rangle$-manifolds is a diffeomorphism of the underlying manifolds with corners such that $f(\partial_j M) = \partial_j N$ for all j.

The following notation is taken from Laures [61]. Let $\underline{2}$ denote the category associated with the partially ordered set $\{0, 1\}$, $0 \leq 1$, *i.e.* the category freely generated by the graph $0 \xrightarrow{*} 1$. Denote by $\underline{2}^n$ the n-fold Cartesian product of $\underline{2}$ and equip its set of objects $\{0, 1\}^n$ with the corresponding partial order. An *$\langle n \rangle$-diagram* is a functor $\underline{2}^n \to \mathbf{Top}$. We use the term $\langle n \rangle$-diagram *of inclusions* for an $\langle n \rangle$-diagram which sends each morphism of $\underline{2}^n$ to an inclusion, and so on.

Every $\langle n\rangle$-manifold M gives rise to an $\langle n\rangle$-diagram $M\colon \underline{2}^n \to \mathbf{Top}$ of inclusions as follows. For the objects $a = (a_0, \ldots, a_{n-1}) \in |\underline{2}^n|$, write $a' := (1 - a_0, \ldots, 1 - a_{n-1})$, and denote the standard basis of $\mathbb{R}^n$ by $(e_0, \ldots, e_{n-1})$. The functor $M\colon \underline{2}^n \to \mathbf{Top}$ is defined on the objects by,

$$M(a) := \bigcap_{i \in \{\, i\colon\ a \leq e_i' \,\}} \partial_i M, \tag{2.5.4}$$

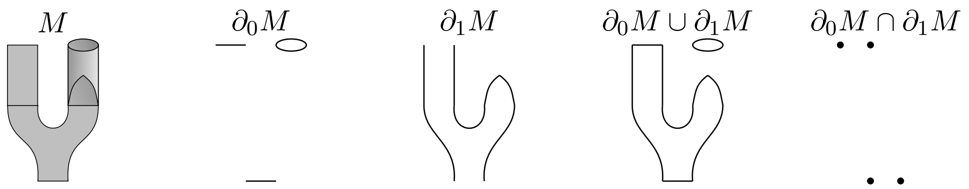
if $a \neq (1, \ldots, 1)$, and by $M((1, \ldots, 1)) := M$. The functor sends the morphisms of $\underline{2}^n$ to the obvious inclusions.

For all $a \in |\underline{2}^n|$, the face $M(a)$ of M forms a $\langle \ell\rangle$-manifold itself for which $\ell = \sum_{i=1}^{n-1} a_i$. An orientation of M induces orientations on the $M(a)$ as usual. The product of an $\langle n\rangle$-manifold M with a $\langle p\rangle$-manifold N forms an $\langle n+p\rangle$-manifold, denoted by $M \times N$. The structure of its faces can be read off from the functor

$$M \times N\colon \underline{2}^{n+p} \simeq \underline{2}^n \times \underline{2}^p \xrightarrow{M \times N} \mathbf{Top} \times \mathbf{Top} \xrightarrow{\times} \mathbf{Top}. \tag{2.5.5}$$

The half-line $\mathbb{R}_+ := [0, \infty)$ is a 1-dimensional manifold with boundary, *i.e.* a 1-dimensional $\langle 1\rangle$-manifold. The product of (2.5.5) then equips $\mathbb{R}_+^n$ with the structure of an n-dimensional $\langle n\rangle$-manifold.

The case that is relevant in the following is that of 2-dimensional $\langle 2\rangle$-manifolds. These are 2-dimensional manifolds with faces with a pair of specified faces $(\partial_0 M, \partial_1 M)$ such that $\partial_0 M \cup \partial_1 M = \partial M$ and $\partial_0 M \cap \partial_1 M$ is a face of both $\partial_0 M$ and $\partial_1 M$. The following diagram shows the faces of one of the typical 2-dimensional $\langle 2\rangle$-manifolds M that are used below.



(2.5.6)

The $\langle 2\rangle$-diagram $M\colon \underline{2}^2 \to \mathbf{Top}$ of inclusions is the following commutative square:

$$\begin{tikzcd}
\partial_0 M \cap \partial_1 M \arrow[r, "M(\mathrm{id}_0 \times *)"] \arrow[d, "M(* \times \mathrm{id}_0)"'] & \partial_0 M \arrow[d, "M(* \times \mathrm{id}_1)"] \\
\partial_1 M \arrow[r, "M(\mathrm{id}_1 \times *)"'] & M
\end{tikzcd} \tag{2.5.7}$$

Another example of a manifold with corners M which is embedded in $\mathbb{R}^3$ is depicted in (1.0.5). It has the structure of a 2-dimensional $\langle 2\rangle$-manifold when one chooses $\partial_0 M$ to be the union of the top and bottom boundaries of the picture, similarly to (2.5.6).

Collars

In order to glue $\langle 2\rangle$-manifolds along specified faces, we need the following technical results.

Lemma 2.5.1 (Lemma 2.1.6 of [61]). Each $\langle n\rangle$-manifold M admits an $\langle n\rangle$-diagram C of embeddings

$$C(a\to b)\colon \mathbb{R}^n_+(a')\times M(a)\hookrightarrow \mathbb{R}^n_+(b')\times M(b) \tag{2.5.8}$$

such that the restriction $C(a\to b)|_{\mathbb{R}^n_+(b')\times M(a)} = \mathrm{id}_{\mathbb{R}^n_+(b')}\times M(a\to b)$ is the inclusion map.

In particular, for every $\langle 2\rangle$-manifold M, there is a commutative square $C\colon \underline{2}^2\to \mathbf{Top}$ of embeddings,

$$\begin{tikzcd}
\mathbb{R}^2_+\times(\partial_0 M\cap\partial_1 M) \arrow[r, "C(\mathrm{id}_0\times *)"] \arrow[d, "C(*\times\mathrm{id}_0)"'] & \partial_0\mathbb{R}^2_+\times\partial_1 M \arrow[d, "C(*\times\mathrm{id}_1)"] \\
\partial_1\mathbb{R}^2_+\times\partial_0 M \arrow[r, "C(\mathrm{id}_1\times *)"'] & \{(0,0)\}\times M
\end{tikzcd} \tag{2.5.9}$$

such that the following restrictions are inclusions,

$$\begin{aligned}
C(\mathrm{id}_0\times *)|_{\partial_0\mathbb{R}^2_+\times(\partial_0 M\cap\partial_1 M)} &= \mathrm{id}_{\partial_0\mathbb{R}^2_+}\times M(\mathrm{id}_0\times *), && (2.5.10)\\
C(*\times\mathrm{id}_0)|_{\partial_1\mathbb{R}^2_+\times(\partial_0 M\cap\partial_1 M)} &= \mathrm{id}_{\partial_1\mathbb{R}^2_+}\times M(*\times\mathrm{id}_0), && (2.5.11)\\
C(*\times\mathrm{id}_1)|_{\{(0,0)\}\times\partial_1 M} &= \mathrm{id}_{\{(0,0)\}}\times M(*\times\mathrm{id}_1), && (2.5.12)\\
C(\mathrm{id}_1\times *)|_{\{(0,0)\}\times\partial_0 M} &= \mathrm{id}_{\{(0,0)\}}\times M(\mathrm{id}_1\times *). && (2.5.13)
\end{aligned}$$

The embedding $C(\mathrm{id}_1\times *)\colon \partial_1\mathbb{R}^2_+\times\partial_0 M\to\{(0,0)\}\times M$ provides us with a diffeomorphism from $([0,\varepsilon]\times\{0\})\times\partial_0 M\subseteq([0,\infty)\times\{0\})\times\partial_0 M=\partial_1\mathbb{R}^2_+\times\partial_0 M$ onto a submanifold of $\{(0,0)\}\times M$ for some $\varepsilon>0$. It restricts to an inclusion on $\{(0,0)\}\times\partial_0 M$ and thereby yields a (smooth) collar neighbourhood for $\partial_0 M$.

2.5.2 Open-closed cobordisms

For a topological space M, we denote by $\Pi_0(M)$ the set of its connected components, and for $p\in M$, we denote by $[p]\in\Pi_0(M)$ its component.

Cobordisms

Definition 2.5.2. Let S be some set. An *S-coloured open-closed cobordism* (M,γ) is a compact oriented 2-dimensional $\langle 2\rangle$-manifold M whose distinguished faces we denote by

$(\partial_0 M, \partial_1 M)$, together with a map $\gamma\colon \Pi_0(\partial_1 M) \to S$. The face $\partial_0 M$ is called the *black boundary*, $\partial_1 M$ the *coloured boundary*, and γ the *colouring*. An *open-closed cobordism* is an S-coloured open-closed cobordism for which S is a one-element set.

Two S-coloured open-closed cobordisms (M, γ_M) and (N, γ_N) are considered *equivalent* if there is an orientation preserving diffeomorphism of 2-dimensional $\langle 2\rangle$-manifolds $f\colon M \to N$ that restricts to the identity on $\partial_0 M$ and that preserves the colouring, *i.e.* $\gamma_N \circ f = \gamma_M$.

The face $\partial_0 M$ is a compact 1-manifold with boundary and therefore diffeomorphic to a free union of circles S^1 and unit intervals $[0,1]$. For each of the unit intervals, there is thus an orientation preserving diffeomorphism $\varphi\colon [0,1] \to \varphi([0,1]) \subseteq \partial_0 M$ onto a component of $\partial_0 M$ such that the boundary points are mapped to the corners, *i.e.* $\varphi(\{0,1\}) \subseteq \partial_0 M \cap \partial_1 M$. We say that the cobordism (M, γ) *equips* the unit interval $[0,1]$ *with the colours* $(\gamma_+, \gamma_-) \in S \times S$ if $\gamma_+ := \gamma([\varphi(1)])$ and $\gamma_- := \gamma([\varphi(0)])$.

Gluing

Let (M, γ_M) and (N, γ_N) be S-coloured open-closed cobordisms and $f\colon {S^1}^* \to M$ and $g\colon S^1 \to N$ be orientation preserving diffeomorphisms onto components of $\partial_0 M$ and $\partial_0 N$, respectively. Here we have equipped the circle S^1 with a fixed orientation, and ${S^1}^*$ denotes the one with opposite orientation. Then we obtain an S-coloured open-closed cobordism $M\,{}_f{\coprod}_g N$ by *gluing* M *and* N *along* S^1 as follows. As a topological manifold, it is the pushout. As mentioned in Section 2.5.1, $\partial_0 M$ and thereby all its components have smooth collar neighbourhoods, and so the standard techniques are available to equip $M\,{}_f{\coprod}_g N$ with the structure of a manifold with corners whose smooth structure is unique up to a diffeomorphism that restricts to the identity on $\partial_0 M \cup \partial_0 N$. It is obvious that $M\,{}_f{\coprod}_g N$ also has the structure of a $\langle 2\rangle$-manifold with $\partial_1(M\,{}_f{\coprod}_g N) = \partial_1 M \cup \partial_1 N$ and, furthermore, that of an S-coloured open-closed cobordism.

Similarly, let $f\colon [0,1]^* \to M$ and $g\colon [0,1] \to N$ be orientation preserving diffeomorphisms onto components of $\partial_0 M$ and $\partial_0 N$, respectively, such that (M, γ_M) equips the interval $f([0,1]^*)$ with the colours $(\gamma_+, \gamma_-) \in S$ and (N, γ_N) equips the interval $g([0,1])$ precisely with the colours (γ_-, γ_+). Then we obtain the gluing of M and N along $[0,1]$ again as the pushout $M\,{}_f{\coprod}_g N$ equipped with the smooth structure that is unique up to a diffeomorphism which restricts to the identity on $\partial_0 M \cup \partial_0 N$. It is easy to see that $M\,{}_f{\coprod}_g N$ also has the structure of a $\langle 2\rangle$-manifold with $\partial_1(M\,{}_f{\coprod}_g N) = \partial_1 M \cup \partial_1 N$ and, moreover, due to the matching of colours, that of an S-coloured open-closed cobordism.

The category $\mathbf{2Cob}^{\mathrm{ext}}(S)$

The following definition of the category of open-closed cobordisms is inspired by that of Baas, Cohen and Ramírez [29]. What we call $\mathbf{2Cob}^{\mathrm{ext}}(S)$ in the following is in fact a skeleton of the category of open-closed cobordisms. For this reason, we choose particular embedded manifolds $C_{\vec{n}}$ as the objects of $\mathbf{2Cob}^{\mathrm{ext}}(S)$. Although these are embedded manifolds, our cobordisms are not, and we consider two cobordisms equivalent once they are related by an orientation preserving diffeomorphism that restricts to the identity on their black boundaries.

Definition 2.5.3. Let S be a set. The category $\mathbf{2Cob}^{\mathrm{ext}}(S)$ is defined as follows. Its objects are triples $(\vec{n},\gamma_+,\gamma_-)$ consisting of a finite sequence $\vec{n} := (n_1,\ldots,n_k)$, $k \in \mathbb{N}_0$, with $n_j \in \{0,1\}$, $1 \leq j \leq k$, and maps $\gamma_\pm\colon \{1,\ldots,k\} \to S \cup \{\emptyset\}$ for which $\gamma_\pm(j) \neq \emptyset$ if $n_j = 1$ and $\gamma_\pm(j) = \emptyset$ if $n_j = 0$[3]. We denote the *length* of such a sequence by $|\vec{n}| := k$.

Each sequence $\vec{n} = (n_1,\ldots,n_k)$ represents the diffeomorphism type of a compact oriented 1-dimensional submanifold of $\mathbb{R}^2$,

$$C_{\vec{n}} := \bigcup_{j=1}^{k} I(j,n_j), \tag{2.5.14}$$

where $I(j,0)$ is the circle of radius $1/4$ centred at $(j,0) \in \mathbb{R}^2$ and $I(j,1) = [j-1/4, j+1/4] \times \{0\}$, both equipped with the induced orientation. Taking the disjoint union of two such manifolds $C_{\vec{n}}$ and $C_{\vec{m}}$ is done as follows,

$$C_{\vec{n}} \coprod C_{\vec{m}} := C_{\vec{n}} \cup T_{(|\vec{n}|,0)}(C_{\vec{m}}), \tag{2.5.15}$$

where $T(x,y)\colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes the translation by (x,y) in $\mathbb{R}^2$.

A morphism $f\colon (\vec{n},\gamma_+,\gamma_-) \to (\vec{n}',\gamma'_+,\gamma'_-)$ is a pair $f = ([f],\Phi)$ consisting of an equivalence class $[f]$ of S-coloured open-closed cobordisms and a specified orientation preserving diffeomorphism

$$\Phi\colon C^*_{\vec{n}} \coprod C_{\vec{n}'} \to \partial_0 f, \tag{2.5.16}$$

such that the following conditions hold:

1. For each $j \in \{1,\ldots,|\vec{n}|\}$ for which $n_j = 1$, the S-coloured open-closed cobordism (f,γ) representing $[f]$ equips the corresponding unit interval $\Phi(I(j,n_j))$ with the colours $(\gamma_+(j),\gamma_-(j))$.

[3] This is done simply because the values $\gamma_\pm(j)$ are never used if $n_j = 0$, but we nevertheless want to keep the indices j in line with those of the sequence $\vec{n}$.

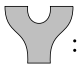
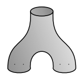
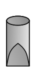
2. For each $j \in \{1, \ldots, |\vec{n}'|\}$ for which $n'_j = 1$, (f, γ) equips the corresponding unit interval $\Phi(I(j, n'_j))$ with the colours $(\gamma'_+(j), \gamma'_-(j))$.

The composition $g \circ f$ of two morphisms $f = ([f], \Phi_f)\colon (\vec{n}, \gamma_+, \gamma_-) \to (\vec{n}', \gamma'_+, \gamma'_-)$ and $g = ([g], \Phi_g)\colon (\vec{n}', \gamma'_+, \gamma'_-) \to (\vec{n}'', \gamma''_+, \gamma''_-)$ is defined as $g \circ f := ([g \coprod_{C_{\vec{n}'}} f], \Phi_{g \circ f})$. Here $[g \coprod_{C_{\vec{n}'}} f]$ is the equivalence class of the S-coloured open-closed cobordism $g \coprod_{C_{\vec{n}'}} f$ which is obtained by successively gluing f and g along all the components of $C_{\vec{n}'}$. $\Phi_{g \circ f}\colon C^*_{\vec{n}} \coprod C_{\vec{n}''} \to \partial_0(g \coprod_{C_{\vec{n}'}} f)$ is the obvious orientation preserving diffeomorphism obtained from restricting $\Phi_f\colon C^*_{\vec{n}} \coprod C_{\vec{n}'} \to \partial_0 f$ and $\Phi_g\colon C^*_{\vec{n}'} \coprod C_{\vec{n}''} \to \partial_0 g$.

For any object $(\vec{n}, \gamma_+, \gamma_-)$, the cylinder $\mathrm{id}_{(\vec{n}, \gamma_+, \gamma_-)} := [0,1] \times C_{\vec{n}}$ forms an S-coloured open-closed cobordism such that $\partial_0 \mathrm{id}_{(\vec{n}, \gamma_+, \gamma_-)} = C^*_{\vec{n}} \coprod C_{\vec{n}}$. It plays the role of the identity morphism.

The category $\mathbf{2Cob}^{\mathrm{ext}}$ is defined as the category $\mathbf{2Cob}^{\mathrm{ext}}(S)$ for the singleton set $S = \{*\}$. When we describe the objects of $\mathbf{2Cob}^{\mathrm{ext}}$, we can suppress the γ_+ and γ_- and simply write the sequences $\vec{n} = (n_1, \ldots, n_{|\vec{n}|})$.

Examples of morphisms of $\mathbf{2Cob}^{\mathrm{ext}}$ are depicted here,

 $: (1,1) \to (1)$,  $: (0) \to (0,0)$,  $: (0) \to (1)$. (2.5.17)

In these pictures, the source of the cobordism is drawn at the top and the target at the bottom. The morphism depicted in (1.0.5) goes from $(1,0,1,1,1)$ to $(0,1,1,0,0)$.

The concatenation $\vec{n} \coprod \vec{m} := (n_1, \ldots, n_{|\vec{n}|}, m_1, \ldots, m_{|\vec{m}|})$ of sequences together with the free union of S-coloured open-closed cobordisms, also denoted by $\coprod$, provides the category $\mathbf{2Cob}^{\mathrm{ext}}(S)$ with the structure of a strict symmetric monoidal category.

Let $k \in \mathbb{N}$. The symmetric group $\mathcal{S}_k$ acts on the subset of objects $(\vec{n}, \gamma_+, \gamma_-) \in |\mathbf{2Cob}^{\mathrm{ext}}(S)|$ for which $|\vec{n}| = k$. This action is defined by,

$$\sigma \triangleright (\vec{n}, \gamma_+, \gamma_-) := ((n_{\sigma^{-1}(1)}, \ldots, n_{\sigma^{-1}(|\vec{n}|)}), \gamma_+ \circ \sigma^{-1}, \gamma_- \circ \sigma^{-1}) \ . \tag{2.5.18}$$

For each object $(\vec{n}, \gamma_+, \gamma_-) \in |\mathbf{2Cob}^{\mathrm{ext}}(S)|$ and any permutation $\sigma \in \mathcal{S}_{|\vec{n}|}$, we define a morphism,

$$\sigma^{(\vec{n}, \gamma_+, \gamma_-)}\colon (\vec{n}, \gamma_+, \gamma_-) \to \sigma \triangleright (\vec{n}, \gamma_+, \gamma_-) \ , \tag{2.5.19}$$

by taking the underlying S-coloured open-closed cobordism of the cylinder $\mathrm{id}_{(\vec{n}, \gamma_+, \gamma_-)}$, and replacing the orientation preserving diffeomorphism

$$\Phi\colon C^*_{\vec{n}} \coprod C_{\vec{n}} \to \partial_0 \mathrm{id}_{(\vec{n}, \gamma_+, \gamma_-)} \tag{2.5.20}$$

by one that has the components of the $C_{\vec{n}}$ for the target permuted accordingly. For example, for $S = \{*\}$, $\vec{n} = (1,0,0,1)$ and $\sigma = (234) \in \mathcal{S}_4$ in cycle notation, we obtain the morphism $\sigma^{(\vec{n})}$ that is depicted in (2.6.13) below. As morphisms of $\mathbf{2Cob}^{\mathrm{ext}}(S)$, *i.e.* up to the appropriate diffeomorphism, these cobordisms satisfy,

$$\tau^{\sigma \triangleright (\vec{n},\gamma_+,\gamma_-)} \circ \sigma^{(\vec{n},\gamma_+,\gamma_-)} = (\tau \circ \sigma)^{(\vec{n},\gamma_+,\gamma_-)} \,. \tag{2.5.21}$$

If the source of $\sigma^{(\vec{n},\gamma_+,\gamma_-)}$ is obvious from the context, we just write σ.

2.5.3 Invariants for open-closed cobordisms

In order to characterize the S-coloured open-closed cobordisms of $\mathbf{2Cob}^{\mathrm{ext}}(S)$ topologically, we need the following invariants. The terminology is taken from Baas, Cohen, and Ramírez [29].

Definition 2.5.4. Let S be a set and $f = ([f], \Phi) \in \mathbf{2Cob}^{\mathrm{ext}}(S)[(\vec{n},\gamma_+,\gamma_-),(\vec{n}',\gamma'_+,\gamma'_-)]$ be a morphism of $\mathbf{2Cob}^{\mathrm{ext}}(S)$ from $(\vec{n},\gamma_+,\gamma_-)$ to $(\vec{n}',\gamma'_+,\gamma'_-)$.

1. The *genus* $g(f)$ is defined to be the genus of the topological 2-manifold underlying f.

2. The *window number* of f is a map $\omega(f)\colon S \to \mathbb{N}_0$ such that $\omega(f)(s)$ is the number of components of the face $\partial_1 f$ that are diffeomorphic to S^1 and that are equipped by $\gamma\colon \Pi_0(\partial_1 f) \to S$ with the colour $s \in S$. In the case $S = \{*\}$, we write $\omega(f) \in \mathbb{N}_0$ instead of $\omega(f)(*)$.

3. Let k be the number of coefficients of $\vec{n} \coprod \vec{n}'$ that are 1, *i.e.* the number of components of the face $\partial_0 f$ that are diffeomorphic to the unit interval. Number these components by $1,\ldots,k$. The *open boundary permutation* $(\sigma(f),\gamma_\partial(f))$ of f consists of a permutation $\sigma(f) \in \mathcal{S}_k$ and a map $\gamma_\partial(f)\colon \{1,\ldots,k\} \to S$. We define $\sigma(f)$ as a product of disjoint cycles as follows. Consider every component X of the boundary ∂f of f viewed as a topological manifold, for which $X \cap \partial_0 f \cap \partial_1 f \neq \emptyset$. These are precisely the components that contain a corner of f. The orientation of f induces an orientation of X and thereby defines a cycle $(i_1,\ldots,i_\ell)$ where the $i_j \in \{1,\ldots,k\}$ are the numbers of the intervals of $\partial_0 f$ that are contained in X. The permutation $\sigma(f)$ is the product of these cycles for all such components X. The map $\gamma_\partial(f)\colon \{1,\ldots,k\} \to S$ is defined such that the S-coloured open-closed cobordism (f,γ) representing $[f]$ equips the interval with the number $j \in \{1,\ldots,k\}$ with the colours $(\gamma_\partial(j), \gamma_\partial(\sigma^{-1}(j)))$.

For example, the morphism depicted in (1.0.5) has 6 components of its black boundary diffeomorphic to the unit interval. Its open boundary permutation is $\sigma(f) = (256)(34) \in \mathcal{S}_6$ if one numbers the intervals in the source (top of the diagram) from left to right by $1, 2, 3, 4$ and those in the target (bottom of the diagram) from left to right by $5, 6$.

2.5.4 Generators

Beginning with this subsection, we restrict ourselves to the case of $\mathbf{2Cob}^{\mathrm{ext}}$ in which there is only one boundary colour, *i.e.* $S = \{*\}$. We use a generalization of Morse theory to manifolds with corners in order to decompose each open-closed cobordism into a composition of open-closed cobordisms each of which contains precisely one critical point. The components of these form the *generators* for the morphisms of the category $\mathbf{2Cob}^{\mathrm{ext}}$.

For the generalization of Morse theory to manifolds with corners, we follow Braess [62]. We first summarize the key definitions and results.

We need a notion of tangent space for a point $p \in \partial M$ if M is a manifold with corners. Every $p \in M$ has a neighbourhood $U \subseteq M$ which forms a submanifold of M and for which there is a diffeomorphism $\varphi\colon U \to \varphi(U)$ onto an open subset $\varphi(U) \subseteq \mathbb{R}^n_+$. Using the fact that $\varphi(p) \in \mathbb{R}^n_+ \subseteq \mathbb{R}^n$, we define the tangent space of p in M as $T_pM := d\varphi_p^{-1}(T_{\varphi(p)}\mathbb{R}^n)$, *i.e.* just identifying it with that of $\varphi(p)$ in $\mathbb{R}^n$ via the isomorphism $d\varphi_p^{-1}$.

Definition 2.5.5. Let M be a manifold with corners.

1. For each $p \in M$, we define the *inwards pointing tangential cone* $C_pM \subseteq T_pM$ as the set of all tangent vectors $v \in T_pM$ for which there exists a smooth path $\gamma\colon [0, \varepsilon] \to M$ for some $\varepsilon > 0$ such that $\gamma(0) = p$ and the one-sided derivative is:

$$\lim_{t \to 0+} (\gamma(t) - \gamma(0))/t = v. \tag{2.5.22}$$

2. Let $f\colon M \to \mathbb{R}$ be smooth. A point $p \in M$ is called a *critical point* and $f(p) \in \mathbb{R}$ its *critical value* if the restriction of the derivative $df_p\colon T_pM \to \mathbb{R}$ to the inwards pointing tangential cone is not surjective, *i.e.* if

$$df_p(C_pM) \neq \mathbb{R}. \tag{2.5.23}$$

The point $p \in M$ is called $(+)$-*critical* if $df_p(C_pM) \subseteq \mathbb{R}_+$ and it is called $(-)$-*critical* if $df_p(C_pM) \subseteq \mathbb{R}_- := -\mathbb{R}_+$.

Note that $df_p\colon T_pM \to \mathbb{R}$ is linear and therefore maps cones to cones, and so $df_p(C_pM)$ is either $\{0\}$, $\mathbb{R}_+$, $\mathbb{R}_-$, or $\mathbb{R}$. If $p \in M$ is a critical point, then $df_p(C_pM)$ is either $\{0\}$, $\mathbb{R}_+$, or $\mathbb{R}_-$. If $p \in M\backslash\partial M$, then $C_pM = T_pM$, and so p is critical if and only if $df_p = 0$. If $p \in \partial M$, $c(p) = 1$, and p is critical, then the restriction of f to ∂M has vanishing derivative, *i.e.* $d(f|_{\partial M})_p = 0$.

Definition 2.5.6. Let M be a manifold with corners and $f\colon M \to \mathbb{R}$ be a smooth function.

1. A critical point $p \in M$ of f is called *non-degenerate* if the Hessian of f at p, restricted to the kernel of df_p, has full rank, *i.e.* if
$$\det \mathrm{Hess}_p(f)|_{\ker df_p \otimes \ker df_p} \neq 0. \tag{2.5.24}$$
2. The function f is called a *Morse function* if all its critical point are non-degenerate.

Note that if $p \in M\backslash\partial M$, then the notion of non-degeneracy is as usual. If $p \in \partial M$, $c(p) = 1$, and p is a non-degenerate critical point, then p is a non-degenerate critical point of the restriction $f|_{\partial M}\colon \partial M \to \mathbb{R}$ in the usual sense. All non-degenerate critical points are isolated [62].

For our open-closed cobordisms, we need a special sort of Morse functions that are compatible with the global structure of the cobordisms.

Definition 2.5.7. Let $M \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n},\vec{m}]$ be an open-closed cobordism with source $C_{\vec{n}}$ and target $C_{\vec{m}}$. Here we have suppressed the diffeomorphisms from $C_{\vec{n}}$ onto a component of $\partial_0 M$, *etc.*, and we write M for any representative of its equivalence class. A *special Morse function* for M is a Morse function $f\colon M \to \mathbb{R}$ such that the following conditions hold:

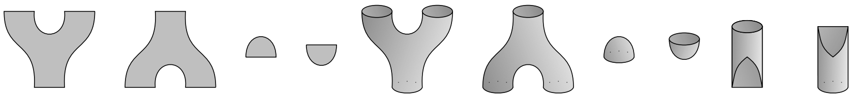
1. $f(M) \subseteq [0,1]$.
2. $f(p) = 0$ if and only if $p \in C_{\vec{n}}$, and $f(p) = 1$ if and only if $p \in C_{\vec{m}}$.
3. Neither $C_{\vec{n}}$ nor $C_{\vec{m}}$ contain any critical points.
4. The critical points of f have pairwise distinct critical values.

Using the standard techniques (see for example [19]), one shows that every open-closed cobordism $M \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n},\vec{m}]$ admits a special Morse function $f\colon M \to \mathbb{R}$. Since M is compact and since all non-degenerate critical points are isolated, the set of critical points of f is a finite set. If neither $a \in \mathbb{R}$ nor $b \in \mathbb{R}$ are critical values of f, the pre-image $N := f^{-1}([a,b])$

forms an open-closed cobordism with $\partial_0 N = f^{-1}(\{a,b\})$ and $\partial_1 N = \partial_1 M \cap N$. If $[a,b]$ does not contain any critical value of f, then $f^{-1}([a,b])$ is diffeomorphic to the cylinder $f^{-1}(\{a\}) \times [0,1]$.

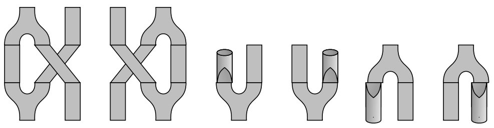
The following proposition classifies in terms of Morse data the non-degenerate critical points that can occur on open-closed cobordisms.

Proposition 2.5.8. Let $M \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be a connected open-closed cobordism and $f\colon M \to \mathbb{R}$ a special Morse function such that f has precisely one critical point. Then M is equivalent to one of the following open-closed cobordisms:



(2.5.25)

or to one of the compositions



. (2.5.26)

All these diagrams show open-closed cobordisms embedded in $\mathbb{R}^3$ and are drawn in such a way that the vertical axis of the drawing plane is $-f$. The source is at the top, and the target at the bottom of the diagram.

Proof. We analyze the properties of the non-degenerate critical point $p \in M$ case by case.

1. If $p \in M \backslash \partial M$, then the critical point is characterized by its index $i(p)$ (the number of negative eigenvalues of $\mathrm{Hess}_p(f)$) as usual; see, for example [63]. There exists a neighbourhood $U \subseteq M$ of p and a coordinate system $x\colon U \to \mathbb{R}^2$ in which the Morse function has the normal form,

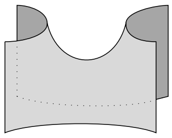
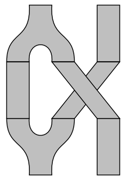
$$f(p) = -\sum_{j=1}^{i(p)} x_j^2(p) + \sum_{j=i(p)+1}^{2} x_j^2(p) \tag{2.5.27}$$

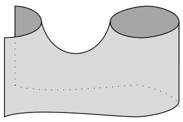
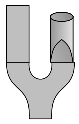
for all $p \in U$.

   (a) If the index is $i(p) = 2$, then the Morse function has a maximum at p, and so the neighbourhood (and thereby the entire open-closed cobordism) is diffeomorphic to ε_C of (2.5.25). Recall that the vertical coordinate of our diagrams is $-f$ rather than $+f$.

   (b) If the index is $i(p) = 1$, then f has a saddle point, and the usual argument shows that the possible cases are either μ_C or Δ_C of (2.5.25).

(c) If the index is $i(p) = 1$, then f has a saddle point. If M were a closed cobordism, *i.e.* $\partial_0 M = \partial M$, the usual argument would show that M is either of the form μ_C or Δ_C of (2.5.25). In the open-closed case, however, the saddle can occur in other cases, too, depending on how the boundary ∂M is decomposed into $\partial_0 M$ and $\partial_1 M$. We proceed with a case by case analysis and show that in each case, this saddle is equivalent to one of the compositions displayed in (2.5.26):

 $\cong$  , (2.5.28)

 $\cong$  . (2.5.29)

Here we show the saddle at the left and the equivalent decomposition as a composition and tensor product of the cobordisms of (2.5.25) with identities on the right. The saddle of (2.5.28) can appear in two orientations and with the intervals in its source and target in any ordering. In any of these cases, it is equivalent to one of the first two compositions displayed in (2.5.26). The saddle of (2.5.29) can appear flipped upside-down or left-right or both, giving rise to the last four compositions displayed in (2.5.26).

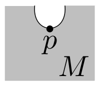
Note that the equivalences of (2.5.28) and (2.5.29) relate cobordisms whose number of critical points differs by an odd number. This is a new feature that dos not occur in the case of closed cobordisms.

2. Otherwise, $p \in \partial_1 M \backslash \partial_0 M$, *i.e.* the critical point is on the coloured boundary, but does not coincide with a corner of M. Consider the restriction $f|_{\partial_1 M} \colon \partial_1 M \to \mathbb{R}$ which then has a non-degenerate critical point at p with index $i'(p) \in \{0, 1\}$.

(a) If $i'(p) = 1$, then $f|_{\partial M}$ has a maximum at p.

i. If p is a $(-)$-critical point of f, the cobordism is diffeomorphic to ε_A of (2.5.25).

ii. If p is a $(+)$-critical point of f, the neighbourhood of p looks as follows,



(2.5.30)

Consider the component of the boundary ∂M of M as a topological manifold. The set of corners $\partial_0 M \cap \partial_1 M$ contains at least two elements. If it contains precisely two elements, then the cobordism is diffeomorphic to $\imath^*$ of (2.5.25). Otherwise, it contains six elements, and the cobordism is diffeomorphic to μ_A of (2.5.25).

iii. Otherwise p is neither (+)-critical nor (−)-critical, and so $df_p = 0$. Non-degeneracy now means that $\mathrm{Hess}_p(f)$ is non-degenerate. Let $i''(p) \in \{0, 1, 2\}$ be the number of negative eigenvalues of $\mathrm{Hess}_p(f)$. The case $i''(p) = 0$ is ruled out by the assumption that $i'(p) = 1$.

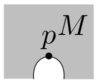
A. If $i''(p) = 2$, then we are in the same situation as in case 2(a)i.

B. Otherwise $i''(p) = 1$, and we are in the same situation as in case 2(a)ii.

(b) If $i'(p) = 0$, then $f|_{\partial M}$ has a minimum at p.

i. If p is a (+)-critical point of f, the cobordism is diffeomorphic to η_A of (2.5.25).

ii. If p is a (−)-critical point of f, the neighbourhood of p looks as follows,



(2.5.31)

Similarly to case 2(a)ii above, the cobordism is either diffeomorphic to Δ_A or to $\imath$ of (2.5.25).

iii. Otherwise $df_p = 0$, and by considering $\mathrm{Hess}_p(f)$ similarly to case 2(a)iii above, we find that we are either in case 2(b)i or 2(b)ii.

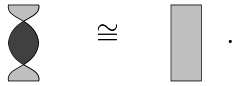
□

The structure of arbitrary open-closed cobordisms can then be established by using a special Morse function and decomposing the cobordism into a composition of pieces that have precisely one critical point each. This result generalizes the conventional handle decomposition to the case of our sort of 2-manifolds with corners.

Proposition 2.5.9. Let $f \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{n}']$ be any morphism. Then $[f] = [f_\ell \circ \cdots \circ f_1]$, *i.e.* f is equivalent to the composition of a finite number of morphisms f_j each of which is of the form $f_j = \mathrm{id}_{\vec{m}_j} \coprod g_j \coprod \mathrm{id}_{\vec{p}_j}$ where g_j is one of the morphisms depicted in (2.5.25) and $\mathrm{id}_{\vec{m}_j}$ and $\mathrm{id}_{\vec{p}_j}$ are identities, *i.e.* cylinders over their source.

Our pictures, for example (1.0.5), indicate how the morphisms are composed from the generators. In order to keep the height of the diagram small, we have already used relations such as $(g \coprod \mathrm{id}_{\vec{n}'}) \circ (\mathrm{id}_{\vec{m}} \coprod f) = g \coprod f$ for $f\colon \vec{n} \to \vec{n}'$ and $g\colon \vec{m} \to \vec{m}'$ which obviously hold in $\mathbf{2Cob}^{\mathrm{ext}}$.

Notice that the flat strip, twisted by 2π when we draw it as embedded in $\mathbb{R}^3$, is nevertheless equivalent to the flat strip:



(2.5.32)

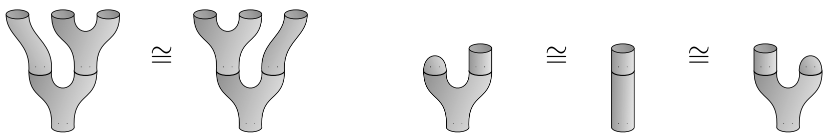
2.5.5 Relations

Below we provide a list of relations that the generators of $\mathbf{2Cob}^{\mathrm{ext}}$ satisfy. In Section 2.5.6, we summarize some consequences of these relations. As already mentioned in the introduction, these relations have emerged from work on boundary conformal field theory. See, for example [27,28,30,32,33]. In Section 2.6, we define a normal form for open-closed cobordisms with a specified genus, window number and open boundary permutation. In Section 2.6.3, we then provide an inductive proof which constructs a finite sequence of diffeomorphisms that puts an arbitrary open-closed cobordism into the normal form using only the relations given below. Hence, we provide a constructive proof that the relations are sufficient to completely describe the category $\mathbf{2Cob}^{\mathrm{ext}}$.

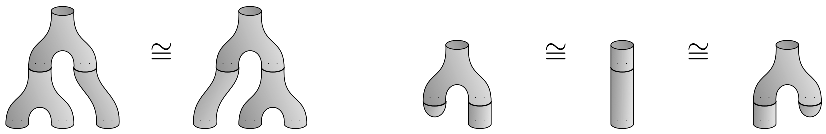
When we use the sign '$\cong$' in our diagrams below, we mean equivalence in the category $\mathbf{2Cob}^{\mathrm{ext}}$.

Proposition 2.5.10. The following relations hold in the symmetric monoidal category $\mathbf{2Cob}^{\mathrm{ext}}$:

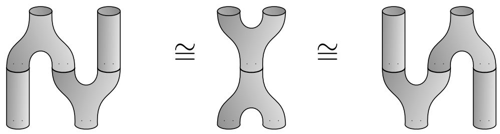
1. The object $\vec{n} = (0)$, *i.e.* the circle $C_{\vec{n}} \cong S^1$, forms a commutative Frobenius algebra object:



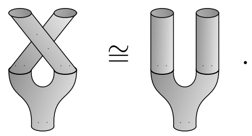
(2.5.33)



(2.5.34)



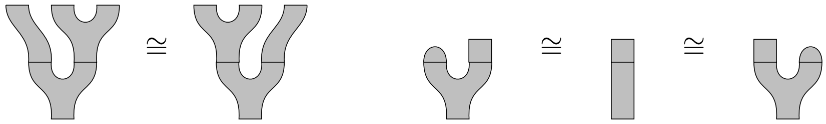
(2.5.35)



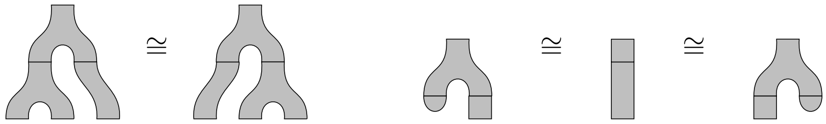
(2.5.36)

2. The object $\vec{n} = (1)$, *i.e.* the interval $C_{\vec{n}} \cong I$, forms a symmetric Frobenius algebra

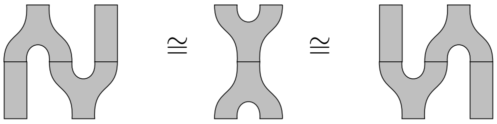
object:



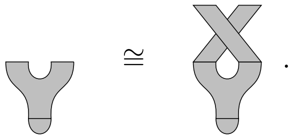
(2.5.37)



(2.5.38)

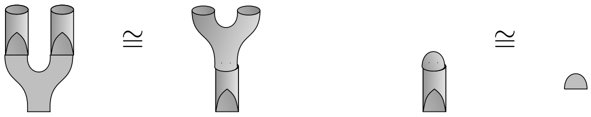


(2.5.39)



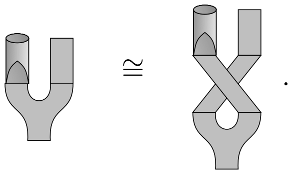
(2.5.40)

3. The 'zipper'  forms an algebra homomorphism:



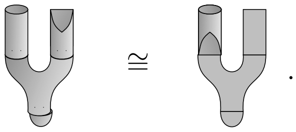
(2.5.41)

4. This relation describes the 'knowledge' about the centre, *c.f.* (2.4.1):



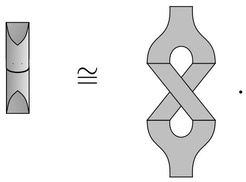
(2.5.42)

5. The 'cozipper'  is dual to the zipper:



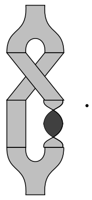
(2.5.43)

6. The Cardy condition:



(2.5.44)

Proof. It can be show in a direct computation that the depicted open-closed cobordisms are equivalent. Writing out this proof would be tremendously laborious, but of rather little insight. For the Cardy condition (2.5.44), the right hand side is most naturally depicted as:



(2.5.45)

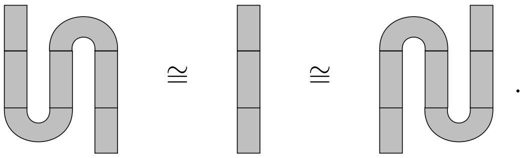
□

2.5.6 Consequences of relations

In this section, we collect some additional diffeomorphisms that can be constructed from the diffeomorphisms in Proposition 2.5.10. To simplify the diagrams, we define:

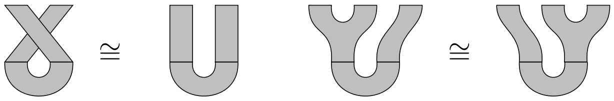
(2.5.46)

These open-closed cobordisms which we sometimes call the *open pairing* and *open copairing*, respectively, satisfy the zig-zag identities:



(2.5.47)

This follows directly from the Frobenius relations, the left and right unit laws, and the left and right counit laws. From Equations (2.5.40) and (2.5.37), the pairing can be shown to be symmetric and invariant,

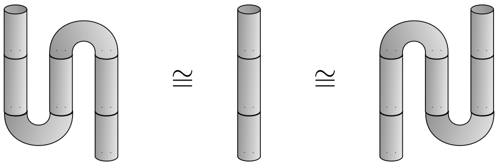


(2.5.48)

and the same holds for the copairing. Similarly, we define the *closed pairing* and the *closed copairing*:

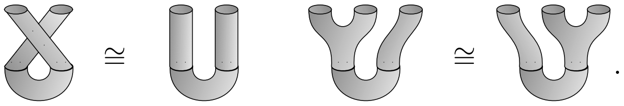
(2.5.49)

These also satisfy the zig-zag identities,



(2.5.50)

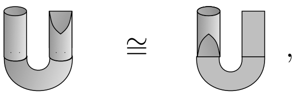
and the closed pairing is symmetric and invariant,

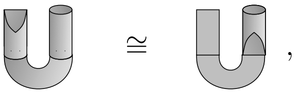


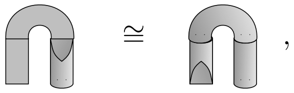
(2.5.51)

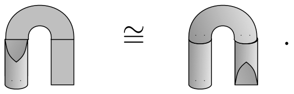
A similar result holds for the closed copairing.

Proposition 2.5.11. The following open-closed cobordisms are diffeomorphic by compositions of the diffeomorphisms of Proposition 2.5.10:

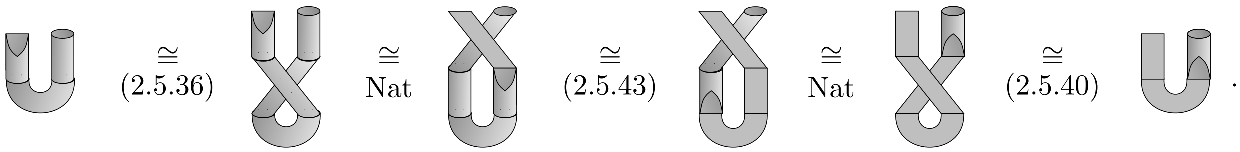

(2.5.52)


(2.5.53)


(2.5.54)

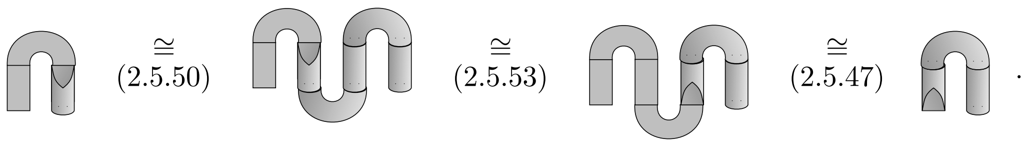

(2.5.55)

Proof. Equation (2.5.52) is just a restatement of the second axiom in Equation (2.5.43). The proof of Equation (2.5.53) is as follows:



(2.5.56)

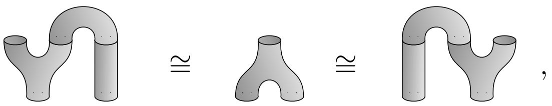
By 'Nat' we have denoted the obvious diffeomorphisms which, algebraically speaking, express the naturality of the symmetric braiding. The proof of Equation (2.5.54) is as follows:

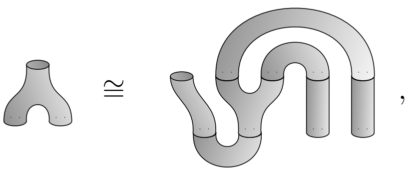


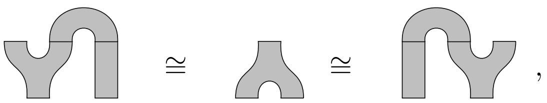
(2.5.57)

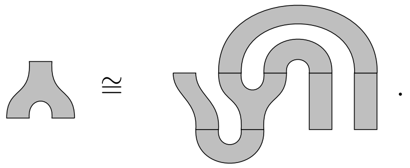
We leave the proof of Equation (2.5.55) as an exercise for the reader. □

Proposition 2.5.12. The following open-closed cobordisms are diffeomorphic:


(2.5.58)

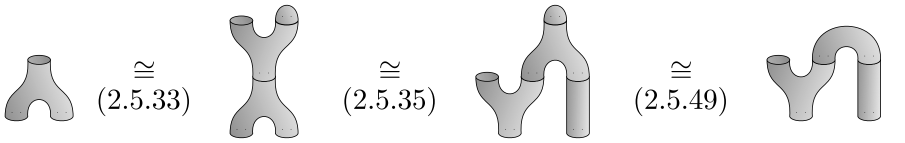

(2.5.59)


(2.5.60)



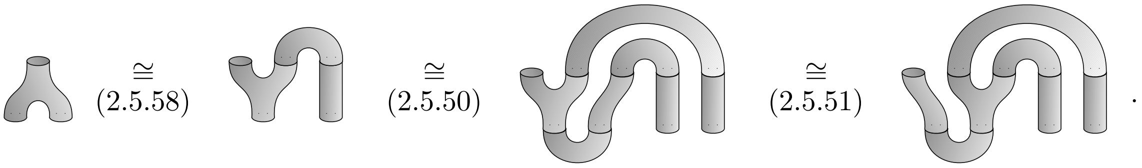
(2.5.61)

Proof. The first diffeomorphism in Equation (2.5.58) is constructed from the following sequence of diffeomorphisms:



(2.5.62)

The second diffeomorphism in Equation (2.5.58) is constructed similarly. The diffeomorphism in Equation (2.5.59) is constructed as follows:

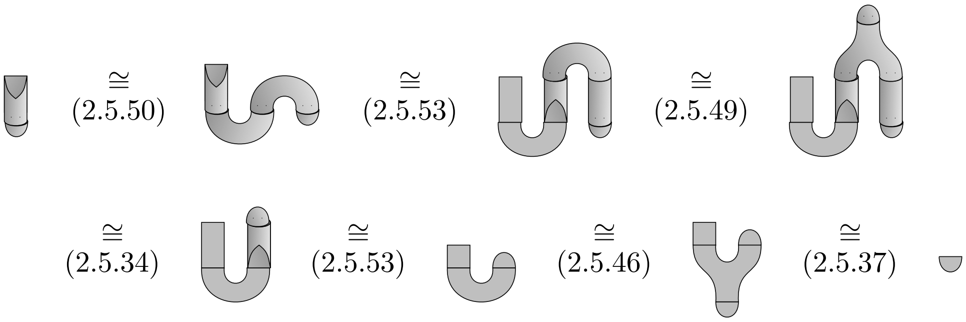


(2.5.63)

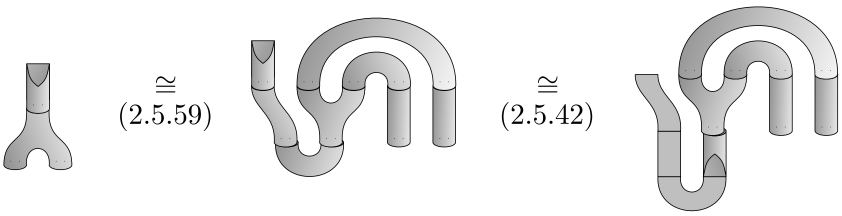
The proofs of Equations (2.5.60) and (2.5.61) are identical to those above with the closed cobordisms replaced by their open counterparts. ☐

Proposition 2.5.13. The cozipper  is a homomorphism of coalgebras.

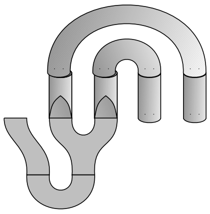
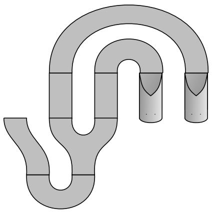
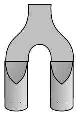
Proof. The proof follows from the following sequence of diffeomorphisms:



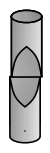
(2.5.64)

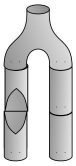
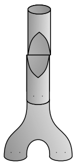
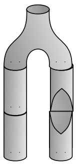


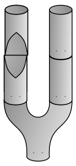
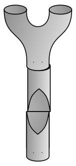
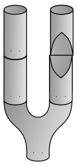
(2.5.65)

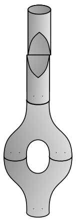
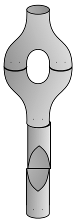
$\underset{(2.5.41)}{\cong}$  $\underset{(2.5.54)}{\cong}$  $\underset{(2.5.61)}{\cong}$  .

∎

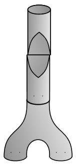
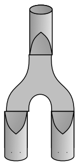
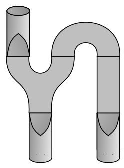
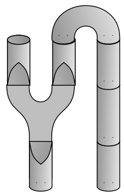
Proposition 2.5.14. Open-closed cobordisms of the form  which we sometimes call *closed windows*, can be moved around freely in any closed diagram. By this we mean that the following open-closed cobordisms are diffeomorphic,

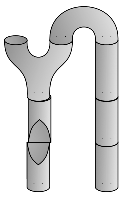
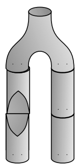
 $\cong$  $\cong$  , (2.5.66)

 $\cong$  $\cong$  , (2.5.67)

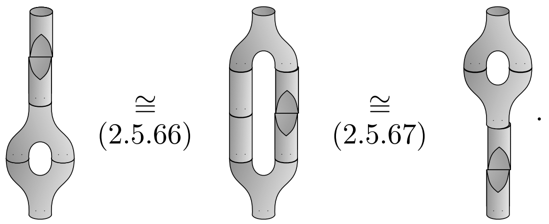
 $\cong$  . (2.5.68)

Proof. Equation (2.5.66) holds because,

 $\underset{(2.5.65)}{\cong}$  $\underset{(2.5.60)}{\cong}$  $\underset{(2.5.54)}{\cong}$ 

$\underset{(2.5.41)}{\cong}$  $\underset{(2.5.58)}{\cong}$  .

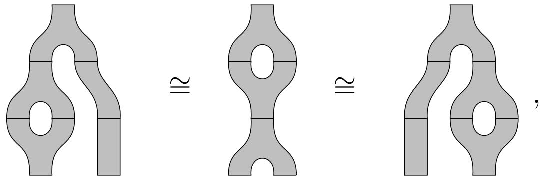
Equation (2.5.67) can be proven similarly. Equation (2.5.68) follows readily from the prior two:



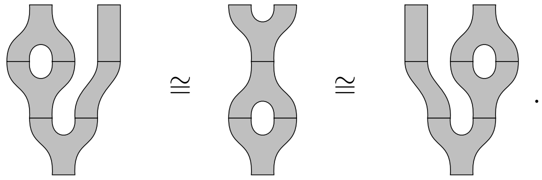
(2.5.69)

□

Proposition 2.5.15. Open-closed cobordisms of the form  which we sometimes call *open windows*, can be moved around freely in any open diagram. More precisely, the following open-closed cobordisms are diffeomorphic,

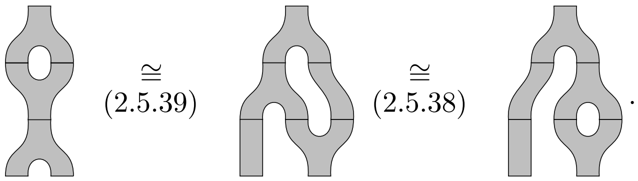


(2.5.70)



(2.5.71)

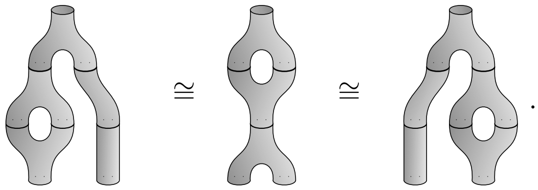
Proof. We show only the proof of one of these relations,



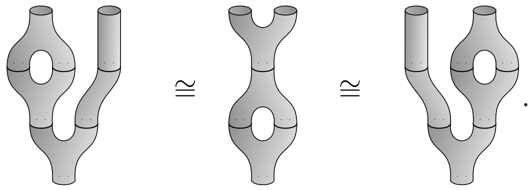
(2.5.72)

□

Proposition 2.5.16. Open-closed cobordisms of the form , also called *genus-one operators*, can be moved around freely in any closed diagram. More precisely,



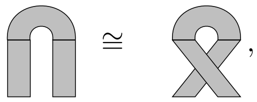
(2.5.73)



(2.5.74)

Proof. The proof is similar to that of Proposition 2.5.15. □

Proposition 2.5.17. The following open-closed cobordisms are diffeomorphic by compositions of the diffeomorphisms of Proposition 2.5.10:

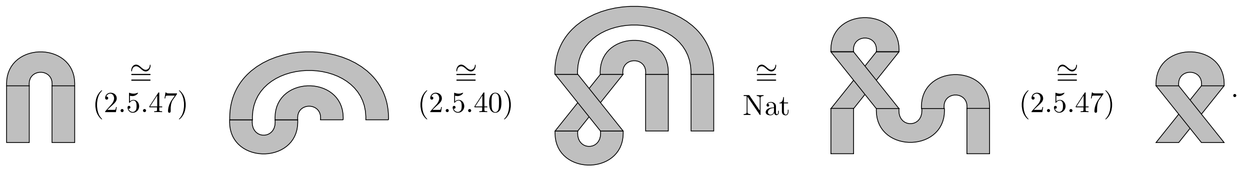


(2.5.75)

(2.5.76)

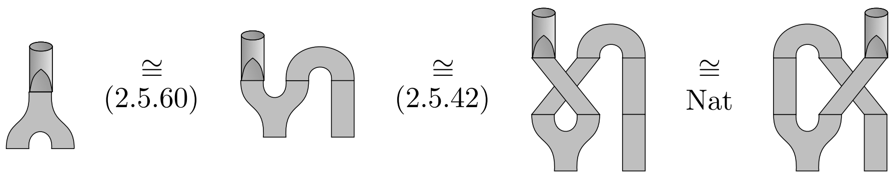
(2.5.77)

Proof. The proof of Equation (2.5.75) is as follows:

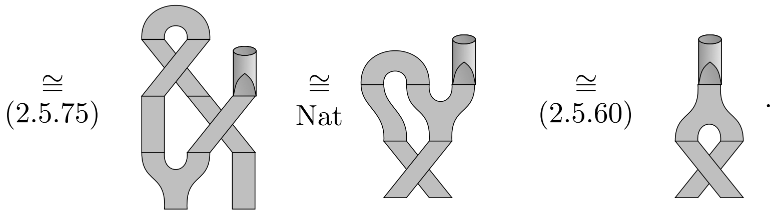


(2.5.78)

The proof of Equation (2.5.76) is as follows:

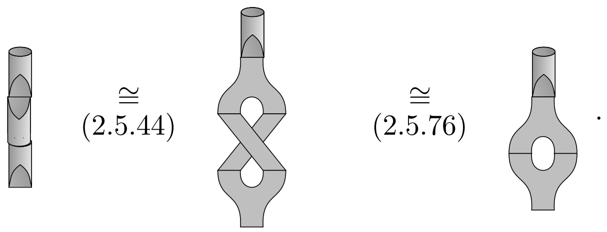


(2.5.79)



(2.5.80)

The second part of Equation (2.5.76) is proven similarly. Finally, Equation (2.5.77) follows from the Cardy condition as shown below:



(2.5.81)

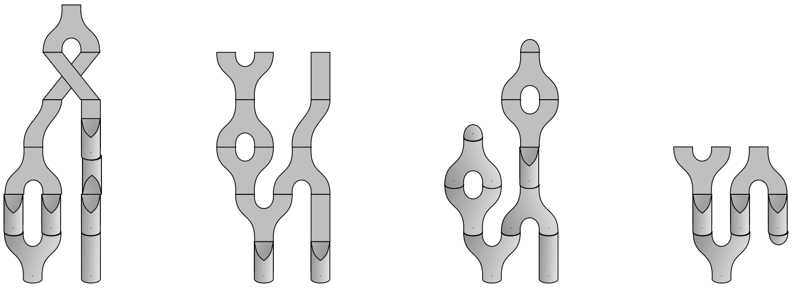
□

2.6 The normal form of an open-closed cobordism

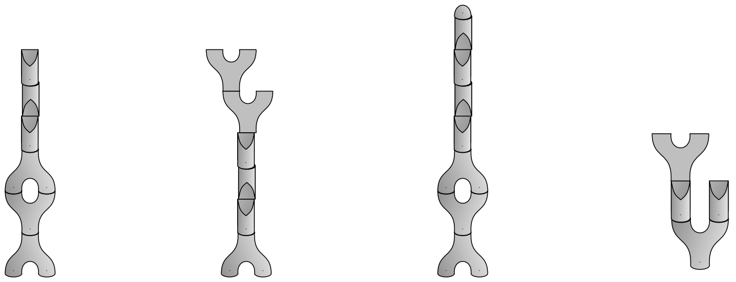
In this section, we describe the normal form of an arbitrary connected open-closed cobordism. This normal form is characterized by its genus, window number, and open boundary permutation (*c.f.* Definition 2.5.4). For non-connected open-closed cobordisms, the normal form has to be taken for each component independently.

2.6.1 The case of open source and closed target

We begin by describing the normal form of a connected open-closed cobordism whose source consists only of intervals and whose target consists only of circles. More precisely, we consider those open-closed cobordisms $f \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ for which $\vec{n} = (1, 1, \ldots, 1)$ and $\vec{m} = (0, 0, \ldots, 0)$ and denote the set of all such cobordisms by $\mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n}, \vec{m}]$. Some examples are shown below:

 . (2.6.1)

Once we have defined the normal form for this class of cobordisms, we describe in Section 2.6.2 the normal form of an arbitrary connected open-closed cobordism by exploiting the duality on the interval and circle, *c.f.* (2.5.47) and (2.5.50). To provide the reader with some intuition about the normal form, the cobordisms of (2.6.1) are shown in normal form below:

 . (2.6.2)

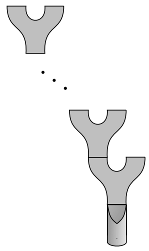
Definition 2.6.1. Let $f \in \mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n}, \vec{m}]$ be connected with open boundary permutation $\sigma(f)$, window number $\omega(f)$, and genus $g(f)$. Write the open boundary permutation as a product $\sigma(f) = \sigma_1 \cdots \sigma_r$, $r \in \mathbb{N}_0$, of disjoint cycles $\sigma_j = (i_1^{(j)}, \ldots, i_{q_j}^{(j)})$ of length $q_j \in \mathbb{N}$,

$1 \leq j \leq r$. The normal form is the composite,

$$\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f) := E_{|\vec{m}|} \circ D_{g(f)} \circ C_{\omega(f)} \circ B_r \circ \Big(\coprod_{j=1}^{r} A(q_j)\Big) \circ \overline{\sigma(f)}, \tag{2.6.3}$$

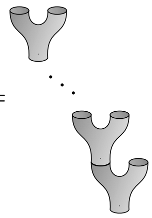
of the following open-closed cobordisms.

- For each cycle σ_j, the open-closed cobordism $A(q_j)$ consists of $q_j - 1$ flat multiplications and then a cozipper,

$$A(q_j) :=$$  . (2.6.4)

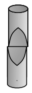
The normal form (2.6.3) contains the free union of such a cobordism for each cycle σ_j, $1 \leq j \leq r$. Cycles of length one are represented by a single cozipper. If $|\vec{n}| = 0$, then we have $r = 0$, and the free union is to be replaced by the empty set.

- If $r \geq 1$, then the open-closed cobordism B_r consists of $r - 1$ closed multiplications,

$$B_r :=$$  (2.6.5)

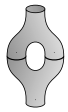
and otherwise $B_0 :=$  .

- The open-closed cobordism $C_{\omega(f)}$ is defined as,

$$C_{\omega(f)} := \underbrace{C' \circ C' \circ \cdots \circ C'}_{\omega(f)}, \qquad C' :=$$  (2.6.6)

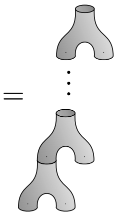
if $\omega(f) \geq 1$ and empty otherwise.

- Similarly,

$$D_{g(f)} := \underbrace{D' \circ D' \circ \cdots \circ D'}_{g(f)}, \qquad D' :=$$  (2.6.7)

if $g(f) \geq 1$ and empty otherwise.

- $E_{|\vec{m}|}$ consists of $|\vec{m}| - 1$ closed comultiplications,

$$E_{|\vec{m}|} :=$$  (2.6.8)

if $|\vec{m}| \geq 1$ and a closed cup $E_0 :=$  otherwise.

- Finally, $\overline{\sigma(f)}$ denotes the open-closed cobordism that represents the permutation $\overline{\sigma(f)}$ (as defined in (2.5.19)) given in the following. Let $\tau(f)$ be the open boundary permutation of the open-closed cobordism

$$E_{|\vec{m}|} \circ D_{g(f)} \circ C_{\omega(f)} \circ B_r \circ \big(\coprod_{j=1}^{r} A(q_j)\big). \tag{2.6.9}$$

Since by construction both $\tau(f)$ and $\sigma(f)$ have the same cycle structure, characterized by the partition $|\vec{n}| = \sum_{j=1}^{r} q_j$, there exists a permutation $\overline{\sigma(f)}$ such that,

$$\sigma(f) = \left(\overline{\sigma(f)}\right)^{-1} \cdot \tau(f) \cdot \overline{\sigma(f)}. \tag{2.6.10}$$

Note that multiplying $\overline{\sigma(f)}$ by an element in the centralizer of $\sigma(f)$ yields the same open-closed cobordism $\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)$ up to equivalence because of the relations (2.5.37) and (2.5.76), and so $\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)$ is well defined.

When we prove the sufficiency of the relations in Section 2.6.3 below, we provide an algorithm which automatically produces the required $\overline{\sigma(f)}$. Figure 2.1 depicts the structure of the normal form up to the $\overline{\sigma(f)}$, *i.e.* it shows a cobordism of the form (2.6.9).

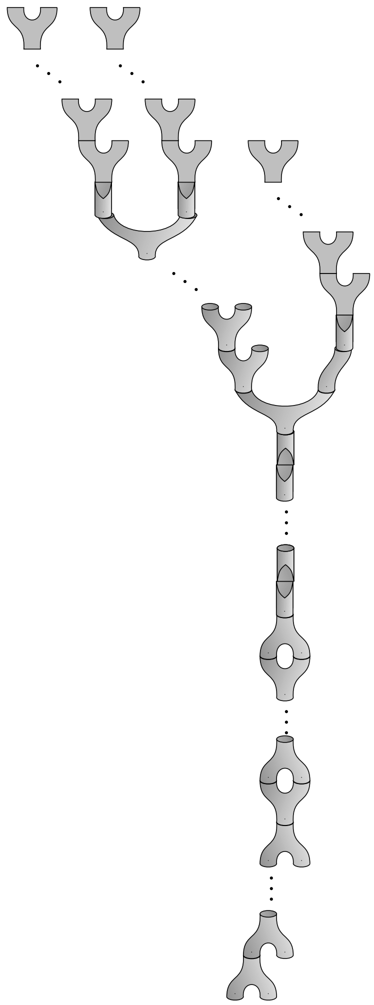


Figure 2.1: This figure depicts the normal form of an open-closed cobordism in $\mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n},\vec{m}]$ without precomposition with a permutation, *i.e.* it shows the open-closed cobordism (2.6.9).

Any cobordism in normal form is invariant (up to equivalence) under composition with certain permutation morphisms as follows.

Proposition 2.6.2. Let $[f] \in \mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n},\vec{m}]$. Then

$$[\sigma^{(\vec{m})} \circ \mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)] = [\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)] \tag{2.6.11}$$

for any $\sigma \in \mathcal{S}_{|\vec{m}|}$, and

$$[\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f) \circ \sigma_j^{(\vec{n})}] = [\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)] \tag{2.6.12}$$

for all cycles $\sigma_j \in \mathcal{S}_{|\vec{n}|}$ from the decomposition of the open boundary permutation $\sigma(f) = \sigma_1 \cdots \sigma_r$ into disjoint cycles.

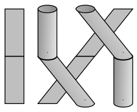
Proof. Equation (2.6.11) follows from (2.5.36), (2.5.50), and (2.5.51) while (2.6.12) follows from (2.5.37) and (2.5.48). □

2.6.2 The case of generic source and target

We now extend the definition of the normal form of connected cobordisms from $\mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n},\vec{m}]$ to $\mathbf{2Cob}^{\mathrm{ext}}[\vec{n},\vec{m}]$. Let f be a representative of the diffeomorphism class of an open closed cobordism $[f]$ in $\mathbf{2Cob}^{\mathrm{ext}}[\vec{n},\vec{m}]$. Let $\vec{n}_0 = (0,0,\ldots,0)$ and $\vec{n}_1 = (1,1,\ldots,1)$ such that $\vec{n}_0 \coprod \vec{n}_1$ is a permutation of $\vec{n}$, and similarly for $\vec{m}$.

We define a map $\Lambda\colon \mathbf{2Cob}^{\mathrm{ext}}[\vec{n},\vec{m}] \to \mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{m}_1 \coprod \vec{n}_1, \vec{m}_0 \coprod \vec{n}_0]$ as follows: Let $\sigma_1 \in \mathcal{S}_{|\vec{n}|}$ denote the permutation that sends $\vec{n}$ to $\vec{n}_1 \coprod \vec{n}_0$. Let $\sigma_2 \in \mathcal{S}_{|\vec{m}|}$ denote the permutation that sends $\vec{m}$ to $\vec{m}_1 \coprod \vec{m}_0$. For example, if $\vec{n} = (1,0,0,1)$, then σ_1 is represented as an open-closed cobordism by:

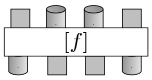
$$\sigma_1 \quad = $$



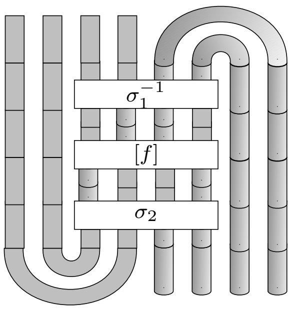
. (2.6.13)

For $[f] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n},\vec{m}]$ we define $\Lambda([f])$ to be the open closed cobordism obtained from $[f]$ by precomposing with σ_1^{-1}, postcomposing with σ_2, gluing closed copairings on each circle in $\vec{n}_0$, and gluing open pairings on each interval in $\vec{m}_1$. For example, let $[f]$ be an arbitrary open closed cobordism from $(1,0,0,1)$ to $(0,1,1,0)$, then $\Lambda([f])$ is illustrated below:

$\Lambda\colon$



$\mapsto$



. (2.6.14)

Up to equivalence, this assignment does not depend on the choice of representative in the class $[f]$; if f' is a different representative then there exists a black boundary preserving diffeomorphism from f to f'. Applying this diffeomorphism in the interior of $\Lambda([f])$ shows that $\Lambda([f'])$ is equivalent to $\Lambda([f])$,

$$[\Lambda([f])] = [\Lambda([f'])]. \tag{2.6.15}$$

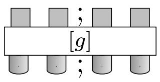
$\Lambda([f])$ is connected if and only if f is.

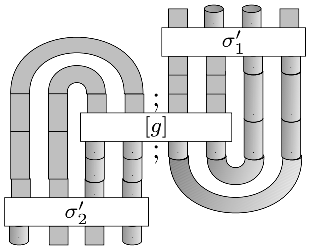
Given some extra structure, an inverse mapping can be defined. Let g be a representative of a diffeomorphism class $[g] \in \mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n}',\vec{m}']$ equipped with:

- a decomposition of its source into a free union $\vec{n}' = \vec{n}'_t \coprod \vec{n}'_s$,
- a decomposition of its target into a free union $\vec{m}' = \vec{m}'_t \coprod \vec{m}'_s$,
- an element of the symmetric group $\sigma'_1 \in \mathcal{S}_{|\vec{n}'_s|+|\vec{m}'_s|}$, and
- an element of the symmetric group $\sigma'_2 \in \mathcal{S}_{|\vec{n}'_t|+|\vec{m}'_t|}$.

Note that the image of an $[f] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ under the mapping Λ is equipped with such structure. The decompositions are given by distinguishing which intervals and circles came from the source and the target. The permutations can be taken to be $\sigma'_1 = \sigma_1$ and $\sigma'_2 = \sigma_2^{-1}$.

We define $\Lambda^{-1}([g])$ to be the open-closed cobordism in $\mathbf{2Cob}^{\mathrm{ext}}[\sigma'_2(\vec{n}'_t \coprod \vec{m}'_t), \sigma'_1(\vec{n}'_s \coprod \vec{m}'_s)]$ given by gluing open copairings to the intervals in $\vec{n}'_t$ and closed pairings to the circles in $\vec{m}'_s$. The result of this gluing is then precomposed with a cobordism representing σ'_1 and postcomposed with a cobordism representing σ'_2.

Λ^{-1}: 

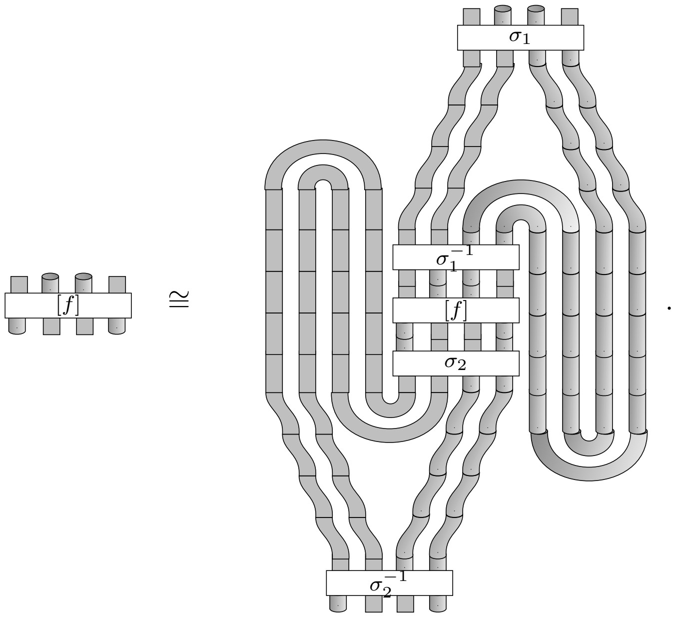
$\mapsto$ 

. (2.6.16)

Again, this assignment does not depend on the choice of representative of the class $[g]$. One can readily verify that this defines a bijection between the equivalence classes of open-closed cobordisms in $\mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ and those of open-closed cobordisms in $\mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{m}_1 \coprod \vec{n}_1, \vec{m}_0 \coprod \vec{n}_0]$ equipped with the extra structure described above. One direction of this verification,

$$[\Lambda^{-1}([\Lambda([f])])] = [f], \tag{2.6.17}$$

is depicted schematically below:



(2.6.18)

In Theorem 2.6.5 below, we show that for any connected $[g] \in \mathbf{2Cob}^{\text{ext}}_{\text{O}\to\text{C}}[\vec{n}, \vec{m}]$, g is equivalent to its normal form, *i.e.*

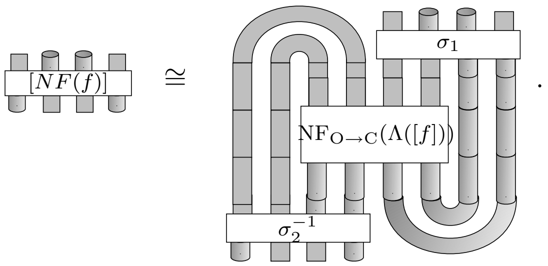
$$[g] = [\text{NF}_{\text{O}\to\text{C}}(g)]. \tag{2.6.19}$$

Applying this result to $[g] := \Lambda([f])$ is the motivation for the definition of the normal form for generic connected $[f] \in \mathbf{2Cob}^{\text{ext}}[\vec{n}, \vec{m}]$.

Definition 2.6.3. Let $[f] \in \mathbf{2Cob}^{\text{ext}}[\vec{n}, \vec{m}]$ be connected. Then we define its normal form by,

$$[\text{NF}(f)] := \Lambda^{-1}([\text{NF}_{\text{O}\to\text{C}}\big(\Lambda([f])\big)]), \tag{2.6.20}$$

which can be depicted as follows:

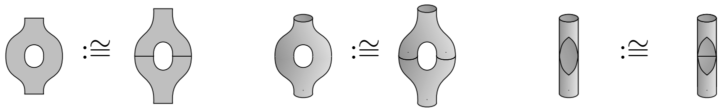


(2.6.21)

2.6.3 Proof of sufficiency of relations

In this section, we show that any connected open-closed cobordism $[f] \in \mathbf{2Cob}^{\text{ext}}_{\text{O}\to\text{C}}[\vec{n}, \vec{m}]$ can be related to its normal form $\text{NF}_{\text{O}\to\text{C}}(f)$ by applying the relations of Proposition 2.5.10 a finite number of times. We know that f is equivalent to an open-closed cobordism of the form stated in Proposition 2.5.9.

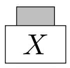
For convenience, we designate the following composites,



(2.6.22)

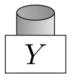
as being distinct generators. This simplifies the proof of the normal form below. We continue to use the shorthand (2.5.46) and (2.5.49). However, we do *not* consider these as distinct generators.

In our diagrams, we denote an arbitrary open-closed cobordism X, whose source is a general object $\vec{n}_X$ that contains at least one 1, as follows:



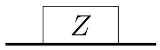
. (2.6.23)

Similarly, to denote an arbitrary open-closed cobordism Y, whose source is a general object $\vec{n}_Y$ containing at least one 0, we use the notation:



. (2.6.24)

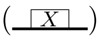
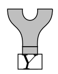
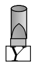
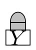
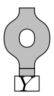
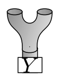
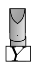
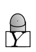
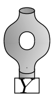
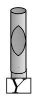
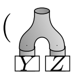
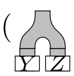
Finally, for an arbitrary open-closed cobordism Z, whose target $\vec{m}$ is not glued to any other cobordism in the decomposition of Σ, we use the following notation:



(2.6.25)

and similarly if the source is not glued to anything.

Definition 2.6.4. Let $[f] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be written in the form of Proposition 2.5.9. The *height* of a generator in the decomposition of f is the following number defined inductively, ignoring all identity morphisms in the decomposition:

- $h($  $) := 0$
- $h($  $) = h($  $) := 0$
- $h($  $) = h($  $) = h($  $) = h($  $) := 1 + h(Y)$
- $h($  $) = h($  $) = h($  $) = h($  $) = h($  $) := 1 + h(Y)$
- $h($  $) = h($  $) := h(Y) + h(Z) + 1.$

Theorem 2.6.5. Let $[f] \in \mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n}, \vec{m}]$ be a connected open-closed cobordism. Then f is equivalent to its normal form, *i.e.*

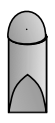
$$[f] = [\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)]. \tag{2.6.26}$$

Proof. We say *decomposition* for a presentation of f as a composition of the generators as in Proposition 2.5.9, *i.e.* for a generalized handle decomposition. We use the term *move* for the application of a diffeomorphism from Proposition 2.5.10, we just say diffeomorphism here meaning diffeomorphism relative to the black boundary, and we use the term *configuration* of a generator in a decomposition to refer to the generators immediately pre- and postcomposed to it.

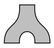
Employing Proposition 2.5.9, let f be given by any arbitrary decomposition. We construct a diffeomorphism from this decomposition to $\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)$ by applying a finite sequence of the moves from Proposition 2.5.10. This proceeds step by step as follows.

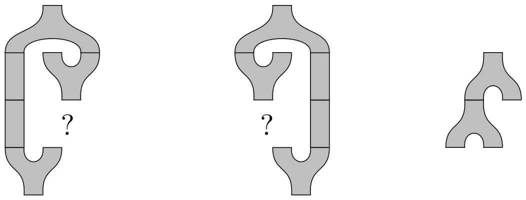
I) The decomposition of f is equivalent to one without any *open cups*  or *open caps* . This is achieved by applying the following moves:

a)  $\xmapsto[(2.5.65)]{}$ 

b)  $\xmapsto[(2.5.41)]{}$ 

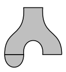
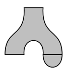
to every instance of the open cup and cap.

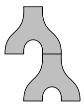
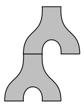
II) The resulting decomposition of f is equivalent to one in which every *open comultiplication*  appears in one of the following three configurations:



(2.6.27)

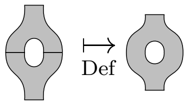
where the '?' may be any open-closed cobordism which may or may not be attached to the multiplication at the bottom. We prove this step-by-step by considering every possible configuration and providing the moves to reduce the decomposition into one of the above mentioned configurations.

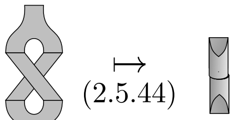
a) The cases  and  are excluded by step (Ia).

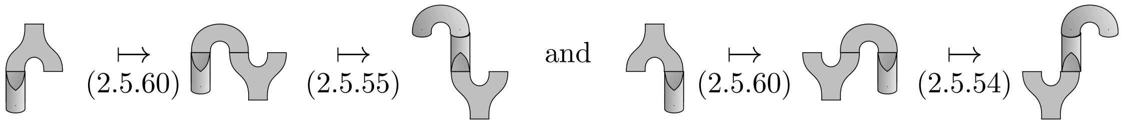
b) Wherever possible apply the move:  $\xmapsto[(2.5.38)]{}$  .

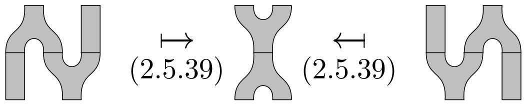
c) We consider all of the remaining possible configurations and provide a list of moves which either remove the open comultiplication or reduce its height. Since there are no longer any open cups after (Ia) and since the target of f is of the form $\vec{m} = (0, \ldots, 0)$, *i.e.* a free union of circles, the open comultiplication is either removed from the diagram or takes the form claimed in (II) before its height is reduced to zero.

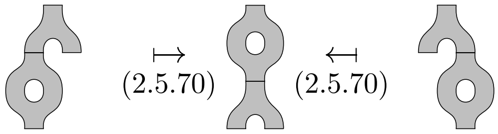
Apply the following moves wherever possible:

1) 

2) 

3) 

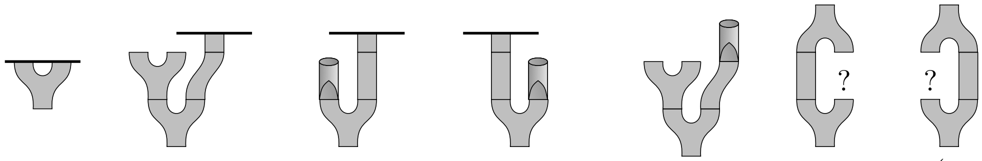
4) 

5) 

d) Iterate steps (IIb) and (IIc). Since each iteration either removes the open comultiplication or reduces its height, this process is guaranteed to terminate with every comultiplication in one of the three configurations of (2.6.27).

III) Now we apply a sequence of moves to the decomposition of f which reduces the number of possible configurations that need to be considered.

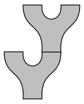
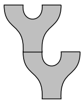
a) To begin, we provide a sequence of moves to put every *open multiplication*  in the decomposition of f into one of the following configurations:

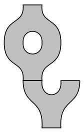
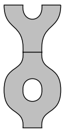
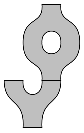


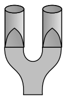
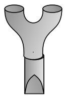
(2.6.28)

Again, we prove this claim by considering all possible configurations of the open multiplication.

Apply the following moves which either removes the open multiplication or increases its height or attains the desired configuration.

1)  $\mapsto$ (2.5.37) 

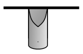
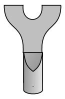
2)  $\mapsto$ (2.5.71)  $\leftarrow\!\dashv$ (2.5.71) 

3)  $\mapsto$ (2.5.41) 

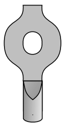
4) The moves of steps II(c)1, II(c)2, II(c)4.

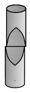
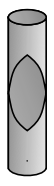
All other configurations are excluded by step I. Since none of these steps increases the number of generators in the decomposition of f, iterating this process either removes all open multiplications or puts them into the configurations in (2.6.28) as claimed above.

b) Now we show that the source of every cozipper  can be put into either of the following configurations:

  (2.6.29)

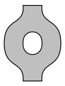
We establish the above claim by applying the following sequence of moves:

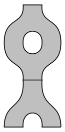
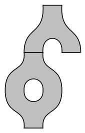
1)  $\mapsto$ (2.5.77) 

2)  $\xmapsto[\text{Def}]{}$ 

3) The moves of step II(c)3.

wherever they are possible. All other configurations are excluded by step I.

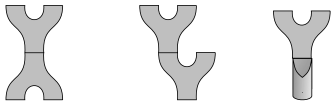
c) In this step we show that every instance of the open window  can be removed. Iterate the following sequence of moves wherever possible.

1)  $\mapsto$ (2.5.70) 

2) The moves of steps III(a)2 and III(b)1.

All other configurations are excluded by step I. Iterating these moves is guaranteed to remove all instances of the open window since each iteration either removes the window or reduces its height. The height of the open window cannot be zero.

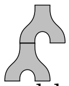
d) From the sequence of moves applied thus far, it follows that the target of every open multiplication is in one of the following configurations:

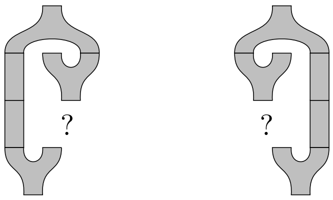


(2.6.30)

All other possibilities are excluded by steps III(a)1, I and IIIc.

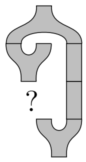
IV) In this step, we apply a sequence of moves that removes all open comultiplications. After step II, we need to consider only three cases. Step III has not changed this situation. From the set of open comultiplications in the decomposition of f, choose one of minimal height.

a) The case  has been excluded by the assumption that the open comultiplication is of minimal height. Hence the only remaining configurations to consider are

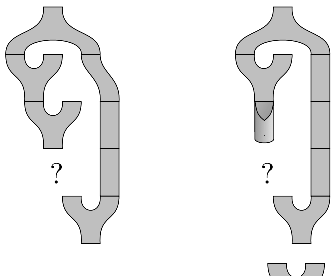


(2.6.31)

where no other open comultiplication occurs in '?' above.

b) By symmetry it suffices to consider one of the remaining configurations, say  .

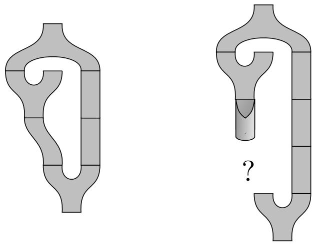
We proceed by considering all possible configurations of '?' above. The first generator in the decomposition of '?' is determined by step IIId and the assumption that the open comultiplication under consideration is of minimal height. Hence, only two situations are possible:



(2.6.32)

1) In the first case, iteratively apply the move $\underset{(2.5.37)}{\mapsto}$ so that the

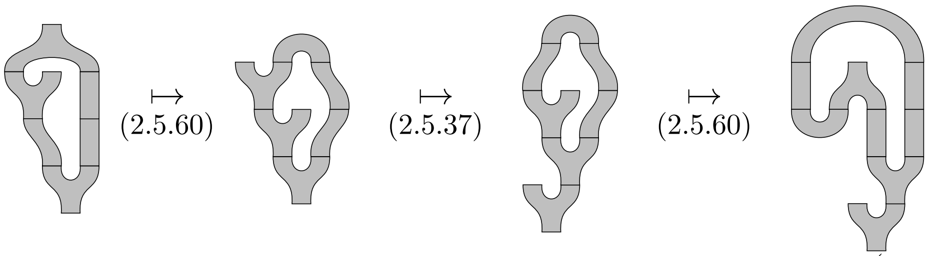
only possible configurations are



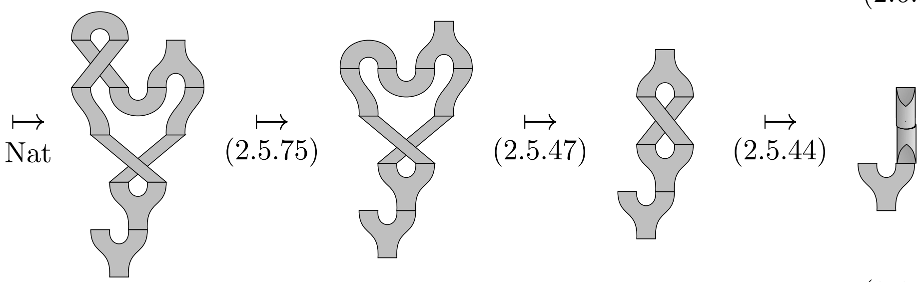
(2.6.33)

In the following two steps we remove the open comultiplication from the above two situations.

2) Consider the first case in (2.6.33) above. The comultiplication is removed by the following sequence of moves:

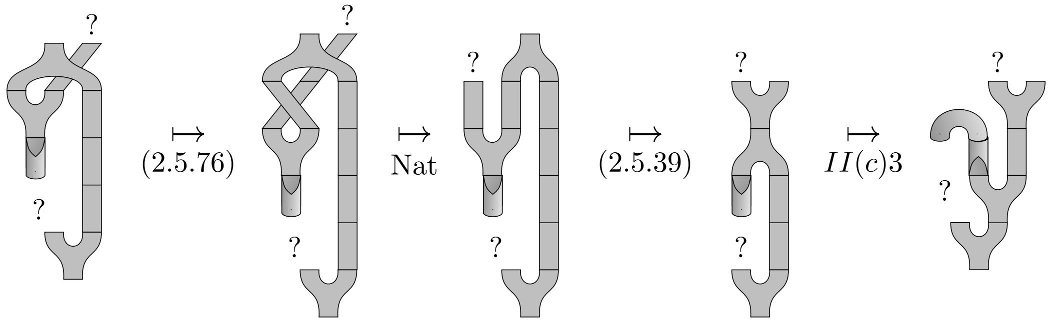


(2.6.34)



(2.6.35)

3) Consider now the second case in (2.6.33). In this case the comultiplication is removed by the following sequence of moves:



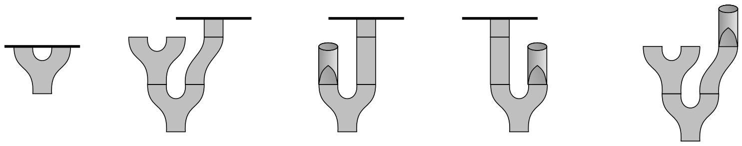
c) Step IVb has changed the cobordism so much that the claims made in steps II and III need not hold any longer. We therefore reapply the steps II and III.

d) Then we iterate the sequence of steps IVb and IVc until all open comultiplications have disappeared. This iteration terminates because neither step II nor step III (which are invoked in IVc) increase the number of open comultiplications, but step IVb always decreases this number by one.

e) When the last open comultiplication has disappeared in step IVd, step IVc ensures that the claims made in steps II and III are satisfied again.

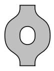
V) At this stage of the proof, all open caps, open cups and open comultiplications have been removed from the decomposition of f. The decomposition has the following further properties.

a) After the step IIIa, it is clear that any open multiplication has its source in one of the following configurations:

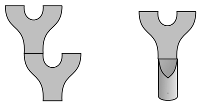


(2.6.36)

b) Every instance of the cozipper  is in the configuration claimed in step IIIb.

c) All instances of  have been removed by step IIIc.

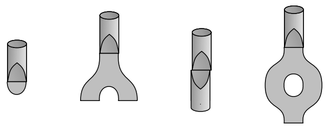
d) From step IIId and step IV, the only possible configurations for the target of an open multiplication are



(2.6.37)

VI) Now we remove every instance of the zipper  in the decomposition of f. We consider all remaining possible configurations involving the zipper and provide the moves to get rid of it.

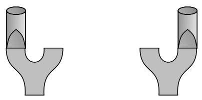
a) The following configurations:



(2.6.38)

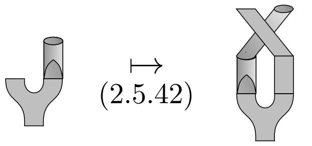
are excluded by steps I, IV, IIIb, and IIIc, respectively.

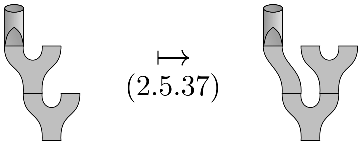
b) The remaining possibilities are

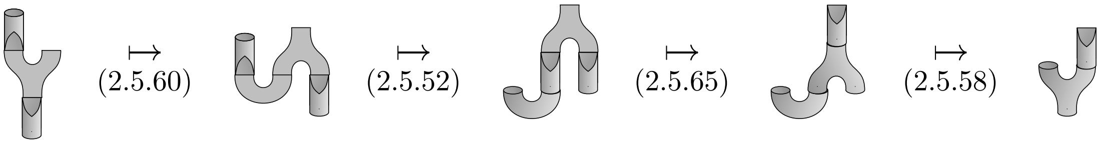


(2.6.39)

Using step Vd together with possibly repeated applications of the following moves:

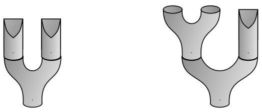
1) 

2) 

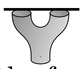
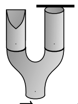
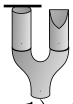
3) 

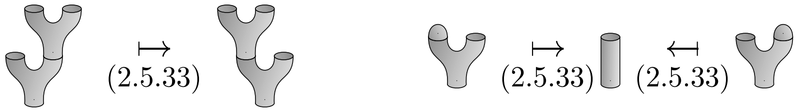
we make sure that all instances of  have disappeared.

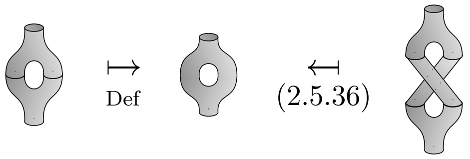
VII) The resulting decomposition of f is equivalent to one in which each *closed multiplication*  has its source in one of these configurations:

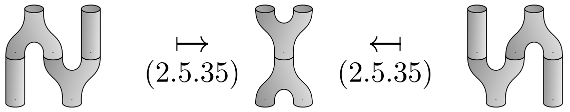


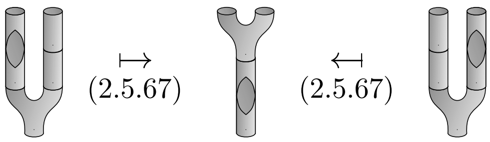
(2.6.40)

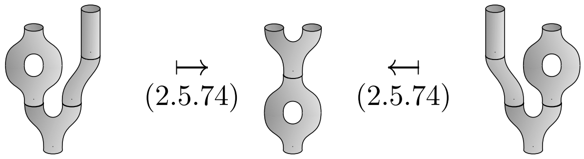
a) The cases ,  and  are excluded by the assumption that the source of f is of the form $\vec{n} = (1, \ldots, 1)$, *i.e.* is a free union of intervals I.

b) 

c) 

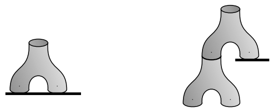
d) 

e) 

f) 

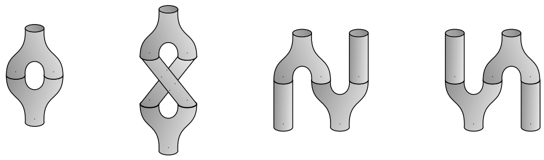
Since each of the above moves either removes the closed multiplication or increases its height while not increasing the number of generators, iterating the above moves is guaranteed to terminate with the closed multiplication in one of the specified configurations.

VIII) The decomposition of f is equivalent to one in which each closed comultiplication is in one of the following two configurations:

 (2.6.41)

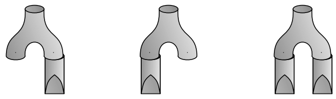
We consider all possible configurations of closed comultiplications.

a) The cases:

 (2.6.42)

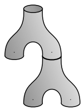
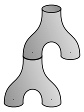
are excluded by step VII.

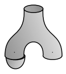
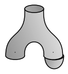
b) The cases:

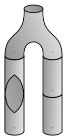
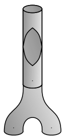
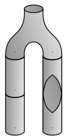
 (2.6.43)

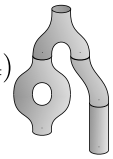
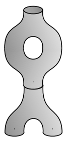
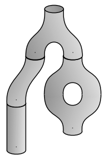
are excluded by step VI.

c) To prove the claim, we iterate the following sequences of moves wherever possible:

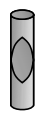
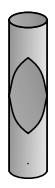
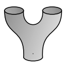
1)  $\mapsto$ (2.5.34) 

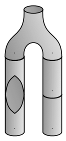
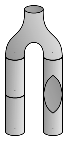
2)  $\mapsto$ (2.5.34)  $\leftarrow\!\shortmid$ (2.5.34) 

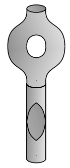
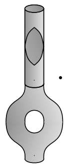
3)  $\mapsto$ (2.5.66)  $\leftarrow\!\shortmid$ (2.5.66) 

4)  $\mapsto$ (2.5.73)  $\leftarrow\!\shortmid$ (2.5.73) 

This iteration is guaranteed to terminate since each move either decreases the height of the closed comultiplication or removes it.

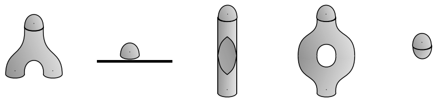
IX) In the resulting decomposition, each instance of the closed window  has above it one of the following: , , , or . There are only two remaining cases to consider.

a) The cases  and  are excluded by step VIII(c)3.

b) The claim follows by iterating the moves  $\underset{(2.5.68)}{\mapsto}$  .

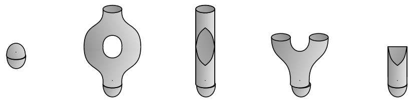
At this point, the decomposition of f is in the normal form desired. In order to see this, we need the claims made in the steps VIII, Va, VI, Vd, VII and the following two results.

X) If a *closed cap*  occurs anywhere in the resulting decomposition of f, then the source of f is the object $\vec{n} = \emptyset$, and the  has its target in one of the following configurations:

 (2.6.44)

This follows since all other possible configurations are excluded by steps VIIb and VI.

XI) If a *closed cup*  occurs anywhere in the resulting decomposition of f then the target of f is the object $\vec{m} = \emptyset$, and the source of the  is in one of the following configurations:

 (2.6.45)

The remaining cases are excluded by step VIIIc.

This concludes the proof. □

The main result for arbitrary connected open-closed cobordisms then follows.

Corollary 2.6.6. Let $[f] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be connected. Then $[f] = [\mathrm{NF}(f)]$.

Proof. Using Definition 2.6.3, then applying Theorem 2.6.5 to $\Lambda(f)$, and then applying (2.6.17), we find,

$$[\mathrm{NF}(f)] = [\Lambda^{-1}([\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(\Lambda([f]))])] = [\Lambda^{-1}([\Lambda([f])])] = [f]. \tag{2.6.46}$$

□

Since the normal form is already characterized by the invariants of Definition 2.5.4, the following corollary is an obvious consequence of the classification of surfaces.

Corollary 2.6.7. Let $[f], [f'] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be connected such that their genus, window number, and open boundary permutation agree, then $[f] = [f']$.

2.7 Open-closed TQFTs

In this section, we define the notion of open-closed TQFTs. We show that the categories $\mathbf{2Cob}^{\mathrm{ext}}$ and $\mathbf{Th}(\mathbf{K\text{-}Frob})$ are equivalent as symmetric monoidal categories which implies that the category of open-closed TQFTs is equivalent to the category of knowledgeable Frobenius algebras.

Definition 2.7.1. Let $\mathcal{C}$ be a symmetric monoidal category. An *open-closed Topological Quantum Field Theory (TQFT)* in $\mathcal{C}$ is a symmetric monoidal functor $\mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$. A *homomorphism* of open-closed TQFTs is a monoidal natural transformation of such functors. By $\mathbf{OC\text{-}TQFT}(\mathcal{C}) := \mathbf{Symm\text{-}Mon}(\mathbf{2Cob}^{\mathrm{ext}}, \mathcal{C})$, we denote the category of open-closed TQFTs.

Theorem 2.7.2. The category $\mathbf{2Cob}^{\mathrm{ext}}$ is equivalent as a symmetric monoidal category to the category $\mathbf{Th}(\mathbf{K\text{-}Frob})$.

This theorem states the precise correspondence between topology (Section 2.5) and algebra (Section 2.4). The second main result of this chapter follows from this theorem and from Proposition 2.4.6.

Corollary 2.7.3. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category. The category $\mathbf{K\text{-}Frob}(\mathcal{C})$ of knowledgeable Frobenius algebras in $\mathcal{C}$ is equivalent as a symmetric monoidal category to the category $\mathbf{OC\text{-}TQFT}(\mathcal{C})$.

These results also guarantee that one can use the generators of Section 2.5.4 and the relations of Section 2.5.5 in order to perform computations in knowledgeable Frobenius algebras. Recall that $\mathbf{2Cob}^{\mathrm{ext}}$ is a strict monoidal category whereas $\mathbf{Th}(\mathbf{K\text{-}Frob})$ is weak. When one translates from diagrams to algebra, one chooses parentheses for all tensor products and then inserts the structure isomorphisms α, λ, ρ as appropriate. The coherence theorem of Mac Lane guarantees that all ways of inserting these isomorphisms yield the same morphisms, and so the morphisms on the algebraic side are well defined by their diagrams.

In particular, we could have presented the second half of Section 2.5, starting with Subsection 2.5.6, entirely in the algebraic rather than in the topological language.

Proof of Theorem 2.7.2. Define a mapping Ξ from the objects of $\mathbf{2Cob}^{\mathrm{ext}}$ to the objects of the category **Th**(**K-Frob**) by mapping the generators as follows:

$$\Xi\colon \emptyset \mapsto \mathbb{1} \tag{2.7.1}$$

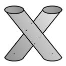
$$\Xi\colon \bigcirc \mapsto C \tag{2.7.2}$$

$$\Xi\colon — \mapsto A \tag{2.7.3}$$

and extending to the general object $\vec{n} \in \mathbf{2Cob}^{\mathrm{ext}}$ by mapping $\vec{n}$ to the tensor product in **Th**(**K-Frob**) of copies of A and C with all parenthesis to the left. More precisely, if $\vec{n} = (n_1, n_2, n_3, \cdots, n_k)$ with each $n_i \in \{0,1\}$, then $\Xi(\vec{n}) = (((\Xi(n_1) \otimes \Xi(n_2)) \otimes \Xi(n_3)) \cdots \Xi(n_k))$ with each $\Xi(0) := C$ and $\Xi(1) := A$. On the generating morphisms in $\mathbf{2Cob}^{\mathrm{ext}}$, Ξ is defined as follows:

 $\mapsto 1_C\colon C \to C$ (2.7.4)

 $\mapsto 1_A\colon A \to A$ (2.7.5)

 $\mapsto \tau_{C,C}\colon C \otimes C \to C \otimes C$ (2.7.6)

 $\mapsto \tau_{A,A}\colon A \otimes A \to A \otimes A$ (2.7.7)

 $\mapsto \tau_{A,C}\colon A \otimes C \to C \otimes A$ (2.7.8)

 $\mapsto \tau_{C,A}\colon C \otimes A \to A \otimes C$ (2.7.9)

 $\mapsto \mu_A\colon A \otimes A \to A$ (2.7.10)

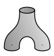
 $\mapsto \eta_A\colon \mathbb{1} \to A$ (2.7.11)

 $\mapsto \Delta_A\colon A \to A \otimes A$ (2.7.12)

 $\mapsto \varepsilon_A\colon A \to \mathbb{1}$ (2.7.13)

 $\mapsto \mu_C\colon C \otimes C \to C$ (2.7.14)

 $\mapsto \eta_C\colon \mathbb{1} \to C$ (2.7.15)

 $\mapsto \Delta_C\colon C \to C \otimes C$ (2.7.16)

 $\mapsto \varepsilon_C\colon C \to \mathbb{1}$ (2.7.17)

 $\mapsto \imath\colon C \to A$ (2.7.18)

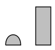
 $\mapsto \imath^*\colon A \to C$ (2.7.19)

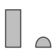
Without loss of generality we can assume that every general morphism f in $\mathbf{2Cob}^{\mathrm{ext}}$ is decomposed into elementary generators in such a way that each critical point in the decomposition of f has a unique critical value. We can then extend Ξ to a map on all the morphisms of

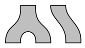
2Cob$^{\text{ext}}$ inductively using the following assignments:

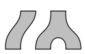

$$\mapsto \mu_A \otimes 1_A \colon (A \otimes A) \otimes A \to A \otimes A \tag{2.7.20}$$

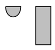

$$\mapsto 1_A \otimes \mu_A \circ \alpha_{A,A,A} \colon (A \otimes A) \otimes A \to A \otimes A \tag{2.7.21}$$

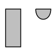

$$\mapsto \eta_A \otimes 1_A \circ \lambda_A^{-1} \colon A \to A \otimes A \tag{2.7.22}$$

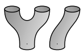

$$\mapsto 1_A \otimes \eta_A \circ \rho_A^{-1} \colon A \to A \otimes A \tag{2.7.23}$$

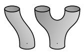

$$\mapsto \Delta_A \otimes 1_A \colon A \otimes A \to (A \otimes A) \otimes A \tag{2.7.24}$$

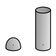

$$\mapsto \alpha_{A,A,A}^{-1} \circ 1_A \otimes \Delta_A \colon A \otimes A \to (A \otimes A) \otimes A \tag{2.7.25}$$

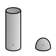

$$\mapsto \lambda_A \circ \varepsilon_A \otimes 1_A \colon A \otimes A \to A \tag{2.7.26}$$

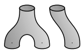

$$\mapsto \rho_A \circ 1_A \otimes \varepsilon_A \colon A \otimes A \to A \tag{2.7.27}$$

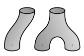

$$\mapsto \mu_C \otimes 1_C \colon (C \otimes C) \otimes C \to C \otimes C \tag{2.7.28}$$

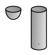

$$\mapsto 1_C \otimes \mu_C \circ \alpha_{C,C,C} \colon (C \otimes C) \otimes C \to C \otimes C \tag{2.7.29}$$

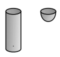

$$\mapsto \eta_C \otimes 1_C \circ \lambda_C^{-1} \colon C \to C \otimes C \tag{2.7.30}$$


$$\mapsto 1_C \otimes \eta_C \circ \rho_C^{-1} \colon C \to C \otimes C \tag{2.7.31}$$


$$\mapsto \Delta_C \otimes 1_C \colon C \otimes C \to (C \otimes C) \otimes C \tag{2.7.32}$$


$$\mapsto \alpha_{C,C,C}^{-1} \circ 1_C \otimes \Delta_C \colon C \otimes C \to (C \otimes C) \otimes C \tag{2.7.33}$$


$$\mapsto \lambda_C \circ \varepsilon_C \otimes 1_C \colon C \otimes C \to C \tag{2.7.34}$$


$$\mapsto \rho_C \circ 1_C \otimes \varepsilon_C \colon C \otimes C \to C \tag{2.7.35}$$

This assignment is well defined and extends to all the general morphisms in **2Cob**$^{\text{ext}}$ by the coherence theorem for symmetric monoidal categories, which ensures that there is a unique morphism from one object to another composed of associativity constraints and unit constraints. The relations in Proposition 2.5.10 and the proof that these are all the required relations in **2Cob**$^{\text{ext}}$ imply that the image of Ξ is in fact a knowledgeable Frobenius algebra in **Th**(**K-Frob**). Hence, Ξ defines a functor **2Cob**$^{\text{ext}} \to$ **Th**(**K-Frob**).

Define a natural isomorphism $\Xi_2 \colon \Xi(\vec{n}) \otimes \Xi(\vec{m}) \to \Xi(\vec{n} \coprod \vec{m})$ for $X, Y \in$ **2Cob**$^{\text{ext}}$ as follows: Let $\vec{n} = (n_1, n_2, n_3, \ldots, n_k)$ and $\vec{m} = (m_1, m_2, m_3, \cdots, m_\ell)$ so that

$$\Xi(\vec{n}) = (((\Xi(n_1) \otimes \Xi(n_2)) \otimes \Xi(n_3)) \cdots \Xi(n_k)), \tag{2.7.36}$$

$$\Xi(\vec{m}) = (((\Xi(m_1) \otimes \Xi(m_2)) \otimes \Xi(m_3)) \cdots \Xi(m_\ell)), \tag{2.7.37}$$

$$\Xi(\vec{n} \coprod \vec{m}) = (((((\Xi(n_1) \otimes \Xi(n_2)) \otimes \Xi(n_3)) \cdots \Xi(n_k)) \otimes \Xi(m_1)) \cdots \otimes \Xi(m_\ell)) \tag{2.7.38}$$

Hence the map $\Xi_2 \colon \Xi(\vec{n}) \otimes \Xi(\vec{m}) \to \Xi(\vec{n} \coprod \vec{m})$ is composed entirely of composites of the natural isomorphism α. By the coherence theorem for monoidal categories, any choice of composites from the source to the target is unique. One can easily verify that if $\Xi_0 := 1_{\mathbb{1}}$, then collection (Ξ, Ξ_2, Ξ_0) defines a monoidal natural transformation. Furthermore, the assignment by Ξ of the open-closed cobordisms generating $\mathbf{2Cob}^{\text{ext}}$'s symmetry ensures that (Ξ, Ξ_2, Ξ_0) is a symmetric monoidal functor.

Using the assignments from equations (2.7.1)-(2.7.19) we see that the generating open-closed cobordisms in $\mathbf{2Cob}^{\text{ext}}$ define a knowledgeable Frobenius algebra structure on the interval and circle. Hence, by the remarks preceding Proposition 2.4.6 we get a strict symmetric monoidal functor $\bar{\Xi}\colon \mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathbf{2Cob}^{\text{ext}}$. In this case, if X is related to Y in $\mathbf{Th}(\mathbf{K\text{-}Frob})$ by a sequence of associators and unit constraints then X and Y are mapped to the same object in $\mathbf{2Cob}^{\text{ext}}$. We now show that Ξ and $\bar{\Xi}$ define an equivalence of categories.

Let $\vec{n}$ be a general object in $\mathbf{2Cob}^{\text{ext}}$. From the discussion above we have that $\bar{\Xi}\Xi(\vec{n}) = \vec{n}$, so that $\bar{\Xi}\Xi(\vec{n}) = 1_{\mathbf{2Cob}^{\text{ext}}}$. If X is an object of $\mathbf{Th}(\mathbf{K\text{-}Frob})$ then X is a parenthesized word consisting of the symbols $\mathbb{1}, A, C, \otimes$. Let $\bar{\Xi}(X) = (n_1, n_2, \cdots, n_n)$ where the ordered sequence $(n_1, n_2, \cdots, n_n)$ corresponds to the ordered sequence of A's and C's in X. Hence, $\Xi\bar{\Xi}(X)$ is the word obtained from X by removing all the symbols $\mathbb{1}$ and putting all parenthesis to the left. Thus, $\Xi\bar{\Xi}(X)$ is isomorphic to X by a sequence of associators and unit constraints. We have therefore established the desired monoidal equivalence of symmetric monoidal categories. □

The following special cases are covered by Corollary 2.7.3.

Definition 2.7.4. Let $\mathbf{2Cob}^{\text{open}}$, $\mathbf{2Cob}^{\text{closed}} = \mathbf{2Cob}$, and $\mathbf{2Cob}^{\text{planar}}$ denote the subcategories of $\mathbf{2Cob}^{\text{ext}}$ consisting only of purely open cobordisms, purely closed cobordisms, and purely open cobordisms that can be embedded into the plane. An open (respectively closed, planar open) TQFT is a functor from $\mathbf{2Cob}^{\text{open}}$ (respectively $\mathbf{2Cob}^{\text{closed}}$, $\mathbf{2Cob}^{\text{planar}}$) into a symmetric monoidal category $\mathcal{C}$ ($\mathcal{C}$ need not be symmetric in the planar open context).

Corollary 2.7.5. *Let $\mathcal{C}$ be a symmetric monoidal category. The category of open TQFTs in $\mathcal{C}$ is equivalent as a symmetric monoidal category to the category of symmetric Frobenius algebras in $\mathcal{C}$.*

The following well-known result on 2-dimensional closed TQFTs [17,19] follows from Corollary 2.7.3, as does the 2-dimensional case of [39].

Corollary 2.7.6. *Let $\mathcal{C}$ be a symmetric monoidal category. The category of closed TQFTs in $\mathcal{C}$ is equivalent as a symmetric monoidal category to the category of commutative Frobenius* algebras in $\mathcal{C}$.

Corollary 2.7.7. Let $\mathcal{C}$ be a monoidal category. The category of planar open topological quantum field theories in $\mathcal{C}$ is equivalent to the category of Frobenius algebras in $\mathcal{C}$.

2.8 Boundary labels

In this section, we generalize the results on knowledgeable Frobenius algebras and on open-closed cobordisms to free boundaries labeled with elements of some set S. The proofs of these results are very similar to the unlabeled case, and so we state only the results.

Definition 2.8.1. Let S be a set. An *S-coloured knowledgeable Frobenius algebra*

$$(\{A_{ab}\}_{a,b\in S}, \{\mu_{abc}\}_{a,b,c\in S}, \{\eta_a\}_{a\in S}, \{\Delta_{abc}\}_{a,b,c\in S}, \{\varepsilon_a\}_{a\in S}, C, \{\imath_a\}_{a\in S}, \{\imath_a^*\}_{a\in S}) \tag{2.8.1}$$

in some symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ consists of,

- a commutative Frobenius algebra object $(C, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$,
- a family of objects A_{ab} of $\mathcal{C}$, $a, b \in S$,
- families of morphisms $\mu_{abc}\colon A_{ab} \otimes A_{bc} \to A_{ac}$, $\eta_a\colon \mathbb{1} \to A_{aa}$, $\Delta_{abc}\colon A_{ac} \to A_{ab} \otimes A_{bc}$, $\varepsilon_a\colon A_{aa} \to \mathbb{1}$, $\imath_a\colon C \to A_{aa}$ and $\imath^*\colon A_{aa} \to C$ of $\mathcal{C}$ for all $a, b, c \in S$ such that the

following conditions are satisfied for all $a, b, c, d \in S$:

$$\mu_{abd} \circ (\mathrm{id}_{A_{ab}} \otimes \mu_{bcd}) \circ \alpha_{A_{ab},A_{bc},A_{cd}} = \mu_{acd} \circ (\mu_{abc} \otimes \mathrm{id}_{A_{cd}}), \tag{2.8.2}$$

$$\mu_{aab} \circ (\eta_{aa} \otimes \mathrm{id}_{A_{ab}}) = \lambda_{A_{ab}}, \tag{2.8.3}$$

$$\mu_{abb} \circ (\mathrm{id}_{A_{ab}} \otimes \eta_{bb}) = \rho_{A_{ab}}, \tag{2.8.4}$$

$$\alpha_{A_{ab},A_{bc},A_{cd}} \circ (\Delta_{abc} \otimes \mathrm{id}_{A_{cd}}) \circ \Delta_{acd} = (\mathrm{id}_{A_{ab}} \otimes \Delta_{bcd}) \circ \Delta_{abd}, \tag{2.8.5}$$

$$(\varepsilon_{aa} \otimes \mathrm{id}_{A_{ab}}) \circ \Delta_{aab} = \lambda^{-1}_{A_{ab}}, \tag{2.8.6}$$

$$(\mathrm{id}_{A_{ab}} \otimes \varepsilon_{bb}) \circ \Delta_{abb} = \rho^{-1}_{A_{ab}}, \tag{2.8.7}$$

$$\begin{aligned}\Delta_{abd} \circ \mu_{acd} &= (\mathrm{id}_{A_{ab}} \otimes \mu_{bcd}) \circ \alpha_{A_{ab},A_{bc},A_{cd}} \circ (\Delta_{abc} \otimes \mathrm{id}_{A_{cd}}) \\ &= (\mu_{abc} \otimes \mathrm{id}_{A_{cd}}) \circ \alpha^{-1}_{A_{ab},A_{bc},A_{cd}} \\ &\quad \circ (\mathrm{id}_{A_{ab}} \otimes \Delta_{bcd}),\end{aligned} \tag{2.8.8}$$

$$\varepsilon_{aa} \circ \mu_{aba} = \varepsilon_{bb} \circ \mu_{bab} \circ \tau_{A_{ab},A_{ba}}, \tag{2.8.9}$$

$$\mu_{aaa} \circ (\imath_a \otimes \imath_a) = \imath_a \circ \mu, \tag{2.8.10}$$

$$\eta_{aa} = \imath_a \circ \eta, \tag{2.8.11}$$

$$\mu_{aab} \circ (\imath_a \otimes \mathrm{id}_{A_{ab}}) = \mu_{abb} \circ \tau_{A_{bb},A_{ab}} \circ (\imath_b \otimes \mathrm{id}_{A_{ab}}), \tag{2.8.12}$$

$$\varepsilon \circ \mu \circ (\mathrm{id}_C \otimes \imath^*_a) = \varepsilon_{aa} \circ \mu_{aaa} \circ (\imath_a \otimes \mathrm{id}_{A_{aa}}), \tag{2.8.13}$$

$$\imath_a \circ \imath^*_b = \mu_{aba} \circ \tau_{A_{ba},A_{ab}} \circ \Delta_{bab}. \tag{2.8.14}$$

It is easy to see that the notion of an S-coloured knowledgeable Frobenius algebra precisely models the topological relations of Proposition 2.5.10 for all possible ways of labeling the free boundaries with elements of the set S. The following consequences of this definition are not difficult to see from the diagrams of Proposition 2.5.10.

Corollary 2.8.2. Let $(\{A_{ab}\}, \{\mu_{abc}\}, \{\eta_a\}, \{\Delta_{abc}\}, \{\varepsilon_a\}, C, \{\imath_a\}, \{\imath^*_a\})$ be an S-coloured knowledgeable Frobenius algebra in some symmetric monoidal category $\mathcal{C}$.

1. Each A_{ab}, $a, b \in S$, is a rigid object of $\mathcal{C}$ whose left- and right-dual is given by A_{ba}.

2. Each A_{aa}, $a \in S$, forms a symmetric Frobenius algebra object in $\mathcal{C}$.

3. Each $\imath_a\colon C \to A_{aa}$, $a \in S$, forms a homomorphism of algebras in $\mathcal{C}$.

4. Each $\imath^*_a\colon A_{aa} \to C$, $a \in S$, forms a homomorphism of coalgebras in $\mathcal{C}$.

5. Each A_{ab} forms an A_{aa}-left-A_{bb}-right-bimodule in $\mathcal{C}$.

6. Each A_{ab} forms an A_{aa}-left-A_{bb}-right-bicomodule in $\mathcal{C}$.

Definition 2.8.3. A *homomorphism*

$$\begin{aligned} f \quad : \quad & (\{A_{ab}\},\{\mu_{abc}\},\{\eta_a\},\{\Delta_{abc}\},\{\varepsilon_a\},C,\{\imath_a\},\{\imath_a^*\}) \\ \to \quad & (\{A'_{ab}\},\{\mu'_{abc}\},\{\eta'_a\},\{\Delta'_{abc}\},\{\varepsilon'_a\},C',\{\imath'_a\},\{\imath'^*_a\}) \end{aligned} \tag{2.8.15}$$

of S-coloured knowledgeable Frobenius algebras is a pair $f = (\{f_{ab}\}_{a,b\in S}, f_C)$ consisting of a homomorphism of Frobenius algebras $f_C\colon C \to C'$ and a family of morphisms $f_{ab}\colon A_{ab} \to A'_{ab}$, $a,b \in S$ that satisfy the following conditions for all $a,b,c \in S$:

$$\mu'_{abc} \circ (f_{ab} \otimes f_{bc}) = f_{ac} \circ \mu_{abc}, \tag{2.8.16}$$

$$\eta'_a = f_{aa} \circ \eta_a, \tag{2.8.17}$$

$$\Delta'_{abc} \circ f_{ac} = (f_{ab} \otimes f_{bc}) \circ \Delta_{abc}, \tag{2.8.18}$$

$$\varepsilon'_a \circ f_{aa} = \varepsilon_a, \tag{2.8.19}$$

$$\imath'_a \circ f_C = f_{aa} \circ \imath_a, \tag{2.8.20}$$

$$\imath'^{\,*}_a \circ f_{aa} = f_C \circ \imath_a^*. \tag{2.8.21}$$

Definition 2.8.4. By $\mathbf{K\text{-}Frob}^{(S)}(\mathcal{C})$ we denote the category of S-coloured knowledgeable Frobenius algebras in some symmetric monoidal category $\mathcal{C}$ and their homomorphisms.

Definition 2.8.5. The category of *open-closed TQFTs* in some symmetric monoidal category $\mathcal{C}$ *with free boundary labels* in some set S is the category

$$\mathbf{OC\text{-}TQFT}^{(S)}(\mathcal{C}) := \mathbf{Symm\text{-}Mon}(\mathbf{2Cob}^{\mathrm{ext}}(S),\mathcal{C}). \tag{2.8.22}$$

In the S-coloured case, the correspondence between the algebraic and the topological category of Corollary 2.7.3 generalizes to the following result.

Theorem 2.8.6. Let S be some set and $\mathcal{C}$ be a symmetric monoidal category. The categories $\mathbf{K\text{-}Frob}^{(S)}(\mathcal{C})$ and $\mathbf{OC\text{-}TQFT}^{(S)}(\mathcal{C})$ are equivalent as symmetric monoidal categories.

In section 3.5 we will see that the groupoid algebra of a finite groupoid gives rise to an S-coloured knowledgeable Frobenius algebra for which S is the set of objects of the groupoid.

2.9 Remarks

In this chapter, we have extended the results of classical cobordism theory to the context of 2-dimensional open-closed cobordisms. Using manifolds with faces with a particular global structure, rather than the full generality of manifolds with corners, we have defined an appropriate category of open-closed cobordisms. Using a generalization of Morse theory to manifolds with corners, we have found a characterization of this category in terms of generators and relations. In order to prove the sufficiency of the relations, we have explicitly constructed the diffeomorphism between an arbitrary cobordism and a normal form which is characterized by topological invariants.

All of the technology outlined above is defined for manifolds with faces of arbitrary dimension. Thus, our work suggests a natural framework for studying extended topological quantum field theories in dimensions three and four. Using 3-manifolds or 4-manifolds with faces, one can imagine defining a category (most likely higher-category) of extended three or four dimensional cobordisms. In both cases, gluing will produce well defined composition operations using the existing technology for manifolds with faces. One could then extract a list of generating cobordisms, again using a suitable generalization of Morse theory.

The main difficulty in obtaining a complete generators and relations description of these higher-dimensional extended cobordism categories is the lack of general theory producing the relations. Specifically, the handlebody theory for manifolds with boundaries and corners is not as advanced as the standard Morse theory for closed manifolds. For the 2-dimensional case, we were able to use relations previously proposed in the literature and to show the sufficiency of these relations by finding the appropriate normal form for 2-dimensional open-closed cobordisms. Our induction proof shows that the proposed relations are in fact necessary and sufficient to reduce an arbitrary open-closed cobordism to the normal form. To extend these results to higher-dimensions, it is expected that a more sophisticated procedure will be required, most likely involving a handlebody theory for manifolds with faces.

We close this chapter by commenting on a different approach to TQFTs with corners. In the literature, for example [48], extended TQFTs are often defined for manifolds with corners in which the basic building blocks have the shape of bigons [48] with only one sort of boundary along which one can always glue. This is a special case of our definition which is obtained if every coloured boundary between two corners is shrunk until it disappears and there is a single corner left that now separates two black boundaries.

Chapter 3

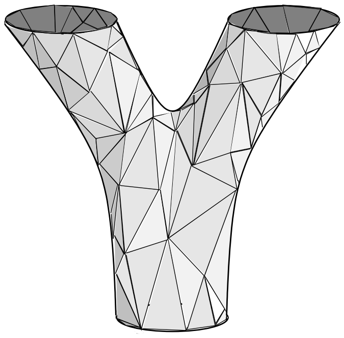
State sum construction of open-closed TQFT

3.1 The Fukuma, Hosono, and Kawai state sum

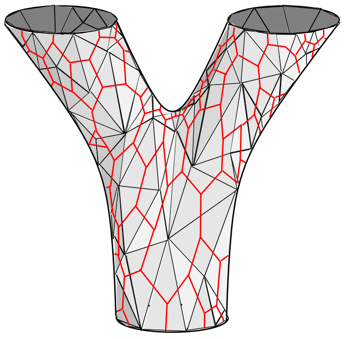
In this section we provide a brief sketch of the Fukuma, Hosono, and Kawai (FHK) state sum construction of 2-dimensional topological quantum field theories. For other expository articles on this 2-dimensional TQFT see [64–67]. The aim of this section is to provide a preview of what is to come as well as to highlight the essential differences between the state sum construction of open-closed cobordisms with the original FHK construction.

Fukuma, Hosono and Kawai found a way to construct 2-dimensional (closed) topological quantum field theories from semisimple algebras over the complex numbers [40]. Though they did not put it this way, their idea amounts to expressing any 2-dimensional cobordism in terms of a string diagram in the symmetric monoidal category $\mathbf{Vect}_{\mathbb{C}}$ of complex vector spaces. Viewed in this light, it is not difficult to see that their construction can be adapted to other symmetric monoidal categories. In particular, the construction works just as well for vector spaces over arbitrary fields k, but in this case the algebras required in the recipe are called *strongly separable*. When the field $k = \mathbb{C}$ the notions of strongly separable algebra and semisimple algebra coincide. The defining feature of strongly separable algebras is the nondegeneracy of a certain canonically associated bilinear form. We will discuss this more in Section 3.2.2.

In order to construct the FHK 2-dimensional TQFT $Z\colon \mathbf{2Cob} \to \mathbf{Vect}_{\mathbb{C}}$ one starts by choosing a triangulation of the cobordism M:



One then takes the Poincaré dual of this triangulation:

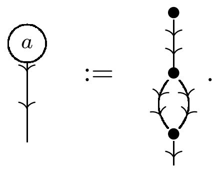


and interprets the result as a string diagram in some symmetric monoidal category $\mathcal{C}$.

This means that each edge in the triangulation of $\partial^{\mathrm{in}}(M)$ corresponds to some object $A \in \mathcal{C}$ and associated to the incoming boundary of M triangulated with m_1 edges we associate the object $A^{\otimes m_1} \in \mathcal{C}$. Likewise, to the outgoing boundary $\partial^{\mathrm{out}}(M)$ triangulated with m_2 edges we associate the object $A^{\otimes m_2}$. The Poincaré dual of the triangulation is then to be interpreted as a morphism in $\mathcal{C}$ mapping $A^{\otimes m_1}$ to $A^{\otimes m_2}$. For ease of exposition we will restrict our discussion to the case of $\mathbf{Vect}_{\mathbb{C}}$ for the remainder of this section.

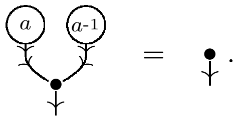
If A is a finite dimensional $\mathbb{C}$-algebra then the left regular representation of A is given by $L\colon A \to \mathrm{End}_{\mathbb{C}}(A)\colon a \mapsto L_a$ with $L_a\colon A \to A\colon b \mapsto ab$. As will be discussed in Section 3.2.2, the trace of the matrices of the left-regular representation equips A with a canonical bilinear form $g_{\mathrm{can}}\colon A \otimes A \to \mathbb{C}\colon a \otimes b \mapsto \mathrm{tr}_A(L_{ab})$. When the algebra A is semisimple (hence strongly separable) this canonical bilinear form is nondegenerate. We will show in Section 3.2.1 that a nondegenerate invariant bilinear form on A equips A with the structure of a Frobenius algebra. When A is strongly separable and the Frobenius structure is induced by the canonical bilinear form the resulting Frobenius structure will be symmetric.

In Section 3.2.3 we will show that the nondegeneracy of the canonical bilinear form is equivalent to the invertibility of what we call the window element:



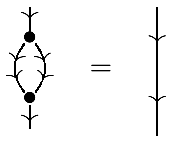
(3.1.1)

One can check that the window element a is in the centre of the algebra A; when A is strongly separable a is a central invertible element. Here we think of an element of the algebra A as a map $\mathbb{C} \to A$. The invertibility of the window element implies there exists an element $a^{-1}\colon \mathbb{C} \to A$ such that



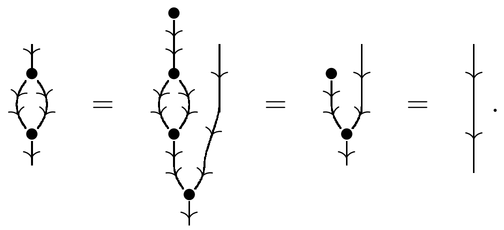
(3.1.2)

When the central invertible element a corresponds with the algebra unit $\eta\colon \mathbb{C} \to A$, then in this case, the bubble move



(3.1.3)

is satisfied because



(3.1.4)

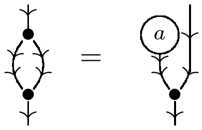
The bubble move was a crucial ingredient used by Fukuma, Hosono, and Kawai to establish the triangulation independence of their closed state sum TQFT $Z(M)$. Furthermore, using the bubble move and the associativity of the algebra A, they showed that the linear map corresponding to the triangulated cylinder is an idempotent p on the algebra A. They then showed that one can recover a commutative Frobenius algebra, hence a functor $Z\colon \mathbf{2Cob} \to \mathbf{Vect}_{\mathbb{C}}$, by projecting the algebra A onto the image of the idempotent $\mathrm{im}\, p$. Indeed, they showed that the image of the idempotent is isomorphic to the centre of the algebra A. By restricting the symmetric Frobenius algebra structure induced by the canonical nondegenerate form on A to the centre of A they obtained a commutative Frobenius algebra and hence a 2-dimensional TQFT.

The generalization of the Fukuma, Hosono, and Kawai state sum to open-closed cobordisms is completely analogous to the construction above with only a few minor alterations required.

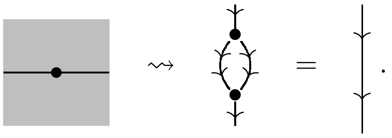
i) Rather than semisimple algebras the refined state sum uses strongly separable algebras. This generalization is needed in order to generalize the target category from $\mathbf{Vect}_{\mathbb{C}}$ to arbitrary symmetric monoidal categories.

ii) In order to generalize the notion of the image of an idempotent, the categories under consideration will have to be abelian so that kernels, cokernels, images, and coimages are all defined.

iii) The refined state sum is such that window element must be chosen to not be the identity, $a \neq \eta$, otherwise all information about the window number of an open-closed cobordism will be lost. This amounts to modifying the bubble move by an invertible central element. In this case, the algebra A will still be strongly separable and the canonical 'bilinear form' will still be nondegenerate.

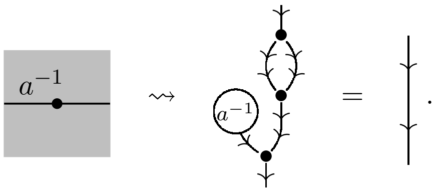


iv) The modified bubble move requires local corrections to the state sum in order to correct for the bubble move no longer holding on the nose. This is achieved by requiring the state sum to assign the inverse of the window element a^{-1} to each interior vertex of the triangulation. To illustrate this point we consider for simplicity a degenerate interior triangulation and the corresponding string diagram generated using the FKH state sum:



(3.1.5)

On the same degenerate triangulation the new state sum assigns a factor of the inverse window element to the interior vertex. Below we illustrate this and show the corresponding string diagram and the modified bubble move in action

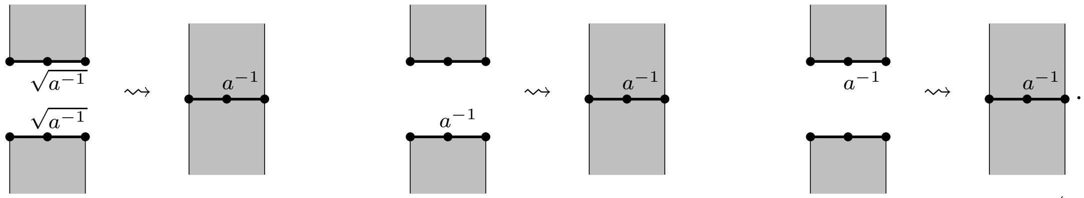


(3.1.6)

v) The triangulations of an open-closed cobordism are arranged so that the corners of the open-closed cobordism correspond to a vertex in the triangulation. These vertices are then given slightly different treatment than the other vertices in the triangulation of the black boundary.

vi) In order to have a well defined gluing of triangulated open-closed cobordisms the inverse of the window element must also be assigned to the interior vertices of the black boundary triangulation. This is because gluing two open-closed cobordisms along their black boundary turns vertices on the black boundary into internal vertices. For the prescription to be well-defined these new internal vertices must have a factor of the inverse window element associated to them.

There are several ways of doing this. Essentially, all that matters is that some fraction of the inverse window element is associated to the incoming and outgoing interior vertices of the black boundary so that when they are glued the product of the factors equals the inverse window element—the factor associated to an internal vertex. Some examples are drawn below:



(3.1.7)

The first of these possibilities requires that the window element have a square root which may not be the case for generic symmetric abelian monoidal categories. To avoid issues like these, we will choose to solve the gluing problem by the third method featured above, that is, by assigning the inverse window element to interior vertices on the outgoing black boundary and doing nothing to the incoming interior vertices. Note that since a is central, so is a^{-1} so that this choice of convention does not contribute anything to the state sum. In Section 3.2.4 (Theorem 3.2.18) we will show that these conventions lead to a natural choice of knowledgeable Frobenius algebra and hence to an open-closed topological field theory.

This chapter is structured as follows. In Section 3.2, we collect the key definitions and facts about symmetric Frobenius algebras, strongly separable symmetric Frobenius algebras, and knowledgeable Frobenius algebras. In Section 3.3, we present a combinatorial treatment of the category $\mathbf{2Cob}^{\mathrm{ext}}$ of open-closed cobordisms. The state sum construction of combinatorial open-closed TQFTs is then presented in Section 3.4.

The groupoid algebra $k[\mathcal{G}]$ of a finite groupoid $\mathcal{G}$ forms an example of a strongly separable algebra for suitable fields k. This chapter culminates by showing that the generalized state sum for this algebra yields an easy example of an open-closed TQFT with D-branes. For a more compact treatment of the material presented in this chapter see [2].

3.2 Strong Separability

The symmetric monoidal categories of interest will be abelian symmetric monoidal categories so that the Hom spaces are abelian groups and the notions of kernels, cokernels, images and coimages are defined. Such categories include the categories of vector spaces, graded vector spaces, R-modules for a commutative ring R, and chain complexes of each of these structures. For convenience we have collected some facts and definitions pertaining to abelian categories in Appendix B. We denote a symmetric monoidal category $\mathcal{C}$ as $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ where $\mathcal{C}$ is a category and $\otimes$ provides $\mathcal{C}$ with a monoidal structure with unit object $\mathbb{1}$ whose associator is denoted α and whose left and right unit constraints are given by λ and ρ. The symmetric braiding is denoted τ.

Recall from Section 2.2 that if $\mathcal{C}$ is locally small, the set $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ forms a commutative monoid. The monoid $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ acts on $\mathrm{Hom}(X, Y)$ for all $X, Y \in |\mathcal{C}|$ by $\xi \cdot f := \lambda_Y \circ (\xi \otimes f) \circ \lambda_X^{-1}$ where $f \in \mathrm{Hom}(X, Y)$ and $\xi \in \mathrm{Hom}(\mathbb{1}, \mathbb{1})$. Using string diagram notation for the morphisms of $\mathcal{C}$, the coherence theorem for monoidal categories allows us to view the elements of $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ as scalars by which the entire diagram is multiplied.

3.2.1 Symmetric Frobenius algebras

In this section, we introduce the notion of a non-degenerate symmetric invariant pairing in order to characterize symmetric Frobenius algebras. In the subsequent sections, we use it to define strongly separable algebras and to classify all symmetric Frobenius algebra structures of a strongly separable algebra.

Definition 3.2.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category and (A, μ, η) be an algebra object in $\mathcal{C}$.

1. A *pairing* on A is a morphism $g\colon A \otimes A \to \mathbb{1}$ of $\mathcal{C}$.

2. A pairing $g\colon A \otimes A \to \mathbb{1}$ is called *non-degenerate* if there exists a morphism $g^*\colon \mathbb{1} \to A \otimes A$ of $\mathcal{C}$ (called *the inverse* of g) such that the *zig-zag identities* hold,

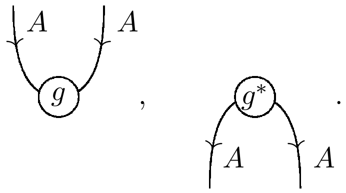
$$\begin{aligned} \rho_A \circ (\mathrm{id}_A \otimes g) \circ \alpha_{A,A,A} \circ (g^* \otimes \mathrm{id}_A) \circ \lambda_A^{-1} &= \mathrm{id}_A, \\ \lambda_A \circ (g \otimes \mathrm{id}_A) \circ \alpha_{A,A,A}^{-1} \circ (\mathrm{id}_A \otimes g^*) \circ \rho_A^{-1} &= \mathrm{id}_A. \end{aligned} \tag{3.2.1}$$

3. A pairing $g\colon A \otimes A \to \mathbb{1}$ is called *symmetric* if $g = g \circ \tau_{A,A}$.

4. A pairing $g\colon A \otimes A \to \mathbb{1}$ is called *invariant*[1] if,

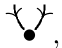
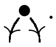
$$g \circ (\mathrm{id}_A \otimes \mu) \circ \alpha_{A,A,A} = g \circ (\mu \otimes \mathrm{id}_A). \tag{3.2.2}$$

The string diagrams for a pairing $g\colon A \otimes A \to \mathbb{1}$ on an algebra object (A, μ, η) in some symmetric monoidal category $\mathcal{C}$ are as follows:

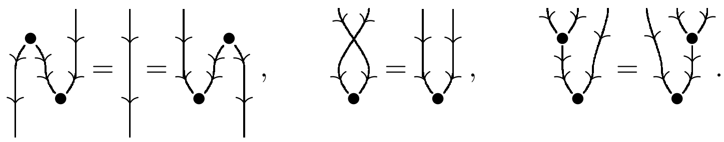


(3.2.3)

Our shorthand notation using blackboard framing then reads:

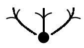
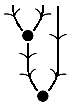
$g =$ , $g^* =$ . (3.2.4)

The conditions of non-degeneracy, symmetry and invariance are depicted as follows:

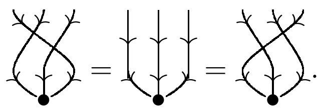


(3.2.5)

For an algebra object (A, μ, η) equipped with a symmetric invariant bilinear pairing g we also use the following shorthand notation for the 'trilinear form' $g^{(3)}\colon (A \otimes A) \otimes A \to \mathbb{1}$ which is defined by:

$g^{(3)} =$  $:=$ , (3.2.6)

and which has the following cyclic symmetry:



(3.2.7)

Lemma 3.2.2. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category. Every symmetric Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ gives rise to a non-degenerate symmetric invariant pairing $g := \varepsilon \circ \mu$ on A with inverse $g^* := \Delta \circ \eta$. Conversely, given an algebra object (A, μ, η) in $\mathcal{C}$ and a non-degenerate symmetric invariant pairing g on A, there is a symmetric Frobenius algebra object $(A, \mu, \eta, \Delta, \eta)$ with $\Delta := (\mu \otimes \mathrm{id}_A) \circ \alpha^{-1}_{A,A,A} \circ (\mathrm{id}_A \otimes g^*) \circ \rho_A^{-1}$ and $\varepsilon := g \circ (\mathrm{id}_A \otimes \eta) \circ \rho_A^{-1}$.

[1]Some authors use the term *associative* rather than *invariant*, see, for example [19].

The defining equations used in this lemma can be read diagrammatically as:


(3.2.8)

Notice that every symmetric Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is a rigid object of $\mathcal{C}$ with left-dual[2] $(A, \varepsilon \circ \mu, \Delta \circ \eta)$.

3.2.2 Strongly separable algebras

Every rigid algebra object in a symmetric monoidal category is equipped with a canonical pairing. Recall first the special case of $\mathcal{C} = \mathbf{Vect}_k$ for an arbitrary field k. Let A be a finite-dimensional algebra over k and denote the left-regular representation by $L\colon A \to \mathrm{End}_k(A), a \mapsto L_a$ with $L_a\colon A \to A, b \mapsto ab$. By associativity, $L_{ab} = L_a \circ L_b$ for all $a, b \in A$. The trace of the matrices of the left-regular representation equips A with a canonical bilinear form,

$$g_{\mathrm{can}}\colon A \otimes A \to k, \quad a \otimes b \mapsto \mathrm{tr}_A(L_{ab}), \tag{3.2.9}$$

which can be shown to be symmetric and invariant. We are interested in those algebras for which this canonical bilinear form is non-degenerate. These are the strongly separable algebras. Let us first recall the definition.

Definition 3.2.3. Let A be an algebra over a commutative ring r. We denote by A^{op} the opposite algebra of A, by $A^e = A \otimes A^{\mathrm{op}}$ its enveloping algebra and by $\mu\colon A^e \to A$, $a \otimes b \mapsto ab$ the augmentation mapping. A is called *separable* if there is an element $e \in A^e$ (called a *separability idempotent*) such that,

1. $(a \otimes 1)e = (1 \otimes a)e$ holds in A^e for all $a \in A$.

2. $\mu(e) = 1$.

A is called *strongly separable* if A is separable with a separability idempotent that satisfies $\tau_{A,A}(e) = e$.

Theorem 3.2.4 (see, for example [68])**.** Let A be an algebra over any field k. Then the following are equivalent:

1. A is finite-dimensional over k, and the canonical bilinear form is non-degenerate.

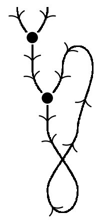
[2] It is right-dual at the same time, but we do not refer to this property in the following.

2. A is strongly separable.

Every strongly separable algebra therefore carries a canonical symmetric Frobenius algebra structure by Lemma 3.2.2. The following definition of a canonical pairing for generic $\mathcal{C}$ reduces to the canonical bilinear form in the case of $\mathcal{C} = \mathbf{Vect}_k$.

Proposition 3.2.5. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category and (A, μ, η) be an algebra object in $\mathcal{C}$ such that the object A is rigid with left-dual $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$. Then there is a symmetric invariant pairing on A given by,

$$g_{\mathrm{can}} := \mathrm{ev}_A \circ \tau_{A,A^*} \circ (\mu \otimes \mathrm{id}_{A^*}) \circ \alpha^{-1}_{A,A,A^*} \circ (\mu \otimes \mathrm{coev}_A) \circ \rho^{-1}_{A\otimes A} =$$



$$. \tag{3.2.10}$$

which we call the *canonical pairing*.

Definition 3.2.6. A rigid algebra object in a symmetric monoidal category is called *strongly separable* if its canonical pairing is non-degenerate.

By Theorem 3.2.4, this notion of a strongly separable algebra object in some symmetric monoidal category agrees with the usual definition in the case $\mathcal{C} = \mathbf{Vect}_k$. We are not aware of any such result for the more general case of modules over a commutative ring. In order to illustrate how strong the condition of strong separability is, we include the following results and examples from [68, 69].

Theorem 3.2.7. Let A be an algebra over some field k.

1. If A is strongly separable, then A is finite-dimensional, separable, and semisimple.

2. If A is separable and commutative, then A is strongly separable.

3. If A is finite-dimensional and semisimple and $\operatorname{char} k = 0$, then A is strongly separable.

4. If A is finite-dimensional and semisimple and k is a perfect field, then A is separable.

Example 3.2.8. Let k be a field and G be a finite group.

1. If $\operatorname{char} k$ does not divide the order of G, then the group algebra $k[G]$ is strongly separable.

2. If $\operatorname{char} k$ divides the order of G, then $k[G]$ is neither semisimple nor separable.

Example 3.2.9. Let k be a field and $M_n(k)$ be the algebra of $(n \times n)$-matrices over k.

1. If char k does not divide n, then $M_n(k)$ is strongly separable.

2. If char k divides n, then $M_n(k)$ is semisimple and separable, but not strongly separable.

In both examples, the non-degeneracy of the canonical bilinear form is a convenient criterion for strong separability. We explain below why in the Examples 3.2.8(2) and 3.2.9(2), the state sum construction fails. In particular, for a finite field of non-zero characteristic p, the original Fukuma–Hosono–Kawai state sum [40] cannot be applied to the $(p \times p)$-matrix algebra $A := M_p(k)$ although k is perfect and $M_p(k)$ is finite-dimensional, separable, and semisimple.

3.2.3 Strongly separable symmetric Frobenius algebras

In this section, we compare the pairing $\varepsilon \circ \mu$ of a generic symmetric Frobenius algebra with the canonical pairing. They differ by multiplication with a central element which we call the *window element*[3].

In a generic locally small symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$, we use the terminology *element of* A for a morphism $a\colon \mathbb{1} \to A$. The set $\mathrm{Hom}(\mathbb{1}, A)$ of elements of an algebra object (A, μ, η) in $\mathcal{C}$ forms a monoid with respect to convolution $a \cdot b := \mu \circ (a \otimes b) \circ \lambda_{\mathbb{1}}^{-1}$ for elements $a, b \in \mathrm{Hom}(\mathbb{1}, A)$ and with unit η. An element $a \in \mathrm{Hom}(\mathbb{1}, A)$ is called *central* if it is contained in the commutative submonoid,

$$\mathcal{Z}(A) := \{\, a \in \mathrm{Hom}(\mathbb{1}, A)\colon \quad \mu \circ (a \otimes \mathrm{id}_A) \circ \lambda_A^{-1} = \mu \circ (\mathrm{id}_A \otimes a) \circ \rho_A^{-1} \,\}. \tag{3.2.11}$$

The set of invertible elements of A forms a group $\mathrm{Hom}(\mathbb{1}, A)^\times \subseteq \mathrm{Hom}(\mathbb{1}, A)$, and the set of invertible central elements $\mathcal{Z}(A)^\times := \mathcal{Z}(A) \cap \mathrm{Hom}(\mathbb{1}, A)^\times \leq \mathrm{Hom}(\mathbb{1}, A)^\times$ a subgroup. This means in particular that the inverse of every central element is central, too. $\mathcal{Z}(A)$ acts on $\mathrm{Hom}(A, A)$ by

$$\mathcal{Z}(A) \times \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, A), \quad (a, f) \mapsto a \cdot f := \mu \circ (a \otimes f) \circ \lambda_A^{-1}. \tag{3.2.12}$$

We also have $(a \cdot \mathrm{id}_A) \circ \eta = a$ and $(a \cdot \mathrm{id}_A) \circ (b \cdot \mathrm{id}_A) = (a \cdot b) \cdot \mathrm{id}_A$ for all $a, b \in \mathcal{Z}(A)$ as well as

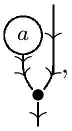
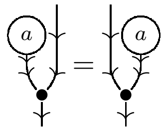
$$\mu \circ ((a \cdot \mathrm{id}_A) \otimes \mathrm{id}_A) = (a \cdot \mathrm{id}_A) \circ \mu = \mu \circ (\mathrm{id}_A \otimes (a \cdot \mathrm{id}_A)), \tag{3.2.13}$$

[3] This terminology is inspired by the open-closed cobordism that is associated with this element.

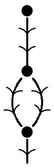
and for a Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ also,

$$((a \cdot \mathrm{id}_A) \otimes \mathrm{id}_A) \circ \Delta = \Delta \circ (a \cdot \mathrm{id}_A) = (\mathrm{id}_A \otimes (a \cdot \mathrm{id}_A)) \circ \Delta. \tag{3.2.14}$$

The following diagrams show an element $a \in \mathrm{Hom}(\mathbb{1}, A)$, the morphism $a \cdot \mathrm{id}_A$, and the centrality condition:

$a =$ , $\quad a \cdot \mathrm{id}_A =$ ,  . (3.2.15)

Definition 3.2.10. Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a symmetric Frobenius algebra object in a locally small symmetric monoidal category $\mathcal{C}$. The *window element* of A is defined by,

$a := \mu \circ \Delta \circ \eta =$  . (3.2.16)

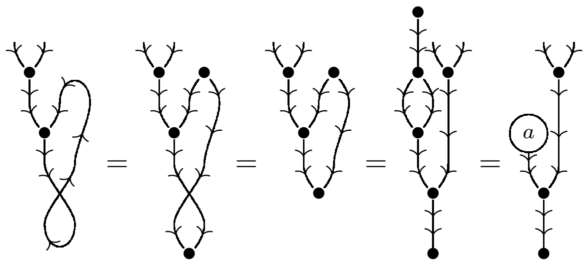
The window element is a central element. The comparison between the pairing $\varepsilon \circ \mu$ of a generic symmetric Frobenius algebra and the canonical pairing can be done as follows.

Proposition 3.2.11. Let $\mathcal{C}$ be a locally small symmetric monoidal category and $(A, \mu, \eta, \Delta, \varepsilon)$ be a symmetric Frobenius algebra object in $\mathcal{C}$. Then the canonical pairing of A is given by,

$$g_{\mathrm{can}} = \varepsilon \circ (a \cdot \mathrm{id}_A) \circ \mu, \tag{3.2.17}$$

where a denotes the window element.

Proof. Notice that A is a rigid object of $\mathcal{C}$, and so it makes sense to study the canonical pairing (3.2.10). We use the diagrams of (3.2.4) for $g := \varepsilon \circ \mu$ and $g^* := \Delta \circ \eta$:

$g_{\mathrm{can}} =$  . (3.2.18)

The first equality is the definition; for the second one, we have exploited the fact that A satisfies the zig-zag identities (2.2.1) both with $(A^*, \mathrm{ev}_A, \mathrm{coev}_A)$ and with (A, g, g^*); the third equality is symmetry; the fourth one follows from the axioms of a Frobenius algebra; and the fifth is the definition of the window element. $\square$

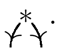
In the above proposition, the pairing $g := \varepsilon \circ \mu$ is always non-degenerate whereas the canonical pairing is non-degenerate if and only if the algebra is strongly separable.

Theorem 3.2.12. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a locally small symmetric monoidal category and $(A, \mu, \eta, \Delta, \varepsilon)$ be a symmetric Frobenius algebra object in $\mathcal{C}$ with window element a. Then the following are equivalent:

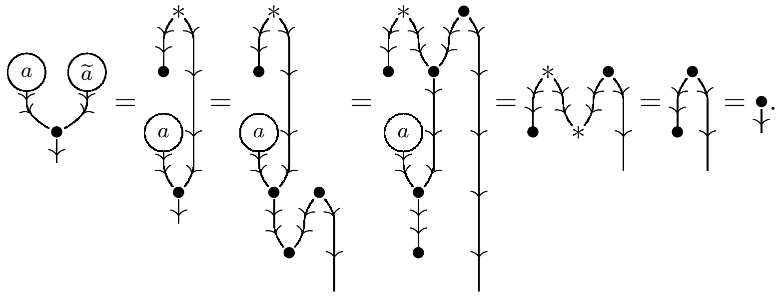
1. The algebra object (A, μ, η) is strongly separable.

2. The window element is invertible.

Proof. Let (A, μ, η) be strongly separable. Then the canonical pairing is non-degenerate and therefore has an inverse $g^*_{\text{can}} \colon \mathbb{1} \to A \otimes A$. We denote the pairing $g := \varepsilon \circ \mu$ and its inverse $g^* := \Delta \circ \eta$ by the diagrams of (3.2.4) and the canonical pairing and its inverse by,

$g_{\text{can}} =$ , $\quad g^*_{\text{can}} =$ . (3.2.19)

Define $\widetilde{a} := \lambda_A \circ (\varepsilon \otimes \mathrm{id}_A) \circ g^*_{\text{can}}$. Then $\widetilde{a}$ is the inverse of a because,



(3.2.20)

The first equality is the definition of $\widetilde{a}$; the second one a zig-zag identity for (A, g, g^*); the third one is a consequence of the axioms of a Frobenius algebra; the fourth one is (3.2.17); the fifth one is a zig-zag identity for $(A, g_{\text{can}}, g^*_{\text{can}})$; and the last equation holds by the axioms of a Frobenius algebra.

Conversely, let the window element a have an inverse a^{-1}. Then a similar computation shows that the canonical pairing satisfies the zig-zag identities with the inverse $g^*_{\text{can}} = \Delta \circ a^{-1}$ and is therefore non-degenerate. □

Combined with Proposition 3.2.11, the preceding theorem implies that the symmetric Frobenius algebra structures of a given strongly separable algebra are characterized by the invertible central elements. The proposition below describes the extent to which the notion of a strongly separable symmetric Frobenius algebra generalizes the notion of a special symmetric Frobenius algebra.

Proposition 3.2.13. Let $\mathcal{C}$ be a locally small symmetric monoidal category and $(A, \mu, \eta, \Delta, \varepsilon)$ be a symmetric Frobenius algebra object in $\mathcal{C}$ such that $\dim A$ is invertible in $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$. Then the following are equivalent:

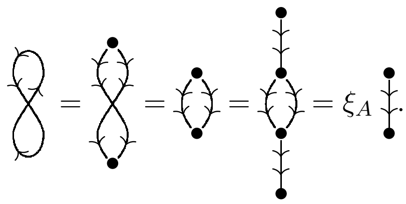
1. The Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ is special with $\varepsilon \circ \eta = \xi_{\mathbb{1}} \cdot \mathrm{id}_{\mathbb{1}}$ and $\mu \circ \Delta = \xi_A \cdot \mathrm{id}_A$ for some invertible $\xi_{\mathbb{1}}, \xi_A \in \mathrm{Hom}(\mathbb{1}, \mathbb{1})$.

2. The algebra object (A, μ, η) is strongly separable, and the window element is of the form $a = \zeta \cdot \eta$ for some invertible $\zeta \in \mathrm{Hom}(\mathbb{1}, \mathbb{1})$.

In this case, $\xi_A = \zeta$ and $\xi_{\mathbb{1}} = \xi_A^{-1} \dim A$.

Proof. If A is special, the window element is $a = \mu \circ \Delta \circ \eta = (\xi_A \cdot \mathrm{id}_A) \circ \eta = \xi_A \cdot \eta$. It is invertible with $a^{-1} = \xi_A^{-1} \cdot \eta$, and so (A, μ, η) is strongly separable.

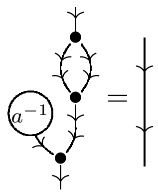
Conversely, if (A, μ, η) is strongly separable with window element $a = \zeta \cdot \eta$ for some invertible $\zeta \in \mathrm{Hom}(\mathbb{1}, \mathbb{1})$, then the second condition of (2.3.10) holds with invertible $\xi_A = \zeta$. For a symmetric Frobenius algebra object in a symmetric monoidal category, the second condition of (2.3.10) implies the first one with $\xi_{\mathbb{1}} = \xi_A^{-1} \dim A$:

$\dim A =$ 

(3.2.21)

Since $\dim A$ is invertible by assumption, so is $\xi_{\mathbb{1}}$. ☐

Remark 3.2.14. Given any strongly separable symmetric Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ with window element a in a locally small symmetric monoidal category $\mathcal{C}$, the identity

$(a^{-1} \cdot \mathrm{id}_A) \circ \mu \circ \Delta =$ 

$= \mathrm{id}_A$ (3.2.22)

generalizes the 'bubble move' of Fukuma–Hosono–Kawai from the canonical symmetric Frobenius algebra structure to the case of a generic symmetric Frobenius algebra structure. In Section 3.4, we explain why this generalization is needed in order to obtain a sharp invariant of open-closed cobordisms from the state sum.

For the algebras of Example 3.2.8(2) and Example 3.2.9(2) which are not strongly separable, the morphism $\mu \circ \Delta$ is zero, and so there is no way of obtaining an analogue of the 'bubble move'.

Example 3.2.15. Let k be a field, $\operatorname{char} k \neq 2$ and $n \in \mathbb{N}$ such that $\operatorname{char} k$ does not divide n. Assume that there exists some $\alpha \in k$ such that $\alpha^2 = -1/2$ (for example $k = \mathbb{C}$).

Let $A = M_n(k)$ be the $n \times n$-matrix algebra over k. Choose a k-basis $(e_{ij})_{1 \le i,j \le n}$ of A such that the multiplication is given by $\mu_A(e_{ij} \otimes e_{k\ell}) = \delta_{jk}\, e_{i\ell}$ and the unit by $\eta_A(1) = \sum_{i=1}^n e_{ii}$. The algebra A forms a symmetric Frobenius algebra with $\Delta_A(e_{ij}) = \alpha^{-1} \sum_{k=1}^n e_{ik} \otimes e_{kj}$ and $\varepsilon_A(e_{ij}) = \alpha\, \delta_{ij}$. We compute $\mu_A \circ \Delta_A = n\alpha^{-1} \cdot \mathrm{id}_A$ and the window element $a_A = n\alpha^{-1} \cdot \eta_A$. It is invertible with $a_A^{-1} = n^{-1}\alpha \cdot \eta_A$, and so A is strongly separable. In fact, A is special with $\xi_A = n\alpha^{-1}$ and $\xi_{\mathbb{1}} = n\alpha$. Obviously, $Z(A) \cong k$.

Let $C = k[x]/(x^2 - 1)$ (see Definition 1.0.2). A k-basis is given by $(1, x)$. C becomes a commutative Frobenius algebra with $\Delta_C(1) = 1 \otimes x + x \otimes 1$, $\Delta_C(x) = 1 \otimes 1 + x \otimes x$, $\varepsilon_C(1) = 0$, and $\varepsilon_C(x) = 1$. We compute $(\mu_C \circ \Delta_C)(c) = 2x\,c$ for all $c \in C$, and the window element is $a_C = 2x$. It is invertible with $a_C^{-1} = x/2$, and so C is strongly separable, too, but it is not special.

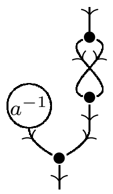
If we define $\imath\colon C \to A$ by $\imath(1) = \eta_A(1)$ and $\imath(x) = -\eta_A(1)$, and $\imath^*\colon A \to C$ by $\imath^*(e_{ij}) = \delta_{ij}\, \alpha(x - 1)$, then $(A, C, \imath, \imath^*)$ forms a knowledgeable Frobenius algebra. Observe that $Z(A)$ is 1-dimensional over k, but C is 2-dimensional, and so $Z(A) \not\cong C$.

3.2.4 Idempotents

In this section, we show that every strongly separable symmetric Frobenius algebra A in an abelian symmetric monoidal category $\mathcal{C}$ gives rise to a knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ in $\mathcal{C}$ and hence an open-closed TQFT by Theorem 2.7.2. In $\mathbf{Vect}_k$, C is isomorphic to the centre of A. In general, it arises as the image of the following canonical idempotent.

Proposition 3.2.16. Let $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ be a strongly separable symmetric Frobenius algebra object in a locally small symmetric monoidal category $\mathcal{C}$ and let a^{-1} denote the inverse of the window element of A. Then the morphism

$$p = p := (a^{-1} \cdot \mathrm{id}_A) \circ \mu_A \circ \tau_{A,A} \circ \Delta_A = \tag{3.2.23}$$



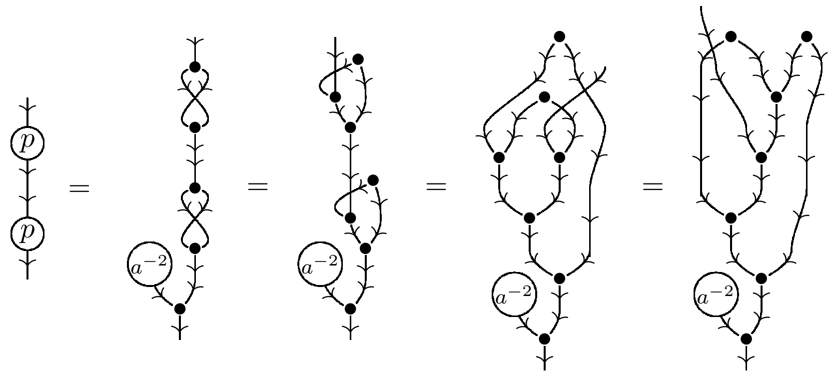
has the following properties,

1. $p^2 = p$,

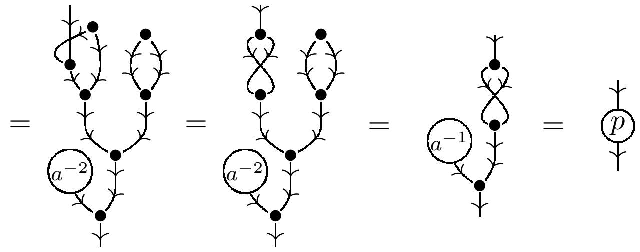
2. $p \circ \eta_A = \eta_A$,

3. $\varepsilon_A \circ p = \varepsilon_A$,
4. $p \circ \mu_A \circ (p \otimes p) = \mu_A \circ (p \otimes p) = p \circ \mu_A \circ (p \otimes \mathrm{id}_A) = p \circ \mu_A \circ (\mathrm{id}_A \otimes p)$,
5. $(p \otimes p) \circ \Delta_A \circ p = (p \otimes p) \circ \Delta_A = (p \otimes \mathrm{id}_A) \circ \Delta_A \circ p = (\mathrm{id}_A \otimes p) \circ \Delta_A \circ p$,
6. $c = p \circ c$ for all $c \in \mathcal{Z}(A)$,
7. $(c \cdot \mathrm{id}_A) \circ p = p \circ (c \cdot \mathrm{id}_A)$ for all $c \in \mathcal{Z}(A)$,
8. $\mu \circ (p \otimes \mathrm{id}_A) = \mu \circ \tau_{A,A} \circ (p \otimes \mathrm{id}_A)$.

Proof. These results are straightforward to prove and many of the proofs can be found in the literature (see for example [67]). In our case, additional bookkeeping is required to account for the powers of the window element. Since the element a^{-1} is in the centre we can move it anywhere in the diagram. For convenience, we will often move the powers of a^{-1} to the bottom of the diagram. For example, the first assertion is proved as follows:



(3.2.24)



(3.2.25)

where the first equality is just the definition, the second equality follows from (3.2.8), the third follows from naturality and associativity, and the fourth follows from the Frobenius identities. The first equality in (3.2.25) follows from associativity, while the second equality follows again from (3.2.8). The second to last equality follows from the definition of a in (3.2.16), and the final equality is the definition of the idempotent p. The other identities can be proved similarly. □

In $\mathbf{Vect}_k$, condition (1) states that p is a projector; condition (8) says that its image is contained in the centre $Z(A)$, and condition (6) says that the centre $Z(A)$ is contained in the image of p, and so p projects onto the centre $Z(A)$. Whereas this $Z(A)$ arises as a subspace $Z(A) = \mathrm{im}\ p \subseteq A$, the *centre* $\mathcal{Z}(A)$ according to (3.2.11) consists of morphisms $\mathbb{1} \to A$. In $\mathbf{Vect}_k$, one can evaluate any such morphism $a \in \mathcal{Z}(A)$ at the unit $1 \in k$ of the field and finds that $a(1) \in Z(A) \subseteq A$.

Note that the idempotent (3.2.23) is precisely $p = \mu_A \circ \tau_{A,A} \circ \Delta_A^{(\mathrm{can})}$ where $\Delta_A^{(\mathrm{can})}$ refers to the canonical symmetric Frobenius algebra structure on A.

In the state sum, the idempotent (3.2.23) appears whenever a unit interval is closed to a circle, *i.e.* it is closely related to the generators $\imath$ and $\imath^*$ of (2.5.25). The image of an idempotent can be defined in any abelian category as follows.

Proposition 3.2.17 (see, for example [70])**.** Let $\mathcal{C}$ be an abelian category and $p\colon A \to A$ be an idempotent. The image factorization of p yields an object $p(A)$, called the *image* of p, which is unique up to isomorphism, together with morphisms $\mathrm{coim}\ p\colon A \to p(A)$ (called the *coimage*) and $\mathrm{im}\ p\colon p(A) \to A$ (called the *image*) such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\mathrm{coim}\ p"] \arrow[dr, "p"'] & p(A) \arrow[d, "\mathrm{im}\ p"] \\ & A \end{tikzcd} \tag{3.2.26}$$

Since $\mathcal{C}$ is abelian, the idempotent p is split. The splitting is given precisely by the two morphisms of the image factorization, and so we have $\mathrm{id}_{p(A)} = \mathrm{coim}\ p \circ \mathrm{im}\ p$. Therefore, the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N_p \arrow[r, "\ker p"] & A \arrow[r, shift left, "\mathrm{coim}\ p"] & p(A) \arrow[l, shift left, "\mathrm{im}\ p"] \arrow[r] & 0 \end{tikzcd}, \tag{3.2.27}$$

is split as indicated. Here N_p denotes the kernel of p. This determines the structure of $A \cong N_p \oplus p(A)$ in terms of the following biproduct:

$$\begin{tikzcd} N_p \arrow[r, shift left, "\ker p"] & N_p \oplus p(A) \arrow[l, shift left, "\mathrm{coker}\ p"] \arrow[r, shift left, "\mathrm{coim}\ p"] & p(A) \arrow[l, shift left, "\mathrm{im}\ p"] \end{tikzcd}. \tag{3.2.28}$$

The sequence from right to left is split exact, too.

Theorem 3.2.18. Let $\mathcal{C}$ be an abelian symmetric monoidal category and $(A, \mu, \eta, \Delta, \varepsilon)$ be a strongly separable symmetric Frobenius algebra object in $\mathcal{C}$ with window element a. Then there exists a knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ where $C = p(A)$ is the image of the idempotent (3.2.23), $\imath = \mathrm{im}\ p$, and $\imath^* = \mathrm{coim}\ p \circ (a \cdot \mathrm{id}_A)$. The commutative Frobenius algebra structure of C is given by,

$$\mu_C = \text{coim } p \circ \mu_A \circ (\text{im } p \otimes \text{im } p), \tag{3.2.29}$$

$$\eta_C = \text{coim } p \circ \eta_A, \tag{3.2.30}$$

$$\Delta_C = (\text{coim } p \otimes \text{coim } p) \circ \Delta_A \circ (a \cdot \mathrm{id}_A) \circ \text{im } p, \tag{3.2.31}$$

$$\varepsilon_C = \varepsilon_A \circ (a^{-1} \cdot \mathrm{id}_A) \circ \text{im } p. \tag{3.2.32}$$

Proof. The proof uses Proposition 3.2.16 and Proposition 3.2.17. □

We show below in Section 3.4 that this knowledgeable Frobenius algebra is precisely the one that is obtained from our generalized state sum for the strongly separable algebra A. The following proposition introduces two families of morphisms that are needed in order to show that the morphisms associated with triangulated open-closed cobordisms do not depend on the triangulation of the boundary.

Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a Frobenius algebra object in a locally small symmetric monoidal category $\mathcal{C}$. For $k \in \mathbb{N}$, we denote by

$$\mu^{(k+1)} := \mu \circ (\mu^{(k)} \otimes \mathrm{id}_A), \qquad \mu^{(2)} := \mu, \qquad \mu^{(1)} := \mathrm{id}_A \tag{3.2.33}$$

and by

$$\Delta^{(k+1)} := (\Delta^{(k)} \otimes \mathrm{id}_A) \circ \Delta, \qquad \Delta^{(2)} := \Delta, \qquad \Delta^{(1)} := \mathrm{id}_A \tag{3.2.34}$$

the iterated multiplication and comultiplication. We also write $A^{\otimes(k+1)} := A^{\otimes k} \otimes A$, $A^{\otimes 1} := A$ and $A^{\otimes 0} := \mathbb{1}$, and for $a \in \mathcal{Z}(A)$, $a^{k+1} \cdot \mathrm{id}_A := (a^k \cdot \mathrm{id}_A) \circ (a \cdot \mathrm{id}_A)$ and $a^0 \cdot \mathrm{id}_A := \mathrm{id}_A$.

Proposition 3.2.19. Let $\mathcal{C}$ be a locally small symmetric monoidal category and $(A, \mu, \eta, \Delta, \varepsilon)$ be a strongly separable symmetric Frobenius algebra object in $\mathcal{C}$ with window element a. Then for $k, \ell \in \mathbb{N}$, the morphisms

$$P_{k\ell} := \Delta^{(k)} \circ (a^{-(k-1)} \cdot \mathrm{id}_A) \circ \mu^{(\ell)} \quad : A^{\otimes \ell} \to A^{\otimes k}, \tag{3.2.35}$$

$$Q_{k\ell} := \Delta^{(k)} \circ (a^{-(k-1)} \cdot \mathrm{id}_A) \circ p \circ \mu^{(\ell)} : A^{\otimes \ell} \to A^{\otimes k}, \tag{3.2.36}$$

satisfy

$$P_{k\ell} \circ P_{\ell m} = P_{km} \qquad \text{and} \qquad Q_{k\ell} \circ Q_{\ell m} = Q_{km} \tag{3.2.37}$$

for all $k, \ell, m \in \mathbb{N}$. Here p denotes the idempotent of (3.2.23). In particular, P_{kk} and Q_{kk} are idempotents, and we have $P_{11} = \mathrm{id}_A$ and $Q_{11} = p$.

Proof. In any symmetric Frobenius algebra, we have

$$\mu^{(k)} \circ \Delta^{(k)} = a^{(k-1)} \cdot \mathrm{id}_A, \tag{3.2.38}$$

which implies both claims. □

Corollary 3.2.20. Let $\mathcal{C}$ be an abelian symmetric monoidal category and $(A, \mu, \eta, \Delta, \varepsilon)$ be a strongly separable symmetric Frobenius algebra object in $\mathcal{C}$. Then there are isomorphisms

$$P_{kk}(A^{\otimes k}) \cong A \qquad \text{and} \qquad Q_{kk}(A^{\otimes k}) \cong p(A) \tag{3.2.39}$$

for all $k \in \mathbb{N}$.

Proof. The isomorphisms with their inverses are given by

$$\Phi_k = \mathrm{coim}\ P_{kk} \circ P_{k1} \colon A \to P_{kk}(A^{\otimes k}), \tag{3.2.40}$$

$$\Phi_k^{-1} = P_{1k} \circ \mathrm{im}\ P_{kk} \colon P_{kk}(A^{\otimes k}) \to A \tag{3.2.41}$$

as well as

$$\Psi_k = \mathrm{coim}\ Q_{kk} \circ Q_{k1} \circ \mathrm{im}\ p \colon p(A) \to Q_{kk}(A^{\otimes k}), \tag{3.2.42}$$

$$\Psi_k^{-1} = \mathrm{coim}\ p \circ Q_{1k} \circ \mathrm{im}\ Q_{kk} \colon Q_{kk}(A^{\otimes k}) \to p(A). \tag{3.2.43}$$

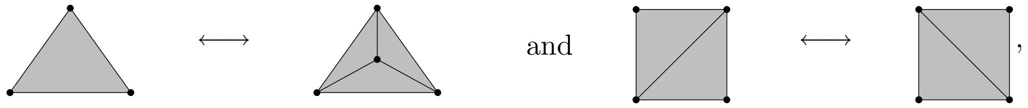
□

3.3 Combinatorial open-closed cobordisms

Open-closed cobordisms can be triangulated as follows. We use the terminology of [71].

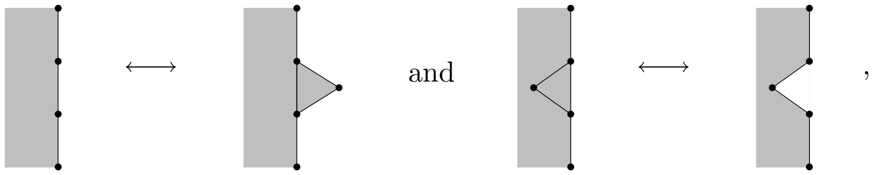
Given an open-closed cobordism M, the underlying topological manifold is a compact oriented 2-manifold with boundary. We therefore have a finite simplicial complex K whose underlying polyhedron we denote by $|K| \subseteq \mathbb{R}^p$ for some p, and a homeomorphism $T_M \colon |K| \to M$ which we call a *triangulation*. The simplicial complex K satisfies the conditions that guarantee that $|K|$ forms an oriented topological 2-manifold, *i.e.* the link of each d-simplex is a $(1-d)$-sphere iff the simplex is in the interior of $|K|$, and it is a $(1-d)$-ball iff the simplex is in the boundary of $|K|$. Furthermore, for each 2-simplex σ, it is specified whether σ or its oppositely oriented simplex σ^* is contained in $|K|$, and each 1-simplex in the interior of $|K|$ appears as a face of precisely two 2-simplices with opposite induced orientations.

If M and N are equivalent open-closed cobordisms, their underlying topological manifolds are homeomorphic. If we have triangulations $T_M \colon |K| \to M$ and $\widetilde{T}_M \colon |L| \to N$ with simplicial complexes K and L, Pachner's theorem [47] says that the simplicial complexes K and L are related by a finite sequence of moves. These moves are the *bistellar moves* (called the 1-3 and 2-2 moves),



(3.3.1)

applicable to all 2-simplices, and the *elementary shellings*



(3.3.2)

applicable to certain 2-simplices some of whose faces coincide with the boundary. The interior of the manifold is indicated by the shading in our pictures. Recall that for finite simplicial complexes which represent compact manifolds with non-empty boundary, each bistellar move can be obtained from a finite sequence of elementary shellings.

The set of corners $\partial_0 M \cap \partial_1 M$ of every open-closed cobordism M is a finite set. Given some triangulation $T_M\colon |K| \to M$, we can apply a finite sequence of elementary shellings in order to subdivide the 1-simplices in the boundary in such a way that to every corner of M, there corresponds a 0-simplex in K, *i.e.* that $\partial_0 M \cap \partial_1 M \subseteq T_M(|K_0|)$ where $K_0 \subseteq K$ denotes the 0-skeleton of K. From now on we assume, without loss of generality, that every triangulation has this property. Given a 1-simplex $\sigma \in K$ in the boundary, we therefore have either $T_M(|\sigma|) \subseteq \partial_0 M$ or $T_M(|\sigma|) \subseteq \partial_1 M$, *i.e.* the 1-simplices in the boundary are either *black* or *coloured*.

Both elementary shellings of (3.3.2) replace two boundary 1-simplices (edges) by a single edge or vice versa. For triangulations with the special property, each of the elementary shellings (3.3.2) belongs to one of the following four types:

1. two black edges $\longleftrightarrow$ one black edge,

2. two coloured edges $\longleftrightarrow$ one coloured edge,

3. one black and one coloured edge $\longleftrightarrow$ one black edge,

4. one black and one coloured edge $\longleftrightarrow$ one coloured edge.

It is not difficult to see that the elementary shellings of types (3.) and (4.) can be obtained from a finite sequence of bistellar moves and elementary shellings of types (1.) and (2.).

When we construct open-closed TQFTs in Section 3.4 below, we consider triangulations of the open-closed cobordisms and then show that the linear map associated with every given cobordism is invariant under the bistellar moves (3.3.1) and under elementary shellings of types (1.) and (2.). Then this linear map is independent of the choice of the triangulation.

3.3.1 Smoothing theory

When one studies smooth manifolds by combinatorial techniques, the relation between combinatorial and smooth manifolds is described by two types of theorems:

- *Triangulation:* Every compact smooth manifold (with boundary) admits a Whitehead triangulation. If two such manifolds are diffeomorphic, then their triangulations are related by a finite sequence of the appropriate Pachner moves.
- *Smoothing:* Given a finite simplicial complex K that satisfies the conditions which ensure that its underlying polyhedron $|K|$ forms a topological manifold (with boundary), one needs to know (a) under which conditions there exists a smooth manifold that has $|K|$ as its triangulation and (b) whether the resulting smooth manifold is unique up to diffeomorphism.

Such theorems are available in order to compare smooth manifolds with boundary and combinatorial manifolds with boundary, but we are not aware of any systematic treatment for manifolds with corners, manifolds with faces, or $\langle 2\rangle$-manifolds.

In the preceding section, we have solved the triangulation problem for open-closed cobordisms by resorting to the underlying topological manifold which is just a topological 2-manifold with boundary. It admits a triangulation, and this triangulation is unique up to combinatorial equivalence, *i.e.* Pachner moves, by the validity of the Combinatorial Triangulation Conjecture and the *Hauptvermutung* for 2-dimensional manifolds, see, for example [72]. We have then dealt with the corner points 'by hand'.

The other direction, a solution to the smoothing problem, is not needed if one is just interested in a combinatorial construction of open-closed TQFTs. For completeness, we nevertheless sketch how one can obtain the corresponding smoothing theorem: let K be a finite simplicial complex that triangulates an open-closed cobordism. Then every 1-simplex in the boundary is either black or coloured as we have explained above. The underlying polyhedron $|K|$ together with this partitioning of the boundary is already sufficient to read off the topological invariants defined in Section 2.5.3. By the normal form of open-closed cobordisms of Definition 2.6.3, there exists an open-closed cobordism with the given invariants, and by Corollary 2.6.6, it is unique up to equivalence.

3.4 State Sum Construction

We begin this section with an overview of the state sum construction in informal language.

Given a strongly separable symmetric Frobenius algebra object $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ in an abelian symmetric monoidal category $\mathcal{C}$ and a connected open-closed cobordism M with triangulation $T_M\colon |K| \to M$, we construct a morphism $Z(M)$ in $\mathcal{C}$.

For the duration of this section let M be a connected open-closed cobordism with source $\partial_0 M^{\mathrm{in}} := \vec{n} = (n_1, \cdots, n_k)$ and target $\partial_0 M^{\mathrm{out}} := \vec{n}' = (n'_1, \cdots, n'_{k'})$. Let j enumerate the black boundary components of M so that h_j denotes the number of 1-simplices in the triangulation of the component n_j for $1 \le j \le k$ or the component n'_j for $k+1 \le j \le k+k'$. The number of 1-simplices of $\partial_0 M^{\mathrm{in}}$ is given by the sum $m_1 := \sum_{j=1}^{k} h_j$, and the number of 1-simplices of $\partial_0 M^{\mathrm{out}}$ by the sum $m_2 := \sum_{j=k+1}^{k+k'} h_j$.

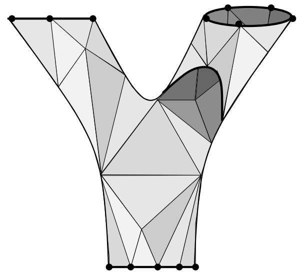


Figure 3.1: $\vec{n} = (1,0)$, $\vec{n}' = (1)$, $h_1 = 2$, $h_2 = 5$, $h_3 = 4$, $m_1 = 7$, and $m_2 = 4$.

As a first step to constructing the morphism $Z(M)$, we construct a morphism $Z_{T_M}(M)\colon A^{\otimes m_1} \to A^{\otimes m_2}$. These morphisms depend on the triangulation of the black boundary, but they are already invariant under bistellar moves and under elementary shellings of type (2.), *i.e.* those in which all the involved boundary edges are coloured.

Define the symbol $A^{(n_j)}$ corresponding to the boundary component n_j to be A if $n_j = 1$ and $p(A)$ if $n_j = 0$ and define $A^{\otimes \vec{n}}$ to be the ordered tensor product $\bigotimes_{j=1}^{k} A^{(n_j)}$. Likewise, we set $A^{\otimes \vec{n}'}$ equal to the ordered tensor product $\bigotimes_{j=k+1}^{k+k'} A^{(n'_j)}$.

In Section 3.4.3, we show that the isomorphisms $P_{kk}(A^{\otimes k}) \cong A$ and $Q_{kk}(A^{\otimes k}) \cong p(A)$ of Corollary 3.2.20 correspond to triangulated cylinders over I or S^1. We construct a map $Z(M)\colon A^{\otimes \vec{n}} \to A^{\otimes \vec{n}'}$ using these isomorphisms and the morphism $Z_{T_M}(M)$. Since the claim of Corollary 3.2.20 is independent of k, and since the isomorphisms used in that corollary correspond to triangulated cylinders over I or S^1, the invariance under bistellar moves and elementary shellings of type (2.) can be used to show independence of the boundary triangulation. The morphism $Z(M)$ is then also invariant under elementary shellings of type (1.), *i.e.* those involving the black boundary. $Z(M)$ is therefore independent of the triangulation and thus well-defined for the open-closed cobordism M.

One can verify explicitly that composition and disjoint union work as required, and so the state sum defines an open-closed TQFT $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$. The objects of $\mathcal{C}$ associated with the interval and the circle are A and $p(A)$, respectively, by construction. What is the knowledgeable Frobenius algebra that characterizes this TQFT?

In order to answer this question, we compute the morphisms of $\mathcal{C}$ associated with the generating open-closed cobordisms (2.5.25) and show that the open-closed TQFT is characterized by the knowledgeable Frobenius algebra of Theorem 3.2.18.

3.4.1 Defining the state sum

We first describe how to construct the morphism $Z_{T_M}(M)\colon A^{\otimes m_1} \to A^{\otimes m_2}$. It is defined by a string diagram in $\mathcal{C}$ obtained from the graph Poincaré dual to the triangulation, see Figure 3.2. By the coherence theorem for symmetric monoidal categories, it does not matter how one projects the Poincaré dual graph onto the drawing plane.

For every 2-simplex (triangle), we put a 'trilinear form' $g^{(3)}$ (*c.f.* (3.2.6)), and for every edge in the interior, we have an inverse bilinear form $g^* = \Delta_A \circ \eta_A$. Note that $g^{(3)}$ has a symmetry under the cyclic group C_3, but not in general under the symmetric group S_3, and so this assignment depends on the orientation.

For every edge on the coloured boundary $\partial_1 M$, we put a unit η_A. For every interior 0-simplex (vertex), we multiply the resulting morphism by the inverse a^{-1} of the window element. Since a^{-1} is central and the cobordism connected, it does not matter where in the diagram we do this.

At this stage, we have a morphism $A^{\otimes(m_1+m_2)} \to \mathbb{1}$ of $\mathcal{C}$. Finally, for every edge in the black out-boundary $\partial_0 M^{\mathrm{out}}$, we put a g^*, too, in order to turn this into a morphism $A^{\otimes m_1} \to A^{\otimes m_2}$. Then, for every vertex in the black out-boundary that is not a corner, we multiply by a^{-1}.

The terminology *sum* in 'state sum' is justified by the following point of view: If $\mathcal{C} = \mathbf{Vect}_k$ and if one chooses a basis of A and expands all linear maps in this basis, the state sum contains a sum over the basis vectors for each edge in the interior of M. This is the *sum* involved in the state sum.

The morphisms specified by the string diagram have two important properties.

- Gluing triangulated open-closed cobordisms along a common black boundary that is triangulated with the same number of edges, corresponds to the composition of morphisms.

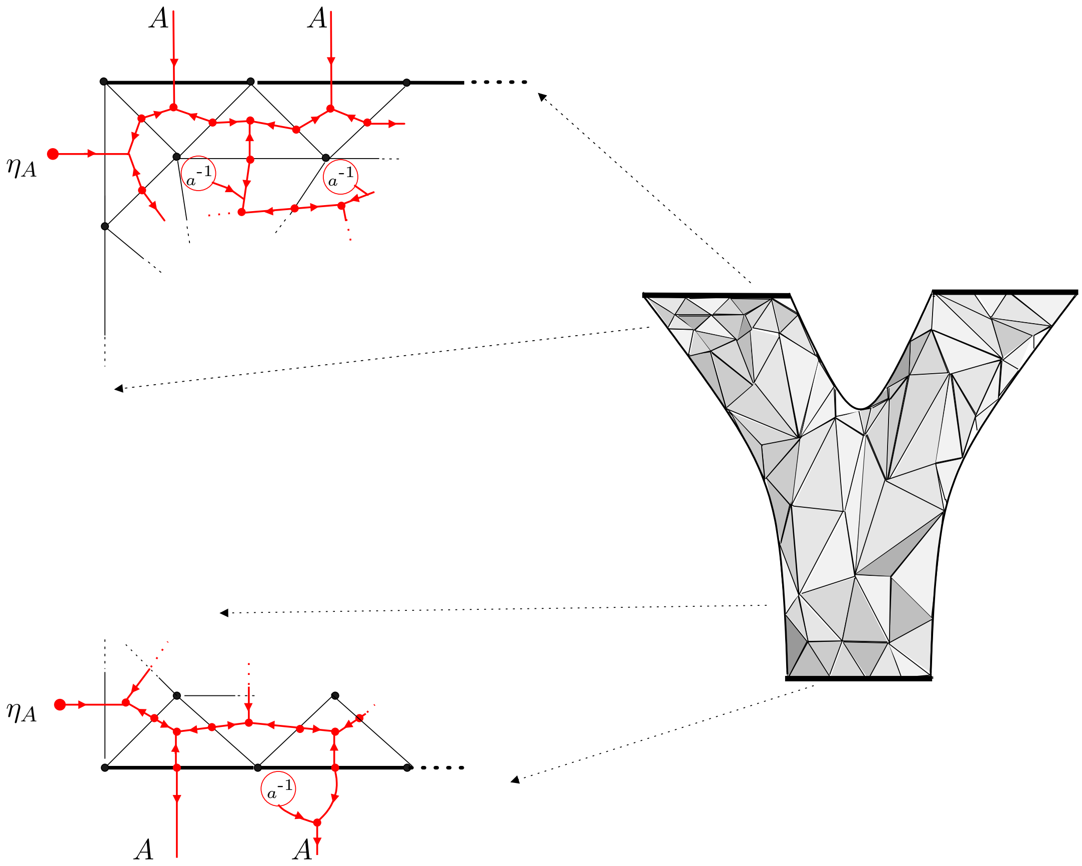


Figure 3.2: This figure illustrates the state sum for an open-closed cobordisms M.

- The disjoint union of open-closed cobordisms gives the tensor product of morphisms.

The definition reads in detail as follows.

Definition 3.4.1. Let $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ be a strongly separable symmetric Frobenius algebra in an abelian symmetric monoidal category $\mathcal{C}$. Let M be an open-closed cobordism with triangulation $T_M \colon |K| \to M$. Let $K^{(j)} \subseteq K$ denote the set of j-simplices, $j \in \{0, 1, 2\}$.

We characterize the edges, *i.e.* the elements $\sigma_{\{i,j\}} \in K^{(1)}$, by two-element sets $\{i, j\} \subseteq K^{(0)}$, $i \neq j$, of vertices. The oriented triangles $\sigma_{(i,j,k)} \in K^{(2)}$ are characterized by triples $(i, j, k) \in K^{(0)} \times K^{(0)} \times K^{(0)}$ of vertices, modulo a permutation by a 3-cycle.

We define the morphism $Z_{T_M}(M) \colon A^{\otimes m_1} \to A^{\otimes m_2}$ as a composition

$$Z_{T_M}(M) := Z^{(2)}_{T_M} \circ (a^{-k} \cdot \mathrm{id}_{A^{\otimes N}}) \circ \tau \circ Z^{(1)}_{T_M}. \tag{3.4.1}$$

where $N = m_2 + |\{\, \sigma \in K^{(1)} \colon \sigma \subseteq \partial M \,\}| + 2|\{\, \sigma \in K^{(1)} \colon \sigma \subseteq M \backslash \partial M \,\}| = m_2 + 3|K^{(2)}|$. The power of the inverse window element in (3.4.1) is $k = |\{\, \sigma \in K^{(0)} \colon \sigma \subseteq M \backslash \partial M \,\}| + |\{\, \sigma \in K^{(0)} \colon \sigma \subseteq \partial_0 M^{\mathrm{out}} \backslash (\partial_0 M \cap \partial_1 M) \,\}|$ — the number of interior vertices plus the number of vertices on the outgoing edge that are not corners. We exploit the coherence theorem for monoidal categories and suppress the associativity and unit constraints of $\mathcal{C}$ and define

$$Z^{(1)}_{T_M} := \left(\bigotimes_{j=1}^{m_1} \mathrm{id}_A\right) \otimes \left(\bigotimes_{j=1}^{m_2} g^*\right) \otimes \left(\bigotimes_{\substack{\sigma\in K^{(1)}:\\ \sigma\subseteq M\setminus\partial M}} g^*\right) \otimes \left(\bigotimes_{\substack{\sigma\in K^{(1)}:\\ \sigma\subseteq\partial_1 M}} \eta_A\right) : A^{\otimes m_1} \to A^{\otimes N}. \quad (3.4.2)$$

and

$$Z^{(2)}_{T_M} := \left(\bigotimes_{j=1}^{m_2} \mathrm{id}_A\right) \otimes \left(\bigotimes_{\sigma\in K^{(2)}} g^{(3)}\right) : A^{\otimes N} \to A^{\otimes m_2}, \quad (3.4.3)$$

The morphism $\tau\colon A^{\otimes N} \to A^{\otimes N}$ permutes the tensor factors. In order to specify this permutation, we associate the factors of the target of (3.4.2) and those of the domain of (3.4.3) with the edges $\sigma_{\{i,j\}} \in K^{(1)}$. This is denoted by superscripts such as $A^{\{i,j\}}$. The permutation τ is specified by requiring that it maps each factor $A^{\{i,j\}}$ to one whose superscript is the same edge.

The superscripts for the A's in the target of (3.4.2) are as follows. We go through the factors of (3.4.2) from left to right.

- For every edge $\sigma_{\{i,j\}}$ in the black in-boundary $\partial_0 M^{\mathrm{in}}$, we have $\mathrm{id}_A\colon A \to A^{\{i,j\}}$. There are m_1 edges of this sort.
- For every edge $\sigma_{\{i,j\}}$ in the black out-boundary $\partial_0 M^{\mathrm{out}}$, we have $g^*\colon \mathbb{1} \to A^{\{i,j\}} \otimes A^{\{i,j\}}$. This edge therefore appears twice as a superscript, but due to the symmetry of g^*, we need not distinguish the two. There are m_2 edges of this sort.
- For every edge $\sigma_{\{i,j\}} \subseteq M\setminus\partial M$ in the interior, we have $g^*\colon \mathbb{1} \to A^{\{i,j\}} \otimes A^{\{i,j\}}$. Again the superscript occurs twice, and we do not distinguish.
- For every edge $\sigma_{\{i,j\}} \subseteq \partial_1 M$ in the coloured boundary, we have $\eta_A\colon \mathbb{1} \to A^{\{i,j\}}$.

The superscripts for the A's in the domain of (3.4.3) are as follows.

- For every edge $\sigma_{\{i,j\}}$ in the black out-boundary $\partial_0 M^{\mathrm{out}}$, we have $\mathrm{id}_A\colon A^{\{i,j\}} \to A$.
- For every oriented triangle $\sigma_{(i,j,k)} \in K^{(2)}$, we have $g^{(3)}\colon A^{\{i,j\}} \otimes A^{\{j,k\}} \otimes A^{\{k,i\}} \to \mathbb{1}$. Due to the cyclic symmetry of the 'trilinear form' $g^{(3)}$, this morphism is invariant under permutations of the triple (i,j,k) by a 3-cycle.

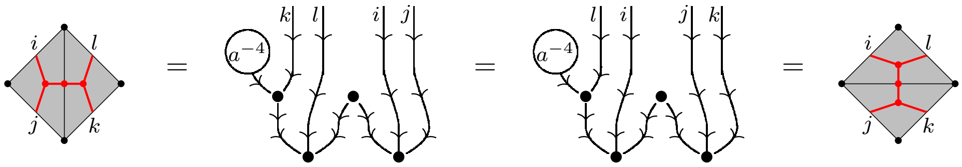
Notice that the edges that appear as superscripts in the target of (3.4.2) and those in the domain of (3.4.3) agree including their multiplicities, and that the permutation τ is well defined.

See (3.4.7) for an example of the diagram produced by the state sum.

3.4.2 Invariance under Pachner moves

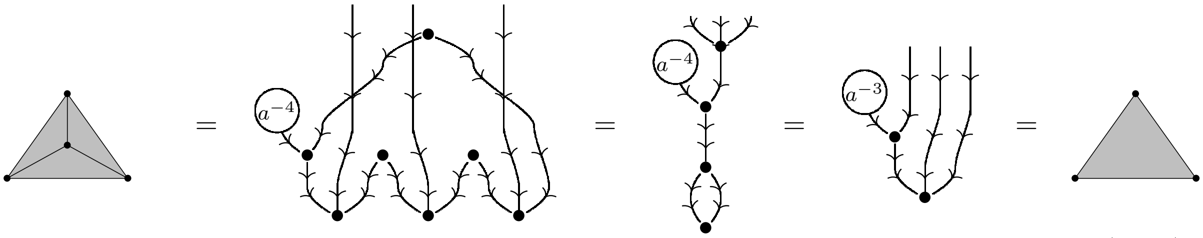
Proposition 3.4.2. For a connected open-closed cobordism M with triangulation T_M, the state sum $Z_{T_M}(M)$ is invariant under the 1-3 and 2-2 Pachner moves and under the elementary shellings of type (2.).

Proof. The 2-2 Pachner move follows from the cyclic symmetry of the 'trilinear form' $g^{(3)}$.



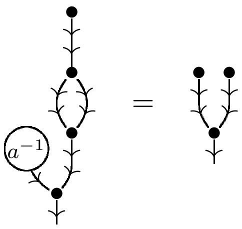
(3.4.4)

The 1-3 Pachner move is slightly more difficult because it involves subdividing a triangle which inserts an additional internal vertex. It makes use of the bubble move (3.2.22):



(3.4.5)

There are two elementary shellings (3.3.2) of type (2.). Recall that the state sum assigns to each edge of the coloured boundary the algebra unit $\eta_A\colon \mathbb{1} \to A$. The first move of (3.3.2) follows directly from the unit axioms. The second move turns an interior vertex into an exterior vertex (featured to the right). This move follows from the bubble move (3.2.22):



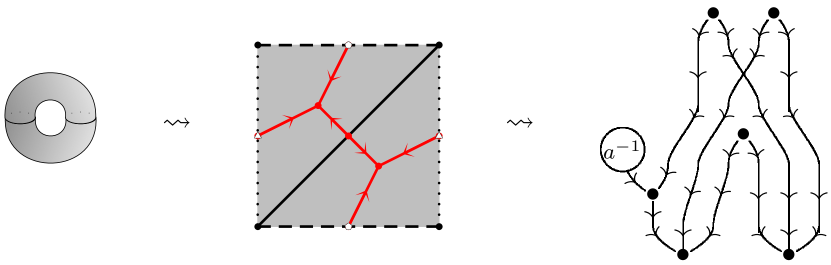
(3.4.6)

□

Note that the bubble move (3.2.22) is required to prove the above proposition. This is the reason why we cannot define the state sum for the non strongly separable algebras of Example 3.2.8(2) and 3.2.9(2).

For convenience, we sometimes use degenerate triangulations in which the two vertices in the boundary of an edge agree. In this case it is always understood that we apply bistellar moves and elementary shellings in order to turn them into proper simplicial complexes.

An example showing the diagram produced by the state sum on the torus T^2 is depicted below:



(3.4.7)

Here we have used the triangulation of the torus as a rectangle where the dotted and dashed lines are identified in the usual way. After the identifications this triangulation has a single interior vertex and hence the single factor of a^{-1} that appears in the string diagram on the right.

3.4.3 Independence of the triangulation of black boundaries

We now define a morphism $Z(M)$ from the morphism $Z_{T_M}(M)$ which does not depend on the choice of triangulation of the black boundary. Observe that to each black boundary component n_j triangulated with h_j edges, we have associated the vector space $A^{\otimes h_j}$.

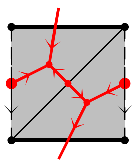
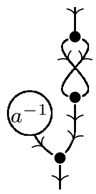
Proposition 3.4.3. For the triangulations $T^{k\ell}_{I\times I}$ and $T^{k\ell}_{S^1\times I}$ of the flat strip $I \times I$ and the cylinder $S^1 \times I$ with ℓ incoming edges and k outgoing edges in their black boundaries, the state sum of Definition 3.4.1 yields the morphisms $P_{k\ell}\colon A^{\otimes \ell} \to A^{\otimes k}$ and $Q_{k\ell}\colon A^{\otimes \ell} \to A^{\otimes k}$ of Proposition 3.2.19. That is,

$$Z_{T^{k\ell}_{I\times I}}(I \times I) \quad = \quad P_{k\ell}, \tag{3.4.8}$$

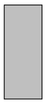
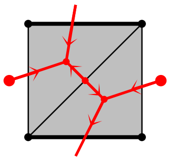
$$Z_{T^{k\ell}_{S^1\times I}}(S^1 \times I) \quad = \quad Q_{k\ell}. \tag{3.4.9}$$

Proof. Write down the string diagram defining the state sum, *c.f.* Figure 3.2, and use the bubble move and the axioms of a symmetric Frobenius algebra.

We here include the simplest triangulations of $S^1 \times I$ and $I \times I$ and the associated morphisms for $k = \ell = 1$:

$Z_{T^{11}_{S^1\times I}}($  $)$ =  = 

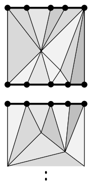
(3.4.10)

$Z_{T^{11}_{I\times I}}($  $)$ $=$  $=$  (3.4.11)

□

P_{44}

M



Given any triangulated open-closed cobordism M with a black boundary component homeomorphic to I and triangulated with ℓ edges, one can now glue a suitably triangulated cylinder $I \times I$ to that boundary. By Proposition 3.4.2, this yields the same morphism $Z_{T_M}(M)$. Similarly, for every black boundary component homeomorphic to S^1 and triangulated with ℓ edges, one can glue a suitably triangulated $S^1 \times I$ to that boundary, again leaving $Z_{T_M}(M)$ unchanged. It is therefore sufficient to consider the restriction of $Z_{T_M}(M)$ to the appropriate images of the idempotents $P_{\ell\ell}$ and $Q_{\ell\ell}$, respectively. We therefore define:

Definition 3.4.4. For every open-closed cobordism M with triangulation T_M, we define the state sum $\widetilde{Z}_{T_M}(M)$ by subsequently pre- and post-composing $Z_{T_M}(M)$ with the following morphisms: for each $n_j \in \vec{n} = \partial_0 M^{\mathrm{in}}$ triangulated with h_j edges, pre-composition with $\mathrm{im}\, P_{h_j h_j}$ if $n_j = 1$ and pre-composition with $\mathrm{im}\, Q_{h_j h_j}$ if $n_j = 0$; for each $n'_j \in \vec{n}' = \partial_0 M^{\mathrm{out}}$ triangulated with h_j edges, post-composition with $\mathrm{coim}\, P_{h_j h_j}$ if $n_j = 1$ and post-composition with $\mathrm{coim}\, Q_{h_j h_j}$ if $n_j = 0$.

If we write $R^{(0)}_{k\ell} := P_{k\ell}$ and $R^{(1)}_{k\ell} := Q_{k\ell}$, then the above composite is the morphism

$$\begin{aligned}\widetilde{Z}_{T_M}(M) \quad &= \quad \Big(\bigotimes_{j=k+1}^{k+k'} \mathrm{coim}\ R^{(n_j)}_{h_j h_j}\Big) \circ Z_{T_M}(M) \circ \Big(\bigotimes_{j=1}^{k} \mathrm{im}\ R^{(n_j)}_{h_j h_j}\Big) \colon \\ &\qquad \bigotimes_{j=1}^{k} R^{(n_j)}_{h_j h_j}(A^{\otimes h_j}) \to \bigotimes_{j=k+1}^{k+k'} R^{(n_j)}_{h_j h_j}(A^{\otimes h_j}).\end{aligned} \tag{3.4.12}$$

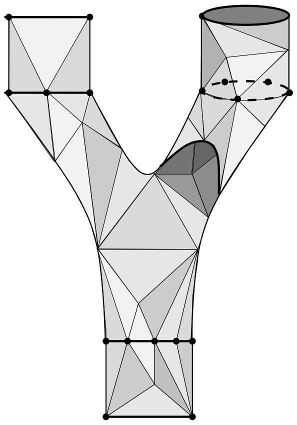
One can now use the isomorphisms of Corollary 3.2.20 in order to relate the $\widetilde{Z}_{T_M}(M)$ for different triangulations of the black boundary as follows. The morphism $\widetilde{Z}_{T_M}(M)$ is completely determined by the triangulation of the boundary $\partial_0 M$ by Proposition 3.4.2. Hence, the morphism $\widetilde{Z}_{T_M}(M)$ associated to a triangulation T_M is related to the morphism $\widetilde{Z}_{T'_M}(M)$ obtained from a different triangulation T'_M by gluing on cylinders whose boundaries are appropriately triangulated. These cylinders yield precisely the morphisms $P_{k\ell}$ and $Q_{k\ell}$.

Definition 3.4.5. For every open-closed cobordism M, we choose a triangulation T_M. We define the state sum $Z(M)$ by subsequently pre- and post-composing $\widetilde{Z}_{T_M}(M)$ with the following morphisms: For each $n_j \in \vec{n} = \partial_0 M^{\mathrm{in}}$ triangulated with h_j edges, pre-composition with Φ_{h_j} if $n_j = 1$ and pre-composition with Ψ_{h_j} if $n_j = 0$; For each $n'_j \in \vec{n}' = \partial_0 M^{\text{out}}$ triangulated with h_j edges, post-composition with $\Phi_{h_j}^{-1}$ if $n_j = 1$ and post-composition with $\Psi_{h_j}^{-1}$ if $n_j = 0$. This yields the morphism

$$Z(M) = \Big(\bigotimes_{j=k+1}^{k+k'} \big(\Xi_{h_j}^{(n_j)}\big)^{-1}\Big) \circ \widetilde{Z}_{T_M}(M) \circ \Big(\bigotimes_{j=1}^{k} \Xi_{h_j}^{(n_j)}\Big) \colon A^{\otimes \vec{n}} \to A^{\otimes \vec{n'}}, \tag{3.4.13}$$

where we write $\Xi_{h_j}^{(0)} := \Psi_{h_j}$ and $\Xi_{h_j}^{(1)} := \Phi_{h_j}$.

The definition of $Z(M)$ is illustrated below:



$$\overset{Z}{\rightsquigarrow} \qquad \begin{array}{c} A \otimes p(A) \\ \big\downarrow {\scriptstyle \Phi_2 \otimes \Psi_5} \\ P_{22}\big(A^{\otimes 2}\big) \otimes Q_{55}\big(A^{\otimes 5}\big) \\ \Big\downarrow {\scriptstyle \widetilde{Z}_{T_M}(M)} \\ P_{44}\big(A^{\otimes 4}\big) \\ \big\downarrow {\scriptstyle \Phi_4^{-1}} \\ A \end{array} \tag{3.4.14}$$

Theorem 3.4.6. The morphism (3.4.13) is well defined, *i.e.* it does not depend on the triangulation T_M of M. In particular, it is independent of the numbers h_j of edges in Definition 3.4.4 and Definition 3.4.5.

Proof. Insert (3.4.12) into (3.4.13) and draw the cylinders over I and over S^1 whose triangulations are given by im $P_{h_j h_j} \circ \Phi_{h_j} = P_{h_j 1}$, *etc.* and glue them to the triangulation used in the state sum $Z_{T_M}(M)$ of Definition 3.4.1. The invariance under bistellar moves and elementary shellings of type (2.) of Proposition 3.4.2 then implies the theorem. □

The following proposition provides a more intuitive description of the state sum $Z(M)$.

Theorem 3.4.7. Given an open-closed cobordism M with triangulation T_M the state sum $Z(M)$ is equal to $Z_{T'_M}(M')$ where M' is the triangulated open-closed cobordism obtained from M by pre- and post-composing with cylinders and strips with the simplest triangulations.

Proof. This is immediate from the definitions of $Z(M)$ and $Z_{T_M}(M)$ and from Propositions 3.2.19 and 3.4.3. □

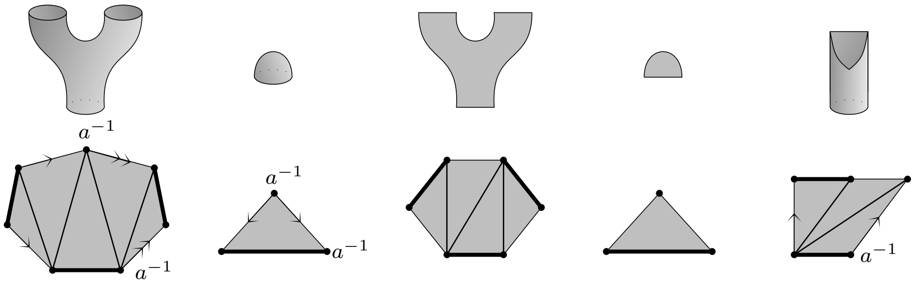
3.4.4 Open-closed Topological Quantum Field Theories

From Definition 3.4.1, it is obvious that the state sum $Z(M)$ associates with the composition of open-closed cobordisms the composition of morphisms of $\mathcal{C}$ and with the disjoint union of open-closed cobordisms the tensor product of morphisms in $\mathcal{C}$. It is not difficult to see that we get a symmetric monoidal functor $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$, *i.e.* an open-closed TQFT.

In this section, we show that this open-closed TQFT is the one characterized by the knowledgeable Frobenius algebra of Theorem 3.2.18.

Generators via the state sum construction

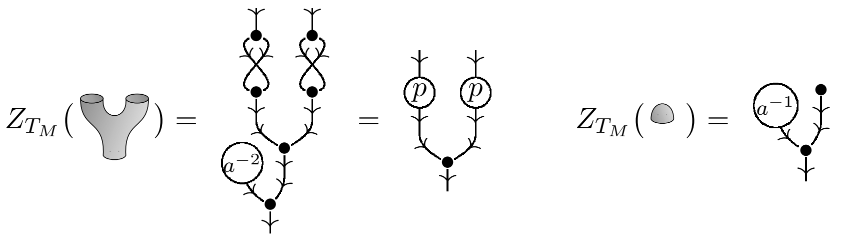
Below we provide a choice of triangulation for some of the generators in $\mathbf{2Cob}^{\mathrm{ext}}$.



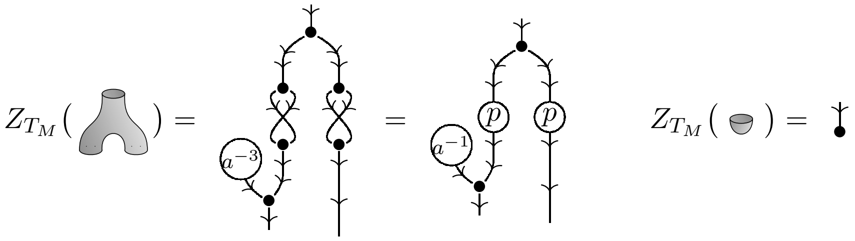
(3.4.15)

Those edges with matching arrow heads on the triangulations are to be identified. The black boundaries are depicted slightly thicker than the coloured boundaries. A choice of triangulation for the remaining generators is immediate from those above. The factors of a^{-1} are meant to remind the reader which vertices in the triangulation contribute factors of a^{-1}.

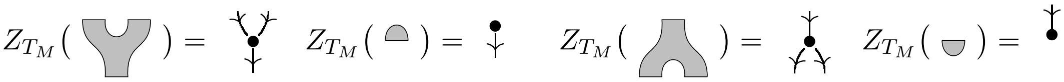
Using these triangulations we can compute the morphisms $Z_{T_M}(M)$ associated to the open-closed cobordisms M generating $\mathbf{2Cob}^{\mathrm{ext}}$. For completeness, we include the triangulation of the cylinders $S^1 \times I$ and $I \times I$ as well.



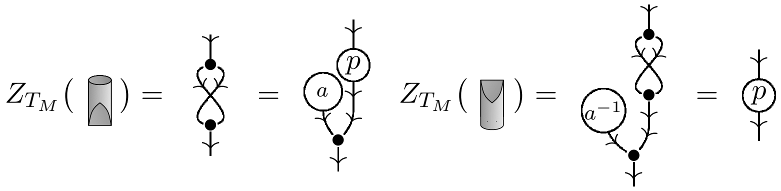
(3.4.16)



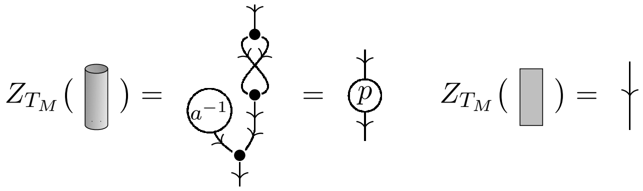
(3.4.17)



(3.4.18)

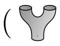
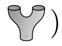


(3.4.19)



(3.4.20)

Theorem 3.4.8. Let $\mathcal{C}$ be an abelian symmetric monoidal category and A be a rigid and strongly separable algebra object in $\mathcal{C}$ that is equipped with the structure of a symmetric Frobenius algebra. Then the state sum (3.4.1) defines an open-closed TQFT $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$. It is characterized by the knowledgeable Frobenius algebra constructed from A in Theorem 3.2.18.

Proof. Using the triangulations of the generators given in (3.4.16)-(3.4.20), compute the morphisms $Z_{T_M}(M)$ for each generator of $\mathbf{2Cob}^{\mathrm{ext}}$. Pre and post composing with the relevant maps specified in Definitions 3.4.4 and 3.4.5 produces the knowledgeable Frobenius algebra $(A, Z(A), \imath, \imath^*)$ defined in Theorem 3.2.18. For example, $Z_{T_M}($  $) = \mu_A \circ (p \otimes p)$ so that $\widetilde{Z}_{T_M}($  $) = \mathrm{coim}\ Q_{11} \circ \mu_A \circ (p \otimes p) \circ (\mathrm{im}\ Q_{11} \otimes \mathrm{im}\ Q_{11})$. Noting that $Q_{11} = p$ and using the image factorization of p (3.2.26) together with the idempotent property $p^2 = p$ it is easy to check that

$Z($  $) = \mathrm{coim}\ p \circ \mu_A \circ (\mathrm{im}\ p \otimes \mathrm{im}\ p)$ (3.4.21) as specified in Theorem 3.2.18.

Since $\mathbf{2Cob}^{\mathrm{ext}}$ is the strict symmetric monoidal category freely generated by a knowledgeable Frobenius algebra object, this uniquely determines a symmetric monoidal functor $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$. □

Recall that given an open-closed TQFT, the algebra object $A := Z(I)$ does not necessarily determine the object $C := Z(S^1)$. Consider, for example, the knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ of Example 3.2.15 in which $C \ncong Z(A)$, and secondly the knowledgeable Frobenius algebra $(A, Z(A), \imath', \imath'^*)$ constructed in Theorem 3.2.18 based on the same A. Both characterize an open-closed TQFT, but only the latter one can be obtained from the state sum.

Conversely, in an open-closed TQFT, the object $Z(S^1)$ does not determine the object $Z(I)$. This can be easily seen from Example 3.4.9 below.

3.4.5 Examples

In Chapter 2 it was shown that connected open-closed cobordisms are determined up to orientation-preserving diffeomorphism preserving the black boundary by a set of topological invariants defined in the work of Baas, Cohen, and Ramírez [29]. These topological invariants are the *genus* (defined as the genus of the underlying topological 2-manifold), the *window number*, defined as the number of components of $\partial_1 M$ diffeomorphic to S^1, and the *boundary permutation*. For a surface M ($\partial_0 M = \emptyset$) only the genus and window number are relevant. In this context we will refer to the window number as the number of punctures in M.

Let $(A, C, \imath, \imath^*)$ be a knowledgeable Frobenius algebra in a symmetric monoidal category $\mathcal{C}$. We call $\mu_C \circ \Delta_C\colon C \to C$ the *genus-one operator* and $\imath^* \circ \imath\colon C \to C$ the *window operator*. The invariant associated to the connected surface M_k^ℓ of genus ℓ with k punctures is determined by evaluating the morphism

$$Z(M_k^\ell) = \varepsilon_C \circ \left(\imath^* \circ \imath\right)^k \circ \left(\mu_C \circ \Delta_C\right)^\ell \circ \eta_C\colon \mathbb{1} \to \mathbb{1} \tag{3.4.22}$$

in $\mathcal{C}$.

In this section, we provide several examples of strongly separable symmetric Frobenius algebras and use the genus-one operator and the window operator to compute the state sum invariant $Z\big(M_k^\ell\big)$.

Example 3.4.9. Let k be a field, $n \in \mathbb{N}$, and $m_1, \ldots, m_n \in \mathbb{N}$, and consider the direct product[4]

$$A := \bigoplus_{j=1}^{n} M_{m_j}(k) \tag{3.4.23}$$

of matrix algebras. We choose a basis $\{e_{pq}^{(j)}\}_{1\leq p,q\leq m_j, 1\leq j\leq n}$ of A such that the multiplication reads $\mu_A(e_{pq}^{(j)} \otimes e_{rs}^{(\ell)}) = \delta_{j\ell}\delta_{rq}e_{ps}^{(j)}$ with unit $\eta_A(1) = \sum_{j=1}^{n}\sum_{p=1}^{m_j} e_{pp}^{(j)}$. The k-algebra (A, μ_A, η_A) is strongly separable if and only if for all j, char k does not divide m_j. From now on we assume that this condition holds.

The centre $Z(A)$ of A has a basis $\{z_j\}_{1\leq j\leq n}$ of orthogonal idempotents $z_j := \sum_{p=1}^{m_j} e_{pp}^{(j)}$, *i.e.* $\mu_A(z_j \otimes z_\ell) = \delta_{j\ell} z_j$. The symmetric Frobenius algebra structures $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ are characterized by the invertible central elements $a = \sum_{j=1}^{n} a_j z_j$, *i.e.* $a_j \in k\backslash\{0\}$ for all j, as follows:

$$\begin{aligned} \Delta_A(e_{pq}^{(j)}) &= a_j m_j^{-1} \sum_{r=1}^{m_j} e_{pr}^{(j)} \otimes e_{rq}^{(j)}, && (3.4.24) \\ \varepsilon_A(e_{pq}^{(j)}) &= \delta_{pq} m_j a_j^{-1}, && (3.4.25) \end{aligned}$$

and indeed one finds $(\mu_A \circ \Delta_A \circ \eta_A)(1) = a$ for the window element. This illustrates further the distinction between special Frobenius algebras and strongly separable Frobenius algebras. A is special if and only if $a_i = a_j$ for all i, j. We compute the idempotent p of (3.2.23) as follows:

$$p(e_{pq}^{(j)}) = \delta_{pq} m_j^{-1} \sum_{r=1}^{m_j} e_{rr}^{(j)}, \tag{3.4.26}$$

and indeed the image is $p(A) \cong Z(A)$ with the splitting

$$\begin{aligned} \mathrm{im}\, p\colon p(A) \to A, &\quad z_j \mapsto \sum_{p=1}^{m_j} e_{pp}^{(j)}, && (3.4.27) \\ \mathrm{coim}\, p\colon A \to p(A), &\quad e_{pq}^{(j)} \mapsto \delta_{pq} m_j^{-1} z_j. && (3.4.28) \end{aligned}$$

where im p is just the inclusion. The knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ of Theorem 3.2.18 for this algebra A is given by the following commutative Frobenius algebra structure

[4] We write $\oplus$ because this is actually the biproduct in the abelian category $\mathbf{Vect}_k$.

$(C,\mu_C,\eta_C,\Delta_C,\varepsilon_C)$ on $C := Z(A)$:

$$\mu_C(z_j\otimes z_\ell) = \delta_{j\ell}z_j, \tag{3.4.29}$$
$$\eta_C(1) = \sum_{j=1}^{n} z_j, \tag{3.4.30}$$
$$\Delta_C(z_j) = a_j^2 m_j^{-2} z_j\otimes z_j, \tag{3.4.31}$$
$$\varepsilon_C(z_j) = m_j^2 a_j^{-2}, \tag{3.4.32}$$

together with

$$\imath\colon C\to A,\qquad z_j\mapsto \sum_{p=1}^{m_j} e^{(j)}_{pp}, \tag{3.4.33}$$

$$\imath^*\colon A\to C,\quad e^{(j)}_{pq}\mapsto a_j m_j^{-1}\delta_{pq}z_j. \tag{3.4.34}$$

We finally compute the genus-one operator $(\mu_C\circ\Delta_C)(z_j) = a_j^2 m_j^{-2} z_j$ and the window operator $(\imath^*\circ\imath)(z_j) = a_j z_j$, and so the invariant (3.4.35) associated with the genus ℓ-surface with k punctures, $k,\ell\in\mathbb{N}_0$, is

$$Z(M_k^\ell)(1) = (\varepsilon_C\circ(\imath^*\circ\imath)^k\circ(\mu_C\circ\Delta_C)^\ell\circ\eta_C)(1) = \sum_{j=1}^{n} a_j^{k+2(\ell-1)} m_j^{-2(\ell-1)}. \tag{3.4.35}$$

Fukuma–Hosono–Kawai [40] choose the canonical Frobenius algebra structure on A, *i.e.* $a=\eta$ and therefore $a_j = 1$ for all j. In this case, the invariant is blind to the *window number* k. With a generic symmetric Frobenius algebra structure, however, one can easily obtain an invariant that can distinguish any two inequivalent connected surfaces.

Example 3.4.10. Let G be a finite group, k a field, and $A := k[G]$ be the group algebra. We choose the basis $\{g\}_{g\in G}$ for A and have $\mu_A(g\otimes h) = gh$ for $g,h\in G$ and $\eta_A(1) = e$. The k-algebra (A,μ_A,η_A) is strongly separable if and only if char k does not divide the order $|G|$ of G. We now assume that this condition holds.

We denote by $[g] := \{\, hgh^{-1}\colon\ h\in G\,\}\subseteq G$ the conjugacy class of $g\in G$ and by $G/\!\sim := \{\,[g]\colon\ g\in G\}$ the set of classes. Then the centre $Z(A)$ has the basis $\{z_{[g]}\}_{[g]\in G/\sim}$ where $z_{[g]} := \sum_{h\in[g]} h$ denotes the class sum. We have the unit $\eta_A(1) = \sum_{[g]\in G/\sim} z_{[g]}$ and $\mu_A(z_{[g]}\otimes z_{[h]}) = \sum_{[\ell]\in G/\sim}\mu^{[\ell]}_{[g],[h]} z_{[\ell]}$ for all $g,h\in G$ with some $\mu^{[\ell]}_{[g],[h]}\in k$.

The $z_{[g]}$ are in general not orthogonal idempotents. Working with a generic invertible central element in the basis $\{z_{[g]}\}_{[g]\in G/\sim}$ is not very instructive. If k is algebraically closed, the irreducible characters $\chi_\rho\colon G\to k$ provide us with a basis $\{z_\rho\}_\rho$ of orthogonal idempotents

$z_\rho := d_\rho |G|^{-1} \sum_{g\in G} \chi_\rho(g) g$, $d_\rho = \chi_\rho(e)$, for $Z(A)$. We then get the same results as for a direct product $\bigoplus_\rho M_{d_\rho}$ of $d_\rho \times d_\rho$-dimensional matrix algebras.

In the following, we restrict ourselves to the symmetric Frobenius algebra structure

$$\Delta_A(g) = \sum_{h\in G} h \otimes h^{-1} g, \tag{3.4.36}$$

$$\varepsilon_A(g) = \begin{cases} 1, & \text{if } g = e \\ 0, & \text{else} \end{cases} \tag{3.4.37}$$

which is characterized by the window element $(\mu_A \circ \Delta_A \circ \eta_A)(1) = |G| e = |G| \eta_A(1)$. The symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ is therefore special in the sense of (2.3.10). In this case

$$g^*(1) = (\Delta_A \circ \mu_A)(1) = \sum_{h\in G} h \otimes h^{-1}, \tag{3.4.38}$$

$$g^{(3)}((g\otimes h)\otimes \ell) = (\varepsilon_A \circ \mu_A \circ (\mu_A \otimes \mathrm{id}_A))((g\otimes h)\otimes \ell) = \begin{cases} 1, & \text{if } gh\ell = e \\ 0, & \text{else} \end{cases} \tag{3.4.39}$$

The state sum $Z(M)$ then agrees with the partition function of a topological gauge theory with gauge group G or, in other words, with the volume of the moduli space of flat G-bundles on M. In the state sum of Definition 3.4.1, the window element $|G|$ is divided out for every vertex in the interior of M (this prefactor of $Z(M)$ is sometimes called the *anomaly*). In the closed TQFT, the meaning of this factor is somewhat mysterious — the factor is merely needed in order to make the 1-3 Pachner move work — but in our extension to the open-closed TQFT, the factor $|G|$ is directly related to the symmetric Frobenius algebra structure of A and thereby to topology.

Remark 3.4.11. Although our state sum of Definition 3.4.1 requires an oriented 2-manifold, the previous example with the group algebra $A = k[G]$ makes sense even for unoriented manifolds (without boundary). This is possible because A also has the structure of an involutory Hopf algebra $(A, \mu_A, \eta_A, \Delta_A^{\mathrm{Hopf}}, \varepsilon_A^{\mathrm{Hopf}}, S_A)$ with

$$\Delta_A^{\mathrm{Hopf}}(g) = g \otimes g, \tag{3.4.40}$$

$$\varepsilon_A^{\mathrm{Hopf}}(g) = 1, \tag{3.4.41}$$

$$S_A(g) = g^{-1}, \tag{3.4.42}$$

with a co-integral $\sum_{g\in G} g$ and an integral $g \mapsto \delta_G(g)$ where $\delta_G(e) = 1$ and $\delta_G(g) = 0$ for all $g \neq e$. For this involutory Hopf algebra, one can evaluate Kuperberg's 3-manifold invariant [51] which does not refer to the 3-simplices and therefore makes sense for (unoriented) 2-manifolds, too. In the oriented case, it agrees with our state sum. The unoriented case is treated in more generality in [31].

3.5 State sums with D-branes

Our next example, the groupoid algebra of a finite groupoid, also yields the state sum of an open-closed TQFT in a straightforward way, but in addition it provides us with an example of an S-coloured open-closed TQFT, *c.f.* 2.8.

A *groupoid* $\mathcal{G} = (X, G, s, t, \imath, \circ, ^{-1})$ consists of sets X (*objects*) and G (*morphisms*) and maps $s\colon G \to X$ (*source*), $t\colon G \to X$ (*target*), $\imath\colon X \to G$ (*identity*), $\circ\colon G{}_t{\times}_s G := \{\,(h_1, h_2) \in G \times G\colon\ t(h_1) = s(h_2)\,\} \to G$ (*composition*, written from left to right) and $^{-1}\colon G \to G$ (*inversion*) such that the following conditions are satisfied,

1. $s(\imath(x)) = x$ and $t(\imath(x)) = x$ for all $x \in X$,
2. $s(h_1 \circ h_2) = s(h_1)$ and $t(h_1 \circ h_2) = t(h_2)$ for all $(h_1, h_2) \in X{}_t{\times}_s X$,
3. $(h_1 \circ h_2) \circ h_3 = h_1 \circ (h_2 \circ h_3)$ for all $h_1, h_2, h_3 \in G$ for which $t(h_1) = s(h_2)$ and $t(h_2) = s(h_3)$,
4. $\imath(s(h)) \circ h = h = h \circ \imath(t(h))$ for all $h \in G$,
5. $s(h^{-1}) = t(h)$ and $t(h^{-1}) = s(h)$ for all $h \in G$,
6. $h^{-1} \circ h = \imath(t(h))$ and $h \circ h^{-1} = \imath(s(h))$ for all $h \in G$.

The groupoid is called *finite* if G is a finite set. For every $x \in X$, we denote its connected component by $[x] := \{\, t(h)\colon\ h \in G, s(h) = x \,\}$. The groupoid is called *connected* if $X = [x]$ for some $x \in X$. For $x \in X$, the *star of* $\mathcal{G}$ *at* x is the set,

$$\mathrm{st}_{\mathcal{G}}(x) = \{\, g \in G\colon\ s(g) = x \,\}. \tag{3.5.1}$$

We denote the order of the star of $\mathcal{G}$ at $x \in X$ by $N_{[x]} := |\mathrm{st}_{\mathcal{G}}(x)|$. It depends only on the connected component $[x]$ of $x \in X$.

Given a finite groupoid $\mathcal{G} = (X, G, s, t, \imath, \circ, ^{-1})$ and a field k, the *groupoid algebra* $(k[G], \mu, \eta)$

is the free vector space $k[G]$ on the set of morphisms with the operations,

$$\mu(h_1 \otimes h_2) = \begin{cases} h_1 \circ h_2, & \text{if } t(h_1) = s(h_2) \\ 0, & \text{else} \end{cases} \tag{3.5.2}$$

$$\eta(1) = \sum_{x \in X} \imath(x), \tag{3.5.3}$$

where $h_1, h_2 \in G$.

Example 3.5.1. Let $(G) = (X, G, s, t, \imath, \circ, {}^{-1})$ be a finite groupoid and consider the groupoid algebra $A := k[G]$. The k-algebra A is strongly separable if and only of char k does not divide $N_{[x]}$ for any $x \in X$. From now on, we assume that this is the case.

We denote by $G^{(0)} := \{\, g \in G\colon\ s(g) = t(g)\,\} \subseteq G$ the set of automorphisms, by $[g] := \{\, h \circ g \circ h^{-1}\colon\ h \in G, t(h) = t(g)\,\}$ the conjugacy class of the automorphism $g \in G^{(0)}$, and by $G^{(0)}/\sim := \{\, [g]\colon\ g \in G^{(0)}\,\}$ the set of conjugacy classes. Choose the basis $\{h\}_{h \in G}$ of A. We find the centre $Z(A) \cong k[G^{(0)}/\sim]$ with a basis $\{z_{[g]}\}_{g \in G^{(0)}/\sim}$ where $z_{[g]} := \sum_{h \in [g]} h$ denotes the class sum.

The canonical symmetric Frobenius algebra structure $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ is given by

$$\varepsilon_A(g) = \begin{cases} N_{[s(g)]}, & \text{if } g = \imath(s(g)) \\ 0, & \text{else} \end{cases} \tag{3.5.4}$$

$$\Delta_A(g) = \frac{1}{N_{[t(g)]}} \sum_{h \in G\colon\ s(h) = s(g)} h \otimes (h^{-1} \circ g), \tag{3.5.5}$$

from which we obtain the canonical idempotent (3.2.23)

$$p(g) = \begin{cases} z_{[g]}/N_{[t(g)]}, & \text{if } t(g) = s(g) \\ 0, & \text{else} \end{cases} \tag{3.5.6}$$

with the image decomposition

$$\operatorname{im} p\colon Z(A) \to A, \qquad z_{[g]} \mapsto \sum_{h \in [g]} h, \tag{3.5.7}$$

$$\operatorname{coim} p\colon A \to Z(A), \qquad g \mapsto \begin{cases} z_{[g]}/N_{[t(g)]}, & \text{if } s(g) = t(g) \\ 0, & \text{else.} \end{cases} \tag{3.5.8}$$

From these data, one can compute the knowledgeable Frobenius algebra $(A, Z(A), \imath, \imath^*)$ that appears in Theorem 3.2.18 with $\imath = \operatorname{im} p$ and $\imath^* = \operatorname{coim} p$. The state sum construction therefore yields the corresponding open-closed TQFT.

There is, however, another point of view according to which the groupoid algebra gives rise to an X-coloured knowledgeable Frobenius algebra (Section 2.8). Although this example is rather trivial, it nicely illustrates where the various structures appear.

Example 3.5.2. Let $\mathcal{G} = (X, G, s, t, \imath, \circ, {}^{-1})$ be a finite groupoid and k be a field such that $\operatorname{char} k$ does not divide $N_{[x]}$ for any $x \in X$. Denote by $\operatorname{Hom}(x,y) = \{\, g \in G\colon s(g) = x, t(g) = y \,\}$ the morphisms from x to y. Then there is a family of vector spaces $A_{xy} := k[\operatorname{Hom}(x,y)]$. By restricting the operations of the groupoid algebra $A = k[G]$ to the A_{xy}, we obtain the following linear maps:

$$\mu_{xyz}\colon A_{xy} \otimes A_{yz} \to A_{xz}, \qquad g_1 \otimes g_2 \mapsto g_1 \circ g_2, \tag{3.5.9}$$

$$\eta_x(1)\colon k \to A_{xx}, \qquad 1 \mapsto \imath(x), \tag{3.5.10}$$

$$\Delta_{xyz}\colon A_{xz} \to A_{xy} \otimes A_{yz}, \qquad g \mapsto \frac{1}{N_{[t(g)]}} \sum_{h \in G\colon\, s(h) = x} h \otimes h^{-1} \circ g, \tag{3.5.11}$$

$$\varepsilon_x\colon A_{xx} \to k, \qquad g \mapsto \begin{cases} N_{[s(g)]}, & \text{if } g = \imath(x) \\ 0, & \text{else} \end{cases} \tag{3.5.12}$$

for $x, y, z \in X$. Similarly by restricting $\imath$ and $\imath^*$, we find for all $x \in X$:

$$\imath_x\colon Z(A) \to A_{xx}, z_{[g]} \mapsto \sum_{h \in [g]\colon\, h \in \operatorname{Hom}(x,x)} h, \tag{3.5.13}$$

$$\imath^*\colon A_{xx} \to Z(A), g \mapsto \frac{1}{N_{[x]}} z_{[g]}. \tag{3.5.14}$$

Then we have an X-coloured knowledgeable Frobenius algebra

$$(\{A_{xy}\}, \{\mu_{xyz}\}, \{\eta_x\}, \{\Delta_{xyz}\}, \{\varepsilon_x\}, Z(A), \{\imath_x\}, \{\imath_x^*\}). \tag{3.5.15}$$

The commutative Frobenius algebra structure of $Z(A)$ is as in Theorem 3.2.18. In particular, each A_{xx}, $x \in X$, forms a symmetric Frobenius algebra, the $\imath_x\colon Z(A) \to A_{xx}$ are algebra homomorphisms, and each A_{xy} forms an (A_{xx}, A_{yy})-bimodule with dual A_{yx}. Observe that the state sum can be evaluated directly for the full groupoid algebra

$$A = \bigoplus_{x,y \in X} A_{xy}, \tag{3.5.16}$$

and so the vector space associated with the unit interval is precisely this direct sum. If one restricts it to the subspaces A_{xy} corresponding to the boundary colours $x, y \in X$ of a given interval, one obtains an X-coloured open-closed TQFT. The full state sum with A, however, contains more than just these homogeneous elements. It includes their linear combinations as well.

This last example is especially relevant in the context where the open-closed cobordisms are interpreted as open and closed string worldsheets. In this case, the colours of an X-coloured knowledgeable Frobenius algebra are interpreted as the set of boundary conditions, or D-branes, for the open strings. The decomposition of the finite groupoid algebra then allows the state sum to compute topological invariants of open and closed string worldsheets equipped with D-brane labels from the set of objects X of the groupoid $\mathcal{G}$.

Chapter 4

Khovanov homology for tangles

4.1 Background

4.1.1 The Jones Polynomial

A *link* L of m components is a submanifold of S^3 that is diffeomorphic to a disjoint union of m simple closed curves. A link with one component is a *knot*. Two links L_1 and L_2 are considered equivalent if they are isotopic. We will be interested in 'local' versions of knot and link diagrams that are confined to a rectangular cube.

Definition 4.1.1. An *unoriented* (n, m)*-tangle* T is a proper, smooth embedding of a 1-manifold with boundary into $\mathbb{R}^2 \times [0, 1]$ such that the boundary ∂T of T satisfies the condition

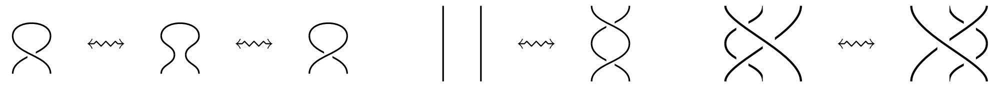
$$\partial T = T \cap \left(\mathbb{R}^2 \times \{0, 1\}\right) = \{1, 2, \dots, n\} \times \{0\} \times \{1\} \cup \{1, 2, \dots, m\} \times \{0\} \times \{0\}, \qquad (4.1.1)$$

and near the endpoints T is perpendicular to the boundary. An oriented (n, m)-tangle is equipped with an orientation of each of its components.

We define the *source* of a (n, m)-tangle T to be the component of ∂T given by $\{1, 2, \dots, n\} \times \{0\} \times \{1\}$ and the *target* of the (n, m)-tangle T to be the component of ∂T given by $\{1, 2, \dots, m\} \times \{0\} \times \{0\}$ (tangles are read from top to bottom). Note that a $(0, 0)$-tangle is just a link. For simplicity we sometimes refer to an (n, m)-tangle as a tangle for short. Two tangles are equivalent if they are isotopic by boundary preserving diffeomorphisms.

Tangles are determined up to equivalence by their *plane diagrams*, which are defined as generic planar projections of the tangle onto the 'back wall' $\mathbb{R} \times [0, 1]$. The projection is *generic* if it has only transversal double points as singularities. Two plane diagrams are called *plane isotopic* if they belong to a 1-parameter family of generic projections, meaning that the combinatorial structure of the tangle is left unchanged.

Theorem 4.1.2. Two tangle diagrams represent equivalent tangles if and only if one can be obtained from the other by a finite sequence of Reidemeister moves:



(4.1.2)

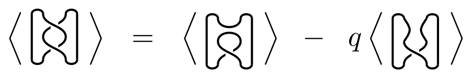
and plane isotopies.

The Jones polynomial $J(L)$ of a link diagram L is a Laurent polynomial in $\mathbb{Z}[q, q^{-1}]$ that is an invariant of the link L. Let L be a plane diagram of an oriented link with n_+ positive crossings $\times$ and n_- negative ones $\times$, $n := n_+ + n_-$. The Kauffman bracket $\langle L \rangle$, from which one can compute the unnormalized Jones polynomial[1] $\widehat{J}(L) := (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$, can be recursively defined as follows:

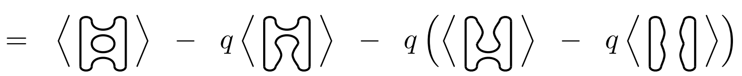
$$\langle \emptyset \rangle = 1; \qquad \langle \bigcirc L \rangle = (q + q^{-1}) \langle L \rangle; \qquad \langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle. \tag{4.1.3}$$

In particular, for every crossing ($\times$), each of the two smoothings, the 0-smoothing ($\asymp$) and the 1-smoothing ($)($) give a contribution to the Kauffman bracket, with a different sign and a different power of q. For the link diagram L with n crossings, there are 2^n smoothings, labeled by sequences $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0,1\}^n$ where $\alpha_j \in \{0,1\}$ indicates whether the j-th crossing was resolved by the 0- or the 1-smoothing. Each of the diagrams S_α, $\alpha \in \{0,1\}^n$, corresponding to the 2^n smoothings, is free of crossings and therefore consists of a disjoint union of a finite number of circles.

Example 4.1.3. Consider the Hopf link . The Kauffman bracket is given by



(4.1.4)



(4.1.5)

$$= (q + q^{-1})^2 - q(q + q^{-1}) - q\big((q + q^{-1}) - q(q + q^{-1})^2\big) \tag{4.1.6}$$

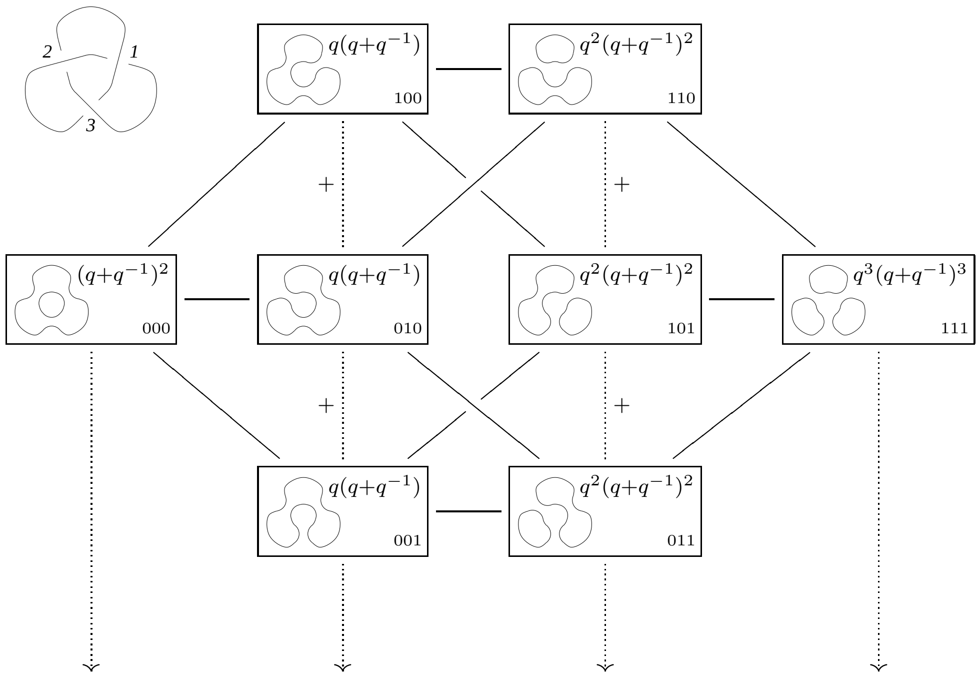
$$= 1 + q^{-2} + q^2 + q^4. \tag{4.1.7}$$

[1] We have adopted the conventions of Khovanov [12]. The usual definition of the Jones polynomial is obtained by substituting $-\sqrt{t}$ for q in the normalized polynomial $J(L) := \widehat{J}(L)/(q + q^{-1})$.

If the Hopf link has the orientation given by so that $n_+ = 0$ and $n_- = 2$, then the unnormalized Jones polynomial is given by

$$\widehat{J}\,(\;) := (-1)^2 q^{-4} \langle\;\rangle = 1 + q^{-2} + q^{-4} + q^{-6}. \tag{4.1.8}$$

Perhaps the first hint that the unnormalized Jones polynomial $\widehat{J}(L)$ might be related to the Euler characteristic of some homology theory comes from the skein relation $\langle \times \rangle = \langle \asymp \rangle - q\langle)\,(\rangle$ in the definition of the Kauffman bracket. If we define the *height* of a resolution S_α of a link diagram L to be the number of 1's in the sequence $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$, then it is apparent from the skein relation that if the resolution S_α contains r 1-smoothing then associated to S_α will be the prefactor $(-q)^r$. Further, *every* resolution S_α at height r will have associated to it this same prefactor. This phenomenon is best illustrated by Bar-Natan's wonderful diagram[2] [14] illustrating how one can compute the unnormalized Jones polynomial of an n crossing link by assigning to each vertex of the hypercube $\{0,1\}^n$ the complete resolution S_α of L where the sequence α corresponds to the edge of the cube $\{0,1\}^n$.



$$\begin{gathered}(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3 \\ = \; q^{-2} + 1 + q^2 - q^6 \xrightarrow[\text{(with } (n_+, n_-) = (3,0))]{\cdot(-1)^{n_-} q^{n_+ - 2n_-}} q + q^3 + q^5 - q^9 \xrightarrow{\cdot(q+q^{-1})^{-1}} J(\;) = q^2 + q^6 - q^8.\end{gathered} \tag{4.1.9}$$

The computation above amounts to assigning to each vertex S_α consisting of k circles the term $(-1)^r q^r (q + q^{-1})^k$ where r is the height of the smoothing S_α. Note that in the

[2]Thanks to Dror Bar-Natan for use of this diagram the two others in the next two sections.

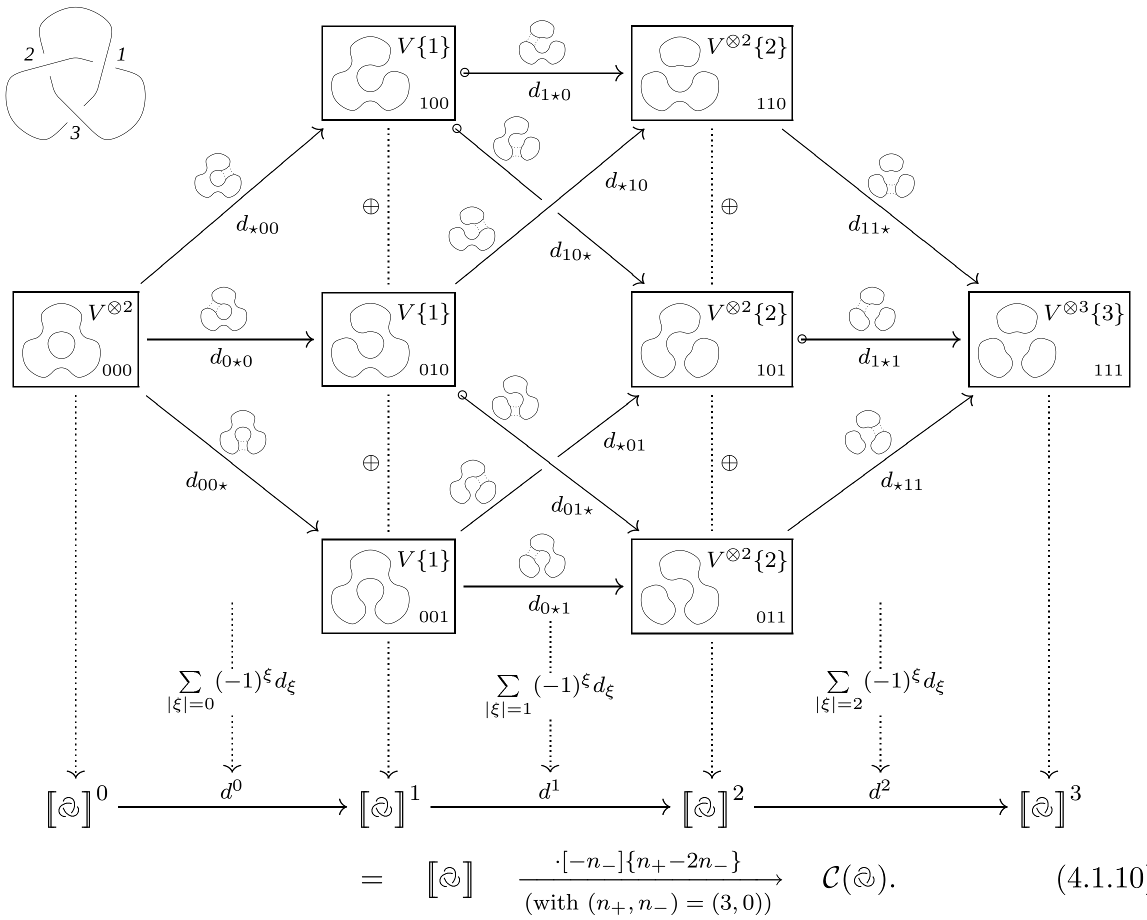
computation above, the terms at a given height r are summed first so that the alternating sum appears at the bottom when we sum over the factors corresponding to each height. Finally, the computation is completed by multiplying by the normalization $(-1)^{n_-}q^{n_+-2n_-}$ determined from the orientation of L.

4.1.2 Khovanov homology for links

The diagram (4.1.9) can be thought of as the jumping off point for describing Khovanov's categorification of the Jones polynomial. Rather than assigning the factor $(-1)^r q^r (q+q^{-1})^k$ to the resolution S_α of height r and consisting of k circles, we instead associate a graded vector space $V^{\otimes k}$ whose grading has been shifted appropriately according to the height r.

More specifically, V is chosen to be a graded vector space with two basis elements v_+ and v_- of degrees ± 1, respectively. This ensures that the graded dimension qdim$V$ of V is equal to $(q+q^{-1})$. To the resolution S_α, we associate the vector space $V^{\otimes k}\{r\}$ where $\cdot\{r\}$ is the degree shift operator on graded vector spaces, defined on the graded vector space $W = \bigoplus_m W_m$ such that $W\{r\}_m := W_{m-r}$. Here we are building into the theory the property that the graded dimension of the vector space associated to S_α corresponds to the factors associated to S_α by the Jones polynomial discussed in the previous section. In particular, the graded dimension of the vector space $\mathrm{qdim} V^{\otimes k}\{r\} = q^r(q+q^{-1})^k$. The appropriate power of (-1) then arises by taking the alternating sum of the graded dimensions of the vector spaces associated to resolutions of a given height.

It is natural to think of the assignment of the vector space $V^{\otimes k}\{r\}$ to the resolution S_α as part of a 2-dimensional TQFT. Indeed, each resolution S_α is a closed 1-manifold consisting of the disjoint union of several circles. For each cycle in the resolution S_α we associate the vector space V and to the disjoint union we have associated the tensor product of such vector spaces. It turns out that 2-dimensional topological quantum field theories play an important role in defining the differential of a complex constructed from the cube of resolutions. This construction is again best illustrated using one of Bar-Natan's diagrams [14].



(4.1.10)

Let $Z_{\mathrm{Kh}} \colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathbf{Vect}_k$ be a 2-dimensional topological quantum field theory such that the image of the circle under Z_{Kh} is the vector space V. For the purposes of obtaining the unnormalized Jones polynomial as the graded Euler characteristic of a homology theory it will be important that the image of the functor Z_{Kh} is actually a graded vector space. Define the chain complex $\llbracket L \rrbracket$ by setting the chain space $\llbracket L \rrbracket^r$ for $0 \leq r \leq n$, to be the direct sum of the vector spaces associated to the resolutions at height r, namely $\llbracket L \rrbracket^r = \bigoplus_{\alpha : r = |\alpha|} Z_{\mathrm{Kh}}(S_\alpha)\{r\}$.

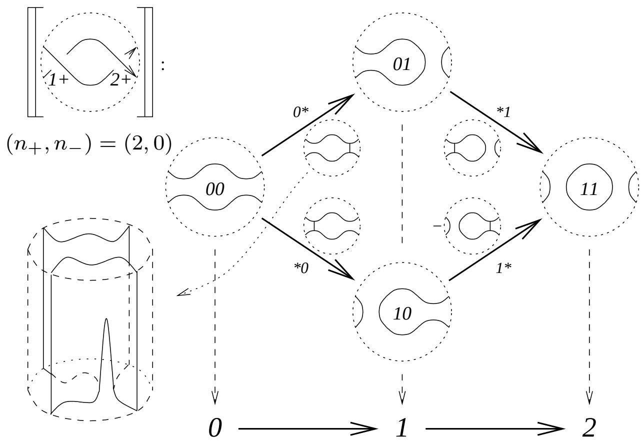
Each edge d_ξ of the n dimensional hypercube $\{0,1\}^n$, illustrated above for the trefoil knot, corresponds to changing between a 0-resolution and a 1-resolution of a link diagram L. In each case, we can associate the edge d_ξ with a 2-dimensional cobordism containing exactly one critical point going between the resolutions. The differentials d^r for the complex $\llbracket L \rrbracket$ are defined by applying Z_{Kh} to the elementary cobordism d_ξ on each edge of the hypercube, summing the maps at each height, and applying minus signs using a convention to be discussed prior to Definition 4.2.3.

The Khovanov homology $\mathrm{Kh}(L)$ is then defined to be the homology of the complex

$$\mathcal{C}(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}, \tag{4.1.11}$$

where $\cdot[s]$ is the operator that shifts complexes s units to the right: $[\![L]\!][s]^r := [\![L]\!]^{r-s}$. The resulting homology groups are bigraded and it was shown in [12] that the graded dimensions of each homology group are an invariant of the link diagram L. In particular, the graded Euler characteristic χ_q of the complex $\mathcal{C}(L)$ produces the unnormalized Jones polynomial. Bar-Natan has shown [13, 14] that the Khovanov homology is a strictly stronger knot invariant meaning that there are knots and links that have the same Jones polynomial, but which can be distinguished by their Khovanov homology.

4.1.3 Bar-Natan's 'picture world'



The natural extension of Khovanov's work is to study the composition properties of the homology theory by defining a similar version for tangles rather than links. Bar-Natan's insight into constructing tangle homologies was to avoid passing into algebraic categories as long as possible and study the composition properties of Khovanov homology using formal complexes of diagrams — the 'picture world'. In the picture world we associate a hypercube of resolutions to a tangle diagram T and formal sums of surfaces to the edges of the cube of resolutions. The degree shift of the resulting complex is kept track of using the dashed arrows pointing to the degree that corresponds to a given height in the cube. As above, the degree of the complex is shifted by $[-n_-]$ where n_- is the number of negative crossings of the tangle diagram. An example taken from [23] is displayed on the right. The key feature is that the components in the resolutions of a tangle consist, not only of circles, but also arcs: the corresponding cobordisms between the resolutions now have corners!

Within this picture world, Bar-Natan demonstrated the nice composition properties present in the general construction of Khovanov homology. He proved the invariance under Reidemeister moves within the formal picture world as well. At the end of the day, one restricts to the subset of tangle diagrams that are links and the picture world is transformed into a computable algebraic invariant by applying a suitably chosen *closed* topological quantum field theory to the surfaces in the formal direct sums. This turns formal direct sums of surfaces into honest direct sums of the linear maps associated to the surfaces by the TQFT.

The only thing missing from Bar-Natan's construction is an algebraic characterization of the TQFTs required to represent all of his surfaces algebraically. By restricting to links, the issue of algebraically representing cobordisms with corners has been avoided. With the ground work laid for topological quantum field theories with corners (Chapter 2), we are now in a position to translate all of the picture world of Bar-Natan into the language of algebra.

In the following sections we describe the required theory needed to turn Bar-Natan's picture world into open-closed cobordisms. Once this is done, the formal direct sums of open-closed cobordisms can be represented algebraically using a suitable choice of open-closed TQFT, or equivalently a suitable choice of knowledgeable Frobenius algebra.

4.2 Tangle homology

The aim of this chapter is to demonstrate an application of open-closed cobordisms by constructing tangle homology theories. To make this application accessible to a broader audience, and to ease the exposition, we will neglect the additional structure of gradings and filtrations on our tangle homology theories. The reader should note that because we are considering the simplified case in which gradings and filtrations are ignored, the tangle homologies defined in this thesis will not have a direct relationship with the Jones polynomial. The reader who is interested in full fledged tangle homology theories arising from open-closed TQFTs should consult the paper [3] where a detailed discussion of gradings and filtrations is supplied and many more examples are given.

Although the tangle homologies described in the chapter lack filtrations and gradings, they do exhibit a novel feature not present in the tangle homology considered by Khovanov [73]. Namely, the tangle homology presented below preserves the monoidal structure of the category of tangles. In particular, the complex corresponding to the disjoint union of two tangles is the tensor product of the complexes associated to the constituents. This property is a crucial feature that must be present if one is interested in representing the full braided monoidal 2-category of 2-tangles via a braided monoidal 2-functor. For more on braided monoidal 2-categories and their relationship to 2-tangles, the reader is referred to the papers [74–78].

We now proceed to adapt Bar-Natan's tangle homology construction to the context of open-closed cobordisms. This section draws heavily on the material presented in [12,13,23,73].

4.2.1 Commutative and skew-commutative cubes

Here we review the definitions of commutative and skew-commutative cubes in order to set up our notation. For more details we refer the reader to Sections 3.2-4.4 of [12], or the review in Section 3.3 of [73].

A commutative cube is a hypercube where every square face forms a commutative diagram. This idea is formalized by letting $\mathcal{I}$ be a finite set of cardinality $|\mathcal{I}|$ and denoting by $r(\mathcal{I})$ the set of all pairs $(\mathcal{L}, a)$ where $\mathcal{L}$ is a subset of $\mathcal{I}$ and $a \in \mathcal{I} \backslash \mathcal{L}$. We write the finite set $\{a, b, \dots, d\}$ as $ab \dots d$ for simplicity and the disjoint union $\mathcal{L}_1 \sqcup \mathcal{L}_2$ of two sets by $\mathcal{L}_1\mathcal{L}_2$ so that $\mathcal{L}a$ denotes the set $\mathcal{L} \sqcup \{a\}$.

Definition 4.2.1. A *commutative $\mathcal{I}$-cube* X over a category $\mathcal{C}$ assigns an object $X(\mathcal{L})$ of $\mathcal{C}$ to each subset $\mathcal{L}$ of $\mathcal{I}$ and a morphism $\xi_a^X(\mathcal{L})\colon X(\mathcal{L}) \to X(\mathcal{L}a)$ to each $(\mathcal{L}, a) \in r(I)$ such that the diagram

$$\begin{tikzcd} X(\mathcal{L}) \arrow[r, "\xi_a^X(\mathcal{L})"] \arrow[d, "\xi_b^X(\mathcal{L})"'] & X(\mathcal{L}a) \arrow[d, "\xi_b^X(\mathcal{L}a)"] \\ X(\mathcal{L}b) \arrow[r, "\xi_a^X(\mathcal{L}b)"'] & X(\mathcal{L}ab) \end{tikzcd} \tag{4.2.1}$$

commutes for any triple $(\mathcal{L}, a, b)$ where $\mathcal{L} \subset \mathcal{I}$, and $a, b \in \mathcal{I} \setminus \mathcal{L}$, $a \neq b$. The morphisms $\xi_a^X(\mathcal{L})$ are called the *structure maps* of X and we will often refer to them as *edges* of the cube X. A *skew-commutative $\mathcal{I}$-cube* over an additive category $\mathcal{C}$ is defined in the same way as a commutative $\mathcal{I}$-cube, except we require that (4.2.1) anti-commutes for every triple $(\mathcal{L}, a, b)$ where $\mathcal{L} \subset \mathcal{I}$, and $a, b \in \mathcal{I} \setminus \mathcal{L}$, $a \neq b$.

If $\mathcal{C}$ is a monoidal category then one can define the internal and external tensor product of commutative $\mathcal{I}$-cubes in the obvious way. The internal tensor product of two $\mathcal{I}$-cubes X and Y is an $\mathcal{I}$-cube, denoted $X \otimes Y$, with $(X \otimes Y)(\mathcal{L}) = X(\mathcal{L}) \otimes Y(\mathcal{L})$ and structural maps defined in the obvious way using the tensor product of morphisms in $\mathcal{C}$. The external tensor product of an $\mathcal{I}$-cube X with an $\mathcal{I}'$-cube Y is an $\mathcal{I}\mathcal{I}'$-cube $(X \boxtimes Y)(\mathcal{L}\mathcal{L}') = X(\mathcal{L}) \otimes Y(\mathcal{L}')$ where $\mathcal{L} \subset \mathcal{I}$ and $\mathcal{L}' \subset \mathcal{I}'$ and the obvious structure maps.

Using the internal tensor product of $\mathcal{I}$-cubes one can turn a commutative I-cube into a skew-commutative $\mathcal{I}$-cube using the skew-commutative $\mathcal{I}$-cube $E_{\mathcal{I}}$ over the category of abelian groups defined as follows: For a finite set $\mathcal{L}$ let $o(\mathcal{L})$ denote the set of total orderings of $\mathcal{L}$. For $x, y \in o(\mathcal{L})$ let $p(x, y) = 0$ if y can be obtained from x by an even number of transpositions of two elements and $p(x, y) = 1$ otherwise. Let $E(\mathcal{L})$ be the free abelian group generated by the set of $x \in o(\mathcal{L})$ modulo the relations $x = (-1)^{p(x,y)}y$ for all pairs $x, y \in o(\mathcal{L})$. Hence, $E(\mathcal{L})$ is isomorphic to $\mathbb{Z}$ and for $a \in \mathcal{I} \setminus \mathcal{L}$ the map $o(\mathcal{L}) \to o(\mathcal{L}a)$ that takes $x \in o(\mathcal{L})$ to $ax \in o(\mathcal{L}a)$ induces an isomorphism $E(\mathcal{L}) \cong E(\mathcal{L}a)$. This definition ensures that setting $E_{\mathcal{I}}(\mathcal{L}) = E(\mathcal{L})$ for $\mathcal{L} \subset \mathcal{I}$ together with the isomorphisms above makes $E_{\mathcal{I}}$ into a skew-commutative $\mathcal{I}$-cube.

Given a commutative $\mathcal{I}$-cube X over an additive category $\mathcal{C}$, taking the internal tensor product $X \otimes E_{\mathcal{I}}$ forms a skew-commutative $\mathcal{I}$-cube over $\mathcal{C}$. Essentially, tensoring a commutative $\mathcal{I}$-cube with $E_{\mathcal{I}}$ has the effect of consistently 'sprinkling' minus signs onto certain edges of the cube so that each square face anti-commutes. The cubes in Section 4.1 are examples of skew-commutative $\mathcal{I}$-cubes. Note that omitting the minus signs placed on each edge would form commutative cubes.

The importance of skew commutative cubes is given by the following:

Definition 4.2.2. Let X be a skew-commutative $\mathcal{I}$-cube over an additive category $\mathcal{C}$. Define a complex $C(X) = \big(C^i(X), d^i\big)$, $i \in \mathbb{Z}$ of objects of $\mathcal{C}$ by

$$C^i(X) = \bigoplus_{\mathcal{L} \subset \mathcal{I}, |\mathcal{L}| = i} X(\mathcal{L}) \tag{4.2.2}$$

with differentials $d^i \colon C^i(X) \to C^{i+1}(X)$ given by the sum of the structure maps of X. Specifically, for $x \in X(\mathcal{L})$, $|\mathcal{L}| = i$, then

$$d^i(x) = \sum_{a \in \mathcal{I} \setminus \mathcal{L}} \xi_a^X(\mathcal{L})x. \tag{4.2.3}$$

Skew-commutativity of X ensures that $d^2 = 0$.

4.2.2 Constructing the complex $[T]$

Let $\overline{\mathbf{2Cob}^{\mathrm{ext}}}$ be the category obtained from $\mathbf{2Cob}^{\mathrm{ext}}$ by formally enriching it over abelian groups. Hence, $\overline{\mathbf{2Cob}^{\mathrm{ext}}}$ is the pre-additive category whose objects are the objects of $\mathbf{2Cob}^{\mathrm{ext}}$ and whose Hom sets $\overline{\mathbf{2Cob}^{\mathrm{ext}}}(\vec{n}, \vec{n'})$ are formal $\mathbb{Z}$-linear combinations of the morphisms $\mathbf{2Cob}^{\mathrm{ext}}(\vec{n}, \vec{n'})$. The additive closure of $\overline{\mathbf{2Cob}^{\mathrm{ext}}}$, denoted by $\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})$, is the additive category whose objects are formal direct sums (possibly empty) $\oplus_{i=1}^n \vec{n}_i$ of objects $\vec{n}_i$ of $\overline{\mathbf{2Cob}^{\mathrm{ext}}}$, and whose morphisms $F \colon \bigoplus_{i=1}^n \vec{n}_i \to \bigoplus_{j=1}^m \vec{n}'_j$ are $n \times m$ matrices of morphisms $F_{ij} \colon \vec{n}_i \to \vec{n}'_j$ of $\overline{\mathbf{2Cob}^{\mathrm{ext}}}$. Morphisms are added using matrix addition and composition is defined analogous to matrix multiplication: if $F\colon \bigoplus_{i=1}^{n} \vec{n}_i \to \bigoplus_{j=1}^{m} \vec{n}'_j$ and $G\colon \bigoplus_{j=1}^{m} \vec{n}'_j \to \bigoplus_{k=1}^{\ell} \vec{n}''_k$, then GF has components given by

$$\big((G_{jk}) \circ (F_{ij})\big)_{ik} := \sum_j G_{jk} \circ F_{ij}. \tag{4.2.4}$$

The aim of this section is to define for every tangle diagram T a commutative cube over the additive category $\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})$. Tensoring this cube with the skew commutative cube $E_{\mathcal{I}}$ allows one to associate a chain complex $[\![T]\!]$ to the tangle diagram T using Definition 4.2.2.

In the context in which our commutative cubes will occur, we can identify $\mathcal{I}$ with the set of n numbered crossings corresponding to a link or tangle diagram T. Each subset $\mathcal{L}$ then corresponds to a subset of the n crossings of T. The object $X(\mathcal{L})$ of $\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})$ corresponding to the subsets $\mathcal{L}$ can be identified with the resolutions S_α of T by assigning to $\mathcal{L} \subset \mathcal{I}$ the resolution of T where each crossing in the set $\mathcal{L}$ has been resolved by the 1-smoothing and those crossings in $\mathcal{I} \setminus \mathcal{L}$ have been resolved by the 0-smoothing. Hence, to each vertex α of the hypercube $\{0,1\}^n$ we associate a complete smoothing S_α of the tangle diagram T. In the next subsection we will explain how to orient and order the resolutions S_α so that they can be identified as objects of $\mathbf{2Cob}^{\mathrm{ext}}$.

The structure maps $\xi_a^X(\mathcal{L})\colon X(\mathcal{L}) \to X(\mathcal{L}a)$ then correspond to maps between the resolutions of T in which the resolution corresponding to $X(\mathcal{L}a)$ differs from $X(\mathcal{L})$ by having exactly one more 1-smoothing than $X(\mathcal{L})$ and being identical everywhere else. Following Bar-Natan we will denote the structure maps as d_ξ for simplicity, where it is understood that ξ is a sequence in $\{0,1,\star\}^n$ with just a single $\star$. The source of the structure map is found by setting $\star$ to 0 and the target is found by setting $\star$ to 1.

This convention also allows for a convenient rule for inserting the appropriate minus signs on the edges of a commutative cube, that is, the effect of tensoring a commutative $\mathcal{I}$-cube with $E_{\mathcal{I}}$. The edge d_ξ gets multiplied by $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$, where j is the location of $\star$ in ξ. This is just (-1) to the power of the number of 1's in the sequence ξ prior to the $\star$. As was explained in the background section of this chapter, if we define the height of the sequence $\alpha = (\alpha_1, \cdots, \alpha_n)$ to be $\sum_i \alpha_i$ and the height $|\xi|$ of the edge ξ to be the height of its source, then the complex defined in Definition 4.2.2 can be presented in simplified form as follows:

Definition 4.2.3. Let $\mathbf{n}$ be the finite set of cardinality n and let X be a commutative $\mathbf{n}$-cube over $\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})$. Define the complex $C(X) = \big(C^i(X), d^i\big)$, $i \in \mathbb{Z}$ of objects in

Mat(**2Cob**$^{\mathrm{ext}}$) by

$$C^i(X) := \bigoplus_{\alpha : i = |\alpha|} S_\alpha, \tag{4.2.5}$$

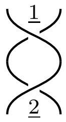
with differential

$$d^i := \sum_{|\xi|=i} (-1)^\xi d_\xi. \tag{4.2.6}$$

Ordering the resolutions of tangle diagram

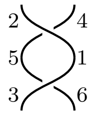
Conventions: For T an oriented tangle diagram with n crossing we denote by n_+ the number of positive crossings and n_- the number of negative ones. Every crossing $(\times)$ has two smoothings, the 0-smoothing $(\asymp)$ and the 1-smoothing $()()$.

Given a tangle diagram T, begin by arbitrarily numbering each crossing.

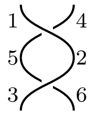


(4.2.7)

This provides an ordering of the resolutions of T and we will show in Proposition 4.2.5 that the complex $[\![T]\!]$ does not depend on this choice of numbering. For notational reasons it is also convenient to label the edges of T. We reserve a separate set of integers for this (not underlined). For example,



or



(4.2.8)

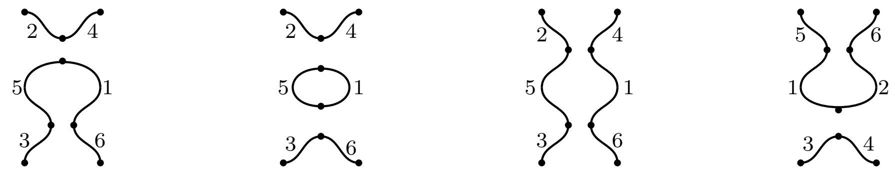
both chosen arbitrarily without any prescribed labeling system (compare with the notational convention in Section 3.3 of [13]).

For a tangle diagram T with n crossings and for $\alpha \in \{0,1\}^n$, the enumeration of the edges and crossings of T as described above is used to provide an ordering of the components of the smoothings S_α of T. A total ordering of the components of S_α is needed in order to interpret S_α as an object of **2Cob**$^{\mathrm{ext}}$. Recall that the objects $\vec{n}$ of **2Cob**$^{\mathrm{ext}}$ are just sequences $(n_1, n_2, \cdots, n_k)$, $k \in \mathbb{N}_0$, with $n_j \in \{0,1\}$ and each 1 in the sequence represents the diffeomorphism type of an interval (arc) and each 0 in the sequence represents the diffeomorphism type of the circle. Once the resolutions S_α have been ordered we will then identify each resolution as an object $\vec{n} \in$ **2Cob**$^{\mathrm{ext}}$ where each arc in the resolution is assigned a 1, each circle in the resolution is assigned a 0, and the sequence is ordered using an ordering convention induced from the enumeration of the tangle diagram T.

The ordering convention is given by labeling each circle and arc in S_α by the minimal label of an edge that appears in it. Using this ordering we can then apply a 2-dimensional open-closed TQFT $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathbf{Vect}_k$ and interpret the resolution S_α as an ordered tensor product of copies of the vector space A associated to the interval and the vector space C associated to the circle.

For example, using the first choice of ordering given in (4.2.8) with the crossings resolved in the order specified by (4.2.7), we have[3]

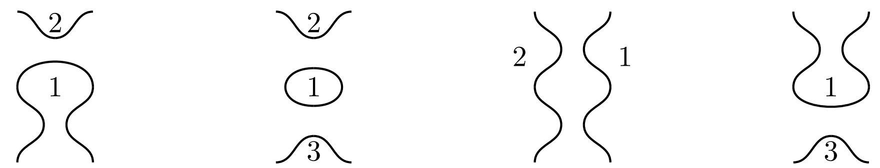
00-smoothing | 01-smoothing | 10-smoothing | 11-smoothing



(4.2.9)

which leads to the following ordering of the components and tensor product of vector spaces

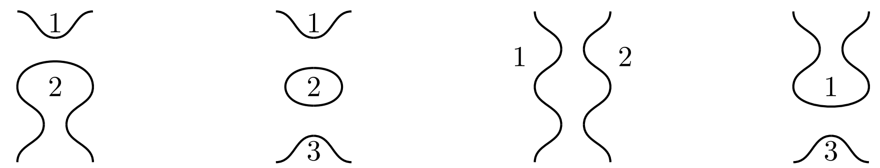
$A_1 \otimes A_2$ | $C_1 \otimes A_2 \otimes A_3$ | $A_1 \otimes A_2$ | $A_1 \otimes A_3$



(4.2.10)

If we had used the numbering given in the second example of (4.2.8) then we would have obtained the ordering

$A_1 \otimes A_2$ | $A_1 \otimes C_2 \otimes A_3$ | $A_1 \otimes A_2$ | $A_1 \otimes A_3$



(4.2.11)

Note that in the above the subscripts are simply for bookkeeping, in particular, $A_1 \otimes C_2 \otimes A_3$ is the same as $A \otimes C \otimes A$. In Proposition 4.2.5 we will also show that this choice of labeling, again, does not effect the isomorphism class of the complex $[T]$.

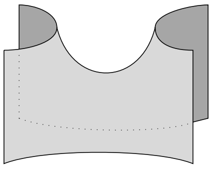
Orienting the resolutions of a tangle diagram

In the previous section we used an open-closed TQFT to assign an ordered tensor product of vector spaces to the resolutions of a tangle diagram. To construct the structure maps of a commutative $\mathcal{I}$-cube whose vertices are the resolutions of the tangle diagram T, we will need to interpret each edge as a cobordism between the various resolutions. To interpret

[3]For tangle homologies derived from state sum TQFTs these resolutions can quite literally be thought of as being glued together from these numbered arcs.

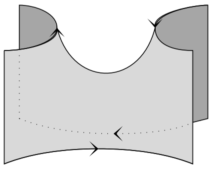
these cobordisms algebraically as maps between the corresponding vector spaces we impose a convention for orienting the resolutions of a tangle diagram T so that the cobordisms between resolutions can be interpreted as morphisms in the category $\mathbf{2Cob}^{\mathrm{ext}}$. Just as the Kauffman bracket does not depend on the orientation of the tangle T, so too this convention will not depend on the orientation of T.

To understand the importance of orienting the resolutions of a tangle diagram consider the cobordism M going between the two resolutions of the elementary right handed crossing (⤫). The cobordism M has boundary consisting of the two smoothings, the 0-smoothing ()() and the 1-smoothing (≍), and is traditionally drawn as a saddle.

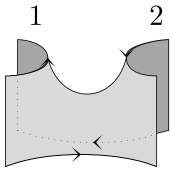


(4.2.12)

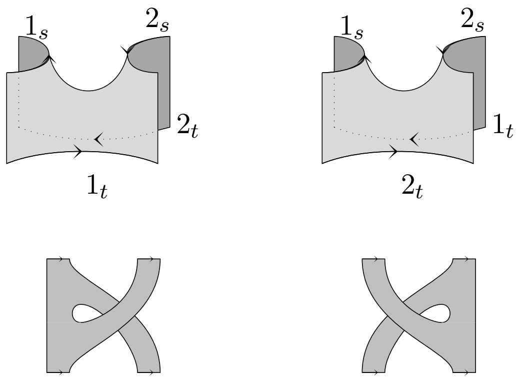
For this saddle to correspond to an open-closed cobordism we must equip it with an orientation,



but the open-closed cobordism that this corresponds to still depends on the embedding used to interpret its boundary as objects of $\mathbf{2Cob}^{\mathrm{ext}}$. By numbering the arcs in the tangle diagram T we have induced an ordering on the components of the resolution. For example, suppose the source is ordered as follows:



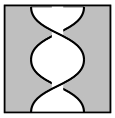
then the corresponding open-closed cobordism depends on the ordering of the target. Specifically, we have



(4.2.13)

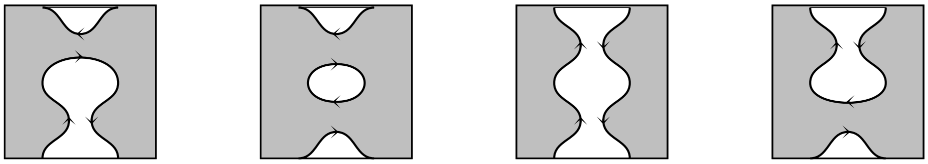
where the choice of open-closed cobordism is determined by whether flowing along the orientation of the source arc labeled 1_s flows into the target arc labeled 1_t or 2_t. Hence, the ordering of the components and the orientation scheme used will affect the choice of open-closed cobordism made later on.

Our convention for orienting the components of the resolutions of the tangle diagram T is most easily explained by colouring the tangle diagram T. The colouring is determined by shading the region to the left of the first arc in the tangle T and shading the rest of the regions in a checkerboard fashion. When there is no first strand we colour the outermost region and continue in a checkerboard fashion.



(4.2.14)

This shading induces a shading on the resolutions of the tangle diagram T, and we orient the components of each resolution counter-clockwise around shaded regions. For example, the resolutions of the tangle in (4.2.7) are oriented as follows:



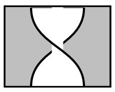
(4.2.15)

Often we will omit the shading and draw only the induced orientations.

Constructing the commutative cube associated to a tangle diagram

Given a tangle diagram T with n crossings equipped with an enumeration of its arcs and crossings, define the vertices of the commutative $\mathcal{I}$-cube $\widehat{X}_T$ by associating to each $\alpha \in \{0,1\}^n$ the smoothing S_α of T equipped with the ordering and orientation of its components. Each edge d_ξ of the commutative cube $\widehat{X}_T$ can then be thought of as a map between objects in $\mathbf{2Cob}^{\mathrm{ext}}$. Each map $d_\xi \colon S_\alpha \to S_{\alpha'}$ maps a resolution S_α to a resolution $S_{\alpha'}$ where α' has exactly one more 1-smoothing than α. Just as in the Bar-Natan tangle homology, we construct such a map using the saddle surface (4.2.12) for each transition from a 0-smoothing to a 1-smoothing; the saddle is translated into an open-closed cobordism using the orientation and ordering of the components in the resolution as in (4.2.13).

Corresponding to the tangle

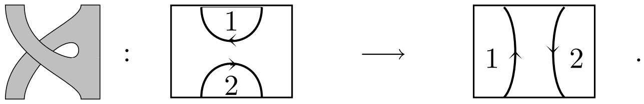


(4.2.16)

with enumeration of the arcs such that the resolutions are ordered

(4.2.17)

we associate the map

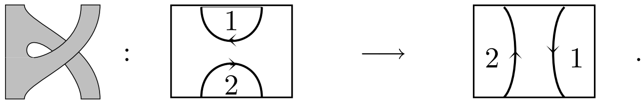


(4.2.18)

For enumerations of the tangle (4.2.16) inducing the ordering

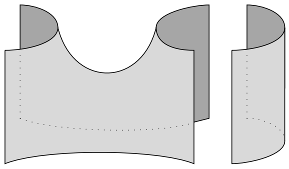
(4.2.19)

we associate the map



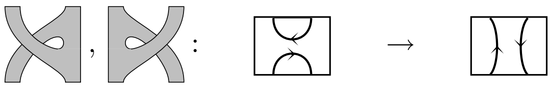
(4.2.20)

For the tangle homologies constructed from state sum open-closed TQFTs (see Section 4.3) only the saddles featured above will be needed. The composition properties of the state sum tangle homologies ensure that when a saddle is part of diagram like the following:

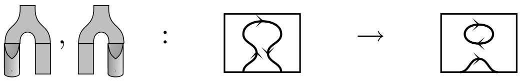


(4.2.21)

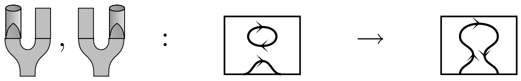
then the associated open-closed cobordism is obtained from one of two saddles above by gluing identities appropriately. For non state sum TQFTs we can construct tangle homology theories that do not admit a composition, see Example (4.5.5), by assigning the following maps d_ξ:



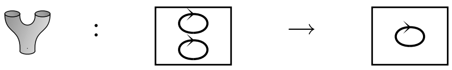
(4.2.22)



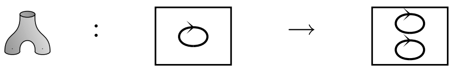
(4.2.23)



(4.2.24)



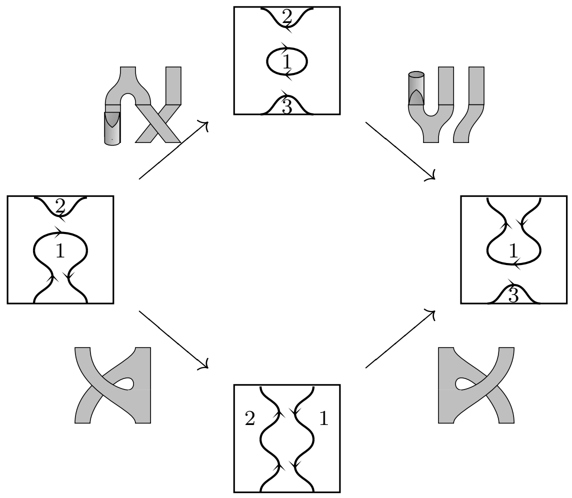
(4.2.25)



(4.2.26)

where appropriate. The first three depend on the ordering of the components.

We illustrate the construction of the commutative $\mathcal{I}$-cube $\widehat{X}_T$ for the tangle diagram given by (4.2.7) with the first enumeration given in (4.2.8)



(4.2.27)

where all open-closed cobordisms are still read from top to bottom, *i.e.*, their source is at the top and their target at the bottom. It is immediate from the invariants of open-closed cobordisms given in [29] that the composites are equal since they have the same genus, window number, and boundary permutation. In fact, Corollary 2.6.7 provides a sequence of diffeomorphisms which relates the two open-closed cobordisms.

The complex $[T]$ and the independence of the enumeration

The complex $[T]$ associated to a tangle diagram is constructed from the skew commutative $\mathcal{I}$-cube $X_T := \widehat{X}_T \otimes E_{\mathcal{I}}$ where $E_{\mathcal{I}}$ is the skew commutative $\mathcal{I}$-cube defined in Section 4.2.1. We denote the vertex $\alpha \in \{0,1\}^n$ as X_T^α which we identify with the resolution S_α of T equipped with its ordering and orientation of its components induced by the enumeration and colouring of T.

We now define the formal complex of open-closed cobordisms that will be shown in Theorem 4.4.4 to be a tangle invariant in an appropriate quotient of $\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})$.

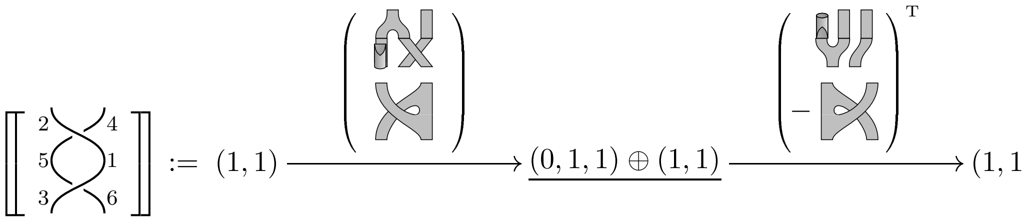
Definition 4.2.4. Let T be an oriented tangle diagram equipped with a numbering of its crossings and arcs. Define the complex $\widetilde{[T]}$ in the category $\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})$ by setting $\widetilde{[T]}^r := \bigoplus_{\alpha:r=|\alpha|} X_T^\alpha$. The differential is given, as in Definition 4.2.3, by $d^r := \sum_{|\xi|=r}(-1)^\xi d_\xi$ where the d_ξ are the edges of X_T determined up to sign by the equations (4.2.22)–(4.2.26). That is, $\widetilde{[T]}$ is just the complex defined in Definition 4.2.3 associated to the commutative cube $\widehat{X}_T$.

The *formal Khovanov bracket*[4] is defined from $\widetilde{[\![T]\!]}$ as

$$[\![T]\!] := \widetilde{[\![T]\!]}[-n_-] \tag{4.2.28}$$

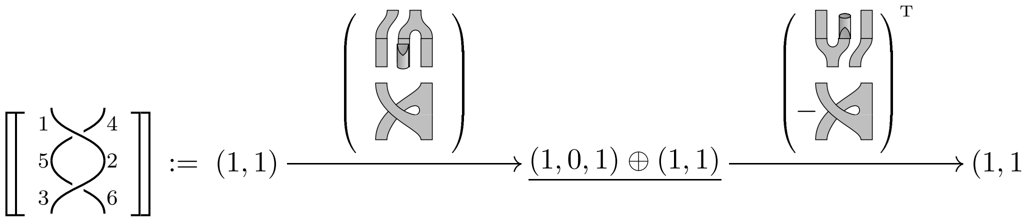
where n_- is the number of negative crossings in the oriented tangle diagram T and where $\cdot[s]$ is the operator that shifts complexes s units to the right: $\widetilde{[\![T]\!]}[s]^r := \widetilde{[\![T]\!]}^{r-s}$.

After identifying the resolutions S_α with objects in $\mathbf{2Cob}^{\mathrm{ext}}$, the complex associate to the cube in example (4.2.27) is then



(4.2.29)

where we have underlined the chain space in degree 0 and $^{\mathrm{T}}$ denotes the matrix transpose. Had we used the second arc numbering in (4.2.8) then we would have got the complex



(4.2.30)

but as the following proposition shows, these two complexes are isomorphic.

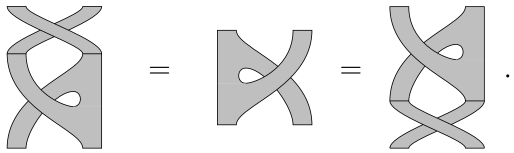
Proposition 4.2.5. The isomorphism class of the complex $[\![T]\!]$ associated to a tangle diagram T does not depend on the numbering of its crossings or on the numbering of its arcs.

Proof. Just as in other presentations of link/tangle homology, permuting the numbering of the crossings merely permutes the resolutions at a given height which does not affect the isomorphism class of the resulting complex.

To see that the labeling of the arcs in the tangle diagram T does not effect the chain isomorphism type of the complex $[\![T]\!]$, suppose that a given labeling leads to an ordering of the components of the resolution S_α and that σ_α is the permutation which relates this ordering to an ordering induced from a different labeling. Then the collection of maps $f^r := \bigoplus_{\alpha: r=|\alpha|-n_-} \sigma_\alpha$ (note the degree shift) define a chain isomorphism between the two complexes. That the

[4]Here we are using the double bracket $[\![\cdot]\!]$ notation in a different manor than in Section 4.1. This is why we are referring to it as the *formal* Khovanov bracket. This is the same convention employed by Bar-Natan in the paper [23].

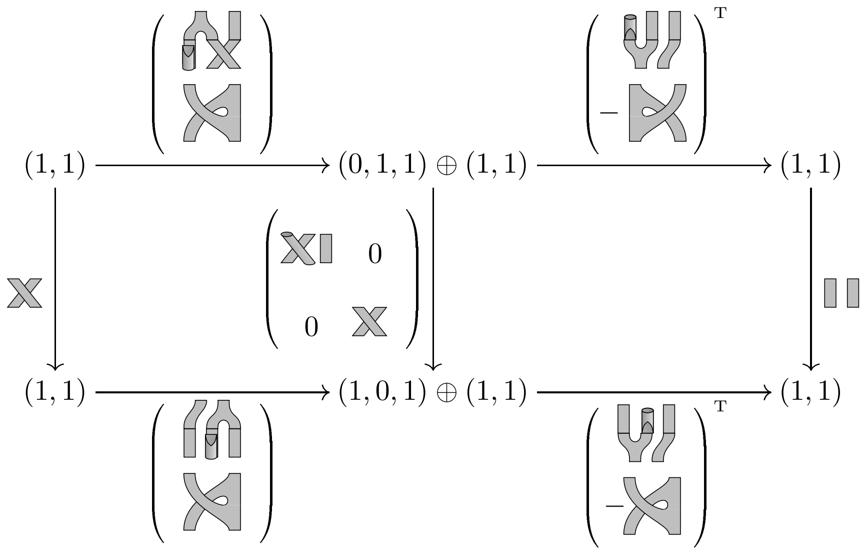
collection f^r do in fact form a chain map follows from the relations of Proposition 2.5.10 and the identities



(4.2.31)

□

Below we exhibit this chain isomorphism between the complexes (4.2.29) and (4.2.30):



(4.2.32)

At this point we have merely translated Bar-Natan's 'picture world' construction of tangle homology into the language of open-closed cobordisms. The advantage of this translation only becomes apparent once the formal complex $[\![T]\!]$ of open-closed cobordisms is evaluated using an open-closed TQFT $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathbf{Vect}_k$. Bar-Natan's proof can be used to show that the complex is Reidemeister move invariant, and this already yields a generalization of Khovanov homology from links to tangles. Whereas Bar-Natan's surfaces can be glued so as to represent the composition of tangles, the resulting chain complexes of vector spaces are not necessarily equipped with an operation representing the composition of tangles. In the next section we will consider a class of open-closed TQFTs that do admit such a gluing operation for the corresponding chain complexes of vector spaces.

4.3 State sum tangle homology

The algebraic operation that represents the composition of tangles, is inspired by the state sum construction of open-closed TQFTs presented in Chapter 3. The mechanism by which one can subdivide edges in the black boundary of an open-closed cobordism, provides us with the blue print for an algebraic operation in order to compose tangles. For example, triangulate the cobordism μ_C corresponding to the pair of pants. Then the three circles in its black boundary are composed from several edges of the triangulation. This composition of the boundary edges to the circle is our guiding example for defining the composition of arcs.

To make this mechanism work, we impose the additional conditions (see Theorem 3.2.18) that hold for those open-closed TQFTs that we can construct from a state sum.

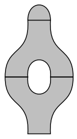
Definition 4.3.1. Let $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathbf{Vect}_k$ be an open-closed TQFT and let $(A, C, \imath, \imath^*)$ be the associated knowledgeable Frobenius algebra, *i.e.* $A := Z((1))$, $C := Z((0))$. The open-closed TQFT Z is called a *state sum open-closed TQFT* if the following conditions are satisfied:

1. The algebra A is strongly separable.

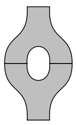
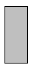
2. The algebra $C = Z(A)$ is the centre of A.

3. The linear map $\imath\colon Z(A) \to A$ is the canonical inclusion.

If the algebra A is strongly separable, then the window element $a = \mu_A \circ \Delta_A \circ \eta_A\colon k \to A$ associated with the surface

$$a = Z(\ \ )$$  (4.3.1)

has a convolution inverse $a^{-1}\colon k \to A$, *i.e.* $\mu_A \circ (a \otimes a^{-1}) = \eta_A = \mu_A \circ (a^{-1} \otimes a)$. Both a and a^{-1} are central, *i.e.* for any $\varphi\colon A \to A$, we have $\mu_A \circ (a \otimes \varphi) = \mu_A \circ (\varphi \otimes a)$. In a strongly separable algebra, multiplication with the inverse of the window element can be used to remove holes (windows). By this, we mean that

$$\mu_A \circ \big(a^{-1} \otimes Z(\ \ )\big) = \mathrm{id}_A = Z(\ \ ),$$   (4.3.2)

i.e. multiplication by a^{-1} on the algebraic side has the same effect as removing a window on the topological side.

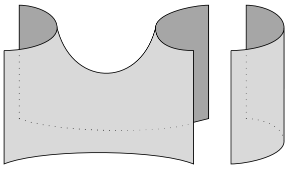
If one wishes to compose tangles, say a crossing (⤫) with an arc (⊃) in order to get (⤫⊃), one needs to compose the extended cobordisms d_ξ of (4.2.22)–(4.2.26) in a suitable fashion.

Let $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathbf{Vect}_k$ be an open-closed TQFT. For a single crossing (⤫), we have the two smoothings $S_0 =)($ and $S_1 = \asymp$. With such a crossing (⤫), we associate the 2-term chain complex of vector spaces $Z(d_\star)\colon Z(S_0) \to Z(S_1)$ whose boundary operation is obtained from the saddle $d_\star$ depicted in (4.2.12). This saddle is now viewed as an open-closed cobordism, and so it gives a map $Z(d_\star)\colon A \otimes A \to A \otimes A$. With the arc), we associate the vector space A, viewed as a 1-term chain complex.

The tensor product of the two chain complexes is the following 2-term chain complex

$$Z(S_0) \otimes A \xrightarrow{Z(d_\star)\otimes \mathrm{id}_A} Z(S_1) \otimes A \tag{4.3.3}$$

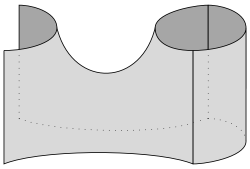
associated with the open-closed cobordism



. (4.3.4)

This diagram shows a disjoint union of two open-closed cobordisms for which the open-closed TQFT yields the tensor product of vector spaces and linear maps of (4.3.3).

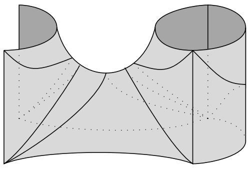
We would like to glue the two open-cobordisms along their coloured boundaries like this,



(4.3.5)

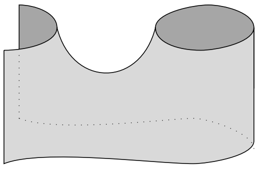
in order to obtain a cobordism from the 0-smoothing ( ) of ( ) to the 1-smoothing ( ) of ( ). In general, however, an open-closed TQFT does not have any operation for such a gluing along coloured boundaries.

The state sum construction of open-closed TQFTs from Chapter 3 suggests the following solution to this problem. We assume that the open-closed TQFT is a state sum open-closed TQFT (Definition 4.3.1). We imagine that the composite surface (4.3.5) is triangulated in such a way that the components of the coloured boundary in (4.3.5) along which we want to glue, coincide with edges of the triangulation. This implies that the circle in the source of the open-closed cobordism (4.3.5) is triangulated with two edges and the arc in its target with three edges. Such a triangulation is displayed here:



. (4.3.6)

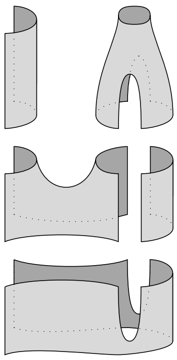
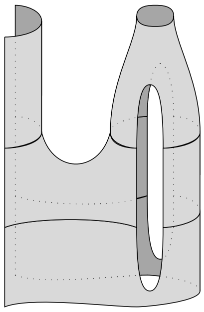
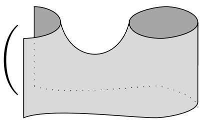
We can now employ the state sum construction presented in Chapter 3 in order to compute the linear map $A \otimes C \to A$ that the open-closed TQFT associates with the open-closed cobordism



$$. \qquad (4.3.7)$$

In the homology theory for generic tangles described in the previous section, this map $A \otimes C \to A$ forms the boundary operation of the 2-term chain complex that relates the two smoothings ()○) and (≍) of the composite tangle diagram (⟩⟨○).

We now present an equivalent way of computing the linear map $A \otimes C \to A$ which makes transparent how this linear map can be computed from the constituents of (4.3.4). In an open-closed TQFT, we cannot compose the two constituents of (4.3.4) by gluing them along their coloured boundary. The idea is to rather pre- and postcompose (4.3.4) with suitable open-closed cobordisms by gluing along their black boundary as follows:

$a^{-2} \cdot Z\Big($  $\Big) = a^{-2} \cdot Z\Big($  $\Big) = Z\Big($  $\Big).$ (4.3.8)

The composition is an open-closed cobordism with two windows, but multiplication by the appropriate power of the inverse window element a^{-1} removes these windows and results in the desired composite open-closed cobordism.

The relationship between the two presentations is best understood by recalling the states sum construction. The state sum construction computes the vector spaces associated to triangulated 1-manifolds by computing the images of a triangulated cylinder over the triangulated 1-manifold. Proposition 3.4.3 implies that this corresponds to the algebraic operation of computing the image of $A \otimes A$ under the image of one of the idempotents P_{kk}, or Q_{kk} defined in Proposition 3.2.19. It may seem less than straightforward to define a rather natural operation like gluing manifolds using such a strange procedure involving idempotents, however, the images of the various idempotents can be characterized by the following universal properties that may be more familiar to the reader.

Proposition 4.3.2. Let $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ be a strongly separable symmetric Frobenius algebra object in some abelian symmetric monoidal category $\mathcal{C}$.

1. The co-image coim $p\colon A \to p(A)$ is the co-equalizer

$$A \otimes A \underset{\mu_A \circ \tau_{A,A}}{\overset{\mu_A}{\rightrightarrows}} A \xrightarrow{\text{coim } p} p(A). \tag{4.3.9}$$

2. The co-image coim $P_{22}\colon A \otimes A \to P_{22}(A \otimes A)$ is the co-equalizer

$$(A \otimes A) \otimes A \underset{(\mathrm{id}_A \otimes \mu_A) \circ \alpha_{A,A,A}}{\overset{\mu_A \otimes \mathrm{id}_A}{\rightrightarrows}} A \otimes A \xrightarrow{\text{coim } P_{22}} P_{22}(A \otimes A). \tag{4.3.10}$$

This means that coim $P_{22} \cong A \otimes_A A$ where A is viewed as an A-left-A-right-bimodule.

The assumption of strong separability is essential here.

Proof. 1. Since coim $p =$ coim $p \circ p$, the morphism coim p satisfies coim $p \circ \mu_A =$ coim $p \circ \mu_A \circ \tau_{A,A}$. Given any morphism $f\colon A \to B$ such that $f \circ \mu_A = f \circ \mu_A \circ \tau_{A,A}$, there is a morphism $\varphi\colon p(A) \to B$ given by $\varphi := f \circ \text{im } p$ such that

$$\varphi \circ \text{coim } p = f \circ p = f \circ \mu_A \circ \tau_{A,A} \circ \Delta_A \circ (a^{-1} \cdot \mathrm{id}_A) = f \circ \mu_A \circ \Delta_A \circ (a^{-1} \cdot \mathrm{id}_A) = f. \tag{4.3.11}$$

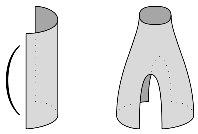
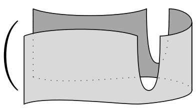
If $\psi\colon p(A) \to B$ also satisfies $f = \psi \circ \text{coim } p$, then $\varphi = f \circ \text{im } p = \psi \circ \mathrm{id}_{p(A)} = \psi$, and so φ is unique with that property.

2. Since coim $P_{22} =$ coim $P_{22} \circ P_{22}$, and because of associativity, we have coim $P_{22} \circ (\mu_A \otimes \mathrm{id}_A) =$ coim $P_{22} \circ (\mathrm{id}_A \otimes \mu_A) \circ \alpha_{A,A,A}$. Given any morphism $f\colon A \otimes A \to B$ such that $f \circ (\mu_A \otimes \mathrm{id}_A) = f \circ (\mathrm{id}_A \otimes \mu_A) \circ \alpha_{A,A,A}$, there is a morphism $\varphi := f \circ \text{im } P_{22}\colon P_{22}(A \otimes A) \to B$ such that $\varphi \circ \text{coim } P_{22} = f$. It can be shown to be unique with that property.

$\square$

This means that the gluing of arcs corresponding to tangle resolutions is simply given by taking the tensor product $A \otimes_A A \cong A$ over A. The benefit of this rather long winded description lies in the fact that gluing the two endpoints of a single arc together naturally leads to the vector space associated to the circle, a property that must be present if our tangle homology theory is to reduce properly to known link homology theories on links.

How do we find the appropriate maps

$\varphi := Z\Bigg($  $\Bigg)$ and $\psi := Z\Bigg($  $\Bigg)$ (4.3.12)

by which to pre- and post-compose? The linear map $Z(d_\star)\colon A \otimes A \to A \otimes A$ associated with the saddle is a morphism of $(A^{\otimes 2}, A^{\otimes 2})$-bimodules, and so the 2-term chain complex associated with the crossing is a chain complex of $(A^{\otimes 2}, A^{\otimes 2})$-bimodules. Similarly, the algebra A

associated with the arc forms an (A, A)-bimodule, and so the 1-term chain complex associated with the arc is a chain complex of (A, A)-bimodules. In general, for each open end of a tangle, we have one action of A.

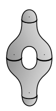
The linear map associated with the composite (4.3.7) is the tensor product of these chain complexes *over* A, using the appropriate left- and right-actions of A that correspond to the open ends of the tangle that are glued. The map ψ is the co-equalizer that defines the tensor product over A, whereas the map φ is the unique map obtained from the universal property of a similar co-equalizer because the boundary map of the chain complex factors through that co-equalizer. (Compare with the definitions of the isomorphisms Ψ_k and Φ_k in Corollary 3.2.20.)

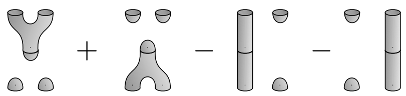
Gluing the saddle diagram, corresponding to the change of a single crossing, to the appropriate identity cobordisms produces the assignments of (4.2.22)–(4.2.26) by construction. The corresponding algebraic operation corresponds to taking the tensor product over A. Hence, for state sum TQFTs the homology of an arbitrary tangle can be computed from its elementary components by taking the tensor product of bimodules.

4.4 Invariance under Reidemeister moves

Denote the category of complexes in the additive category $\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})$ as $\mathbf{Kom}\big(\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})\big)$. The objects of this category are chains of finite length $\cdots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \cdots$ for which $d^i \circ d^{i-1} = 0$ for all i. The morphisms of this category are chain maps. We will only be interested in complexes up to chain homotopy equivalence. Recall that two chain maps $F, G\colon (C^i, d^i) \to (D^i, d^i)$ are homotopic if there exists a collection of maps $h^i\colon C^i \to D^{i-1}$ such that $F^i - G^i = h^{i+1} d^i + d^{i-1} h^i$ for all i. The aim of this section is to show that the complex $[T]$ is a tangle invariant in the category $\mathbf{Kom}\big(\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})\big)$ quotiented out by homotopy equivalence and the following three relations:

 $= 0,$ (4.4.1)

 $= 2,$ (4.4.2)

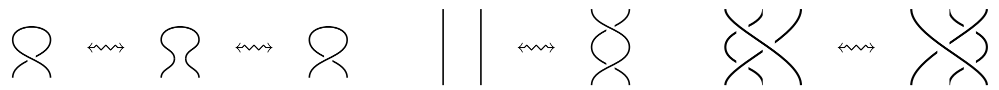
 $= 0.$ (4.4.3)

We denote the resulting quotient as **Kob** for simplicity.

These relations provide sufficient conditions for Reidemeister move invariance. To construct algebraic tangle invariants the knowledgeable Frobenius algebras must preserve these relations. Examples of such knowledgeable Frobenius algebras are given in Section 4.5.1.

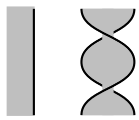
By restricting our attention to state sum TQFTs, we ensure that the corresponding tangle homology can be computed locally. The gluing operations present in state sum tangle homologies ensure that no global information is required to compute the complex associated to a tangle diagram T. We will now adapt Bar-Natan's picture world proof of Reidemeister invariance into the language of open-closed cobordisms. We emphasize that the proof presented below is due to Bar-Natan. Only our presentation in terms of open-closed cobordisms is new. We provide this proof to further illustrate how Bar-Natan's picture world can be translated into open-closed cobordisms.

Because of the nice gluing properties of of the category of tangles we will only need to check Reidemeister invariance for the elementary tangles



(4.4.4)

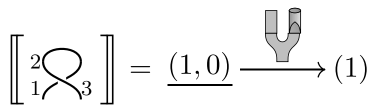
in the skein theoretic sense, meaning that any of the above tangles can appear anywhere in a possibly larger diagram. Note that because our convention for orienting the tangle resolutions involves colouring the tangle, we must take account for both possible colourings of the above diagrams to account for situations in which the region to the left of the first strand is not coloured as in the following example:



(4.4.5)

4.4.1 Invariance under Reidemeister move one

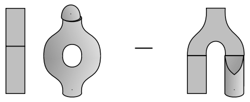
We must show that the formal complex [] $= \left(0 \longrightarrow \underline{(1)} \longrightarrow 0\right)$ is homotopy equivalent to the formal complex



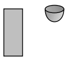
(4.4.6)

where we have underlined the degree zero term of each complex. Note that both of these complex are unchanged by changing the colouring used to orient the resolutions. Hence, we need only consider the one case.

Define the chain map $F\colon$ [] $\to$ [] as follows:

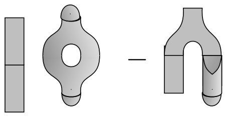
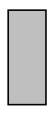
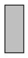
$F^0 :=$  , $\quad F^1 := 0.$ (4.4.7)

It can easily be checked using the topological invariants of open-closed cobordisms that $dF^0 = 0$. Again, a specified sequence of diffeomorphisms relating the two is provided Corollary 2.6.7. Next define the chain map $G\colon$ [] $\to$ [] by

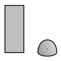
$G^0 :=$ , $\quad G^1 := 0.$ (4.4.8)

there is nothing to check to see that this is a chain map.

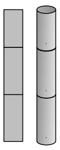
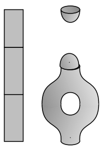
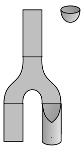
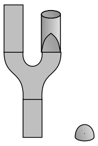
The chain maps F and G define a homotopy equivalence between the complexes [] and []. To see this, first note that $GF = \mathrm{id}$ by (4.4.1) since

$G^0F^0 =$  $= 2$  $-$  $=$  $= \mathrm{id}^0.$ (4.4.9)

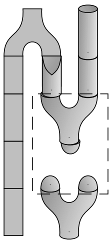
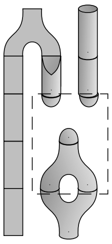
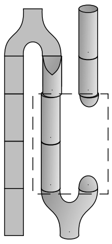
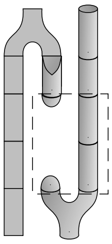
Define the chain homotopy $h\colon \mathrm{id} \to FG$ whose only nontrivial component is given by

$h^1 :=$  . (4.4.10)

The equation $\mathrm{id}^0 - F^0G^0 = h^1d^0$ is depicted as

 $-$  $+$  $=$  . (4.4.11)

After applying appropriate diffeomorphisms and rearranging the terms, this equality follows from the relation (4.4.3)

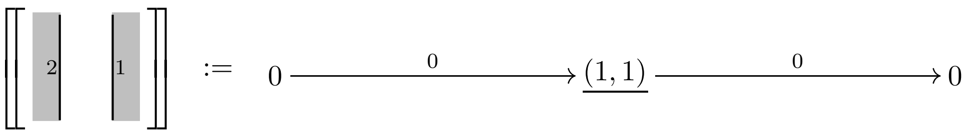
 $+$  $=$  $+$  . (4.4.12)

The equation $\mathrm{id}^1 - F^1G^1 = d^0h^1$ follows from (2.5.41) and (2.5.37).

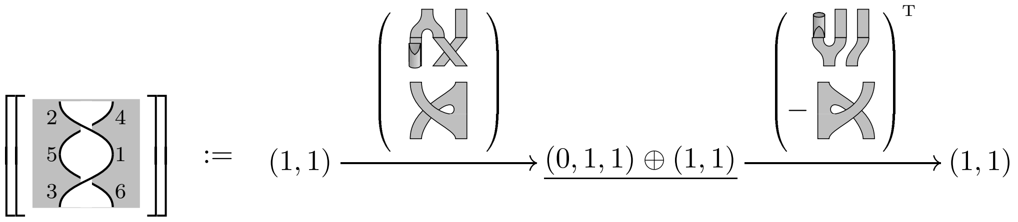
The other version of the first Reidemeister move is proven similarly.

4.4.2 Invariance under Reidemeister move two

We must show that the following complexes

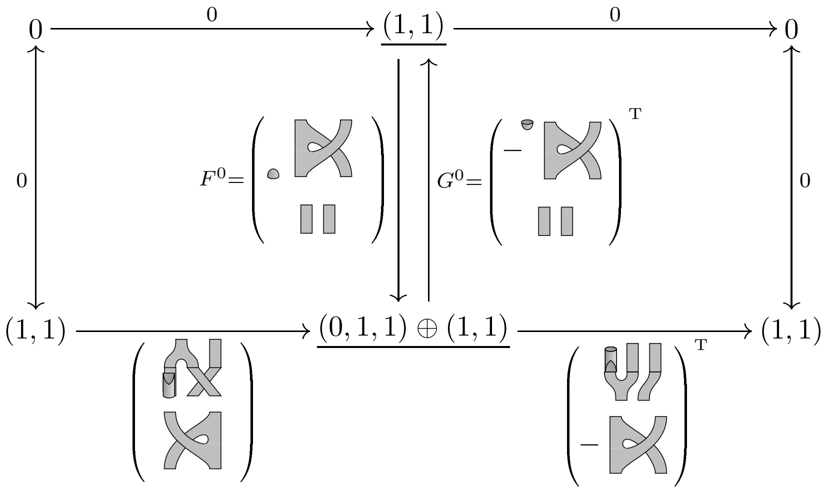


(4.4.13)



(4.4.14)

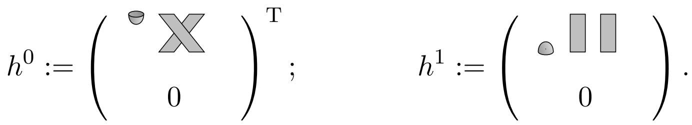
are chain homotopy equivalent. We begin by defining chain maps $F\colon \llbracket\,|\,|\,\rrbracket \to \llbracket \asymp \rrbracket$ and $G\colon \llbracket \asymp \rrbracket \to \llbracket\,|\,|\,\rrbracket$ whose only nontrivial components are given by the following diagram:



(4.4.15)

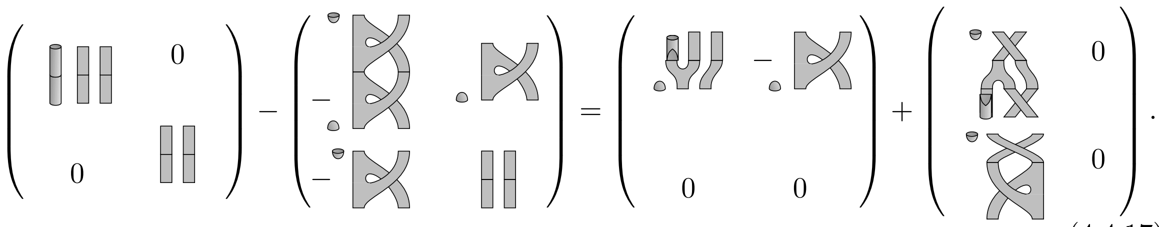
One can readily check that these maps define chain maps. The equation $dF = 0$ uses the fact that the zipper $\imath$ is an algebra homomorphism (preserves the unit) and the left unit axiom for the algebra A. The equation $Gd = 0$ follows from (4.2.31).

Note that $GF = \mathrm{id}$ by the relation (4.4.1) so the homotopy equivalence is established by defining a chain homotopy $h\colon \mathrm{id} \to FG$. The nonzero components of this chain homotopy h are given by



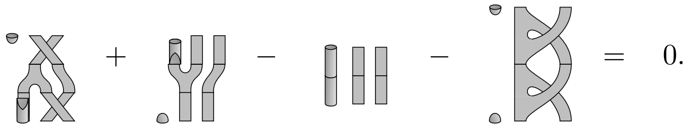
(4.4.16)

The equation $\mathrm{id}^0 - F^0G^0 = h^1d^0 + d^{-1}h^0$ is depicted below



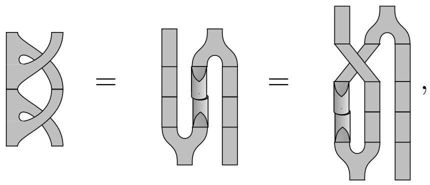
(4.4.17)

Only the first and third component of this matrix equation are non-trivial. The third component from (4.2.31). The first component can be written



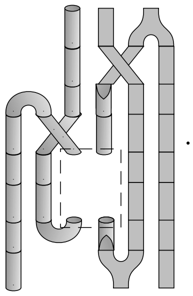
(4.4.18)

Noting that



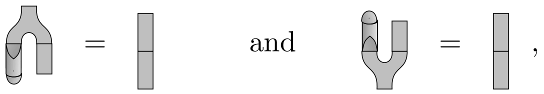
(4.4.19)

then (4.4.18) is just (4.4.3) applied to the cobordism



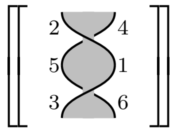
(4.4.20)

The equalities $\mathrm{id}^{-1} - F^{-1}G^{-1} = h^0d^{-1}$ and $\mathrm{id}^1 - F^1G^1 = h^0d^{-1} + d^0h^1$ follow from the identities



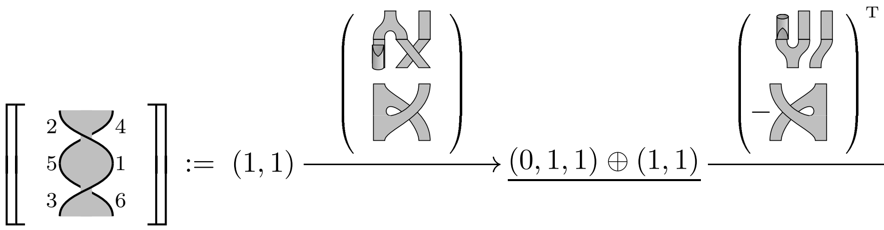
respectively.

For the opposite colouring of the complexes in (4.4.13) and (4.4.14) only the complex



(4.4.21)

would change. In particular, one can verify that the complex becomes



$$:= (1,1) \longrightarrow \underline{(0,1,1)\oplus(1,1)} \longrightarrow (1,1) \qquad (4.4.22)$$

where only the saddles have now been switched. Performing a similar saddle switch for the chain maps F and G produces the required chain homotopy equivalence.

Hence, the complex $[\![T]\!]$ is invariant under the second Reidemeister move up to chain homotopy equivalence. In fact, the chain map $G\colon [\![\asymp]\!] \to [\![\,|\,|\,]\!]$ is actually a *strong deformation retract* since $hF = 0$ by (4.4.1) and we have already shown that $GF = I$ and $I - FG = dh + hd$. Following the standard terminology we say that F is the *inclusion in a strong deformation retract.*

4.4.3 Invariance under Reidemeister move three

To prove that the complex $[\![T]\!]$ is invariant up to homotopy under the third Reidemeister move, we require the following results from homological algebra.

Definition 4.4.1. Let $\Psi\colon (C_0^r, d_0) \to (C_1^r, d_1)$ be a morphism of chain complexes. The *cone* $\Gamma(\Psi)$ of Ψ is the complex with chain spaces $\Gamma^r(\Psi) = C_0^{r+1} \oplus C_1^r$ and the differentials $d^r = \begin{pmatrix} -d_0^{r+1} & 0 \\ \Psi^{r+1} & d_1^r \end{pmatrix}$.

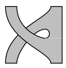
$$\begin{tikzcd} C_{0a} \arrow[r, shift left, "G_0"] \arrow[d, "\Psi"'] & C_{0b} \arrow[l, shift left, "F_0"] \\ C_{1a} \arrow[r, shift left, "F_1"] & C_{1a} \arrow[l, shift left, "G_1"] \end{tikzcd}$$

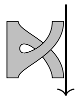
Lemma 4.4.2 (Lemma 4.5 [23])**.** The cone construction is invariant up to homotopy under compositions with the inclusions in strong deformation retracts. That is, consider the diagram of morphisms and complexes on the right. If in that diagram G^0 is a strong deformation retract with inclusion F_0, then the cones $\Gamma(\Psi)$ and $\Gamma(\Psi F_0)$ are homotopy equivalent, and if G_1 is a strong deformation retract with inclusion F_1, then the cones $\Gamma(\Psi)$ and $\Gamma(F_1\Psi)$ are homotopy equivalent. Likewise, if $F_{0,1}$ are strong deformation retracts and $G_{0,1}$ the corresponding inclusions the above statements remain true.

We comment here that the proof of the above lemma is constructive, so that one can explicitly obtain the chain homotopies defining the above homotopy equivalence. In fact, Bar-Natan also gives an explicit proof of invariance under Reidemeister move three which we encourage the reader to translate into the language of open-closed cobordisms. Having such an explicit description of the chain homotopies will become important if one is interested in constructing a braided monoidal 2-category from the formal Khovanov bracket $[\![T]\!]$.

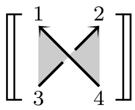
Define the two chain maps $\mathcal{S}_1$ and $\mathcal{S}_2$ between the 1-term complex $(0 \to \underline{(1,1)} \to 0)$ whose only nontrivial components are given by the diagram below:

$$\mathcal{S}_1 := \begin{tikzcd} 0 \arrow[r, "0"] & \underline{(1,1)} \arrow[r, "0"] \arrow[d] & 0 \\ 0 \arrow[r, "0"'] & \underline{(1,1)} \arrow[r, "0"'] & 0 \end{tikzcd} \qquad \mathcal{S}_2 := \begin{tikzcd} 0 \arrow[r, "0"] & \underline{(1,1)} \arrow[r, "0"] \arrow[d] & 0 \\ 0 \arrow[r, "0"'] & \underline{(1,1)} \arrow[r, "0"'] & 0 \end{tikzcd} \tag{4.4.23}$$

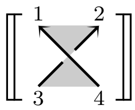




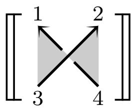
Lemma 4.4.3. With $\mathcal{S}_1$ and $\mathcal{S}_2$ as in (4.4.23) we have that



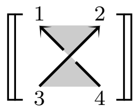
$= \Gamma(\mathcal{S}_1)$



$= \Gamma(\mathcal{S}_2)$



$= \Gamma(\mathcal{S}_1)[1]$

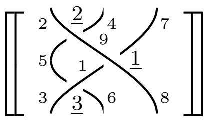


$= \Gamma(\mathcal{S}_1)[1]$ (4.4.24)

where $\cdot[s]$ is the operator that shifts complexes s units to the right: $C[s]^r := C^{r-s}$.

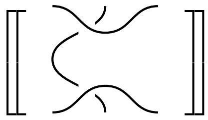
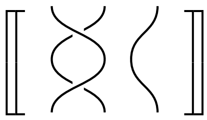
Here we have treated both possible colourings of the elementary tangles so that this lemma remains true in a skein theoretic sense where each crossing represents just a small disk neighbourhood inside a possibly larger tangle.

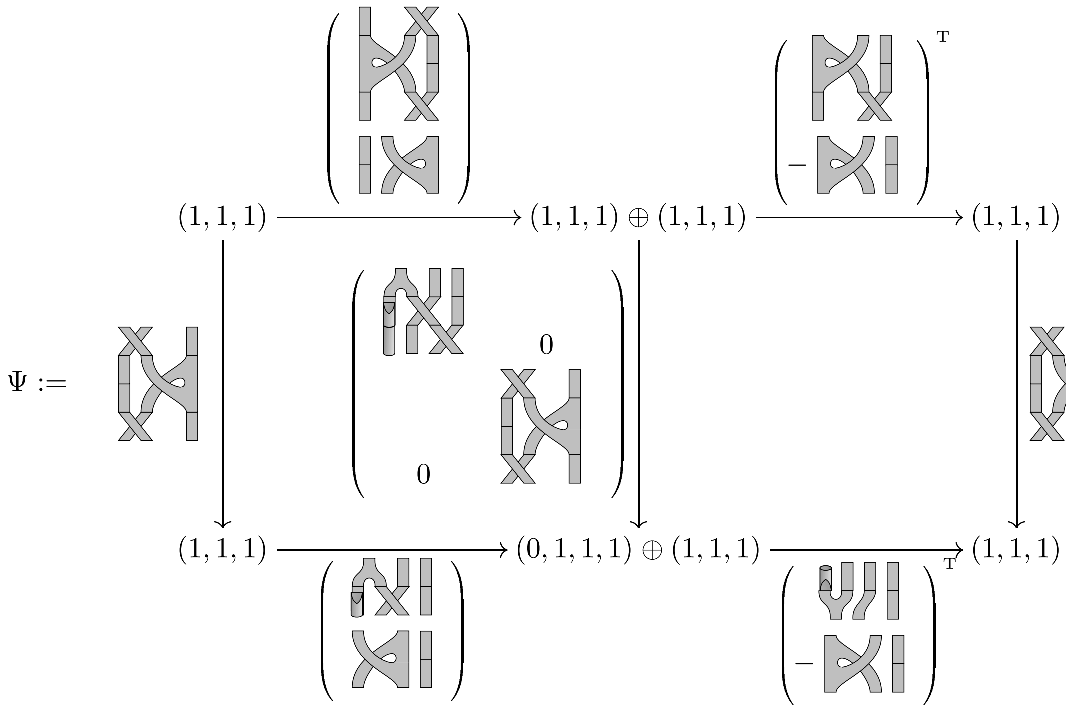
By the discussion above the following complex



, (4.4.25)

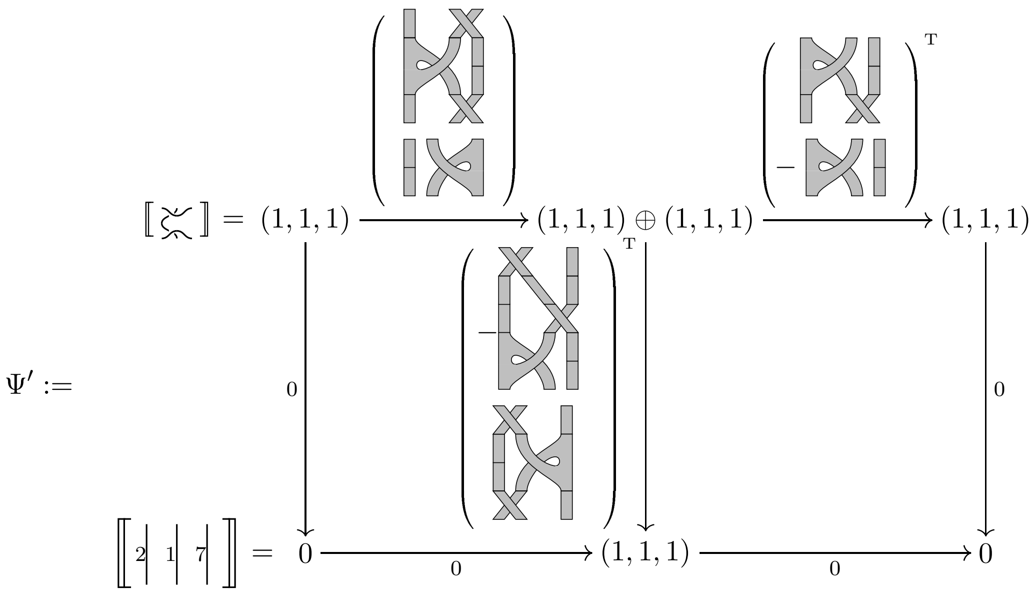
with the region to the right of the first strand shaded, is homotopy equivalent to the cone over the chain map

$\Psi\colon$  $\to$  (4.4.26) given by



(4.4.27)

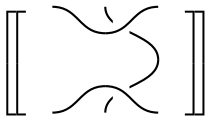
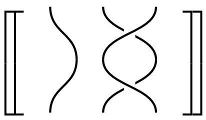
but by Lemma 4.4.2 the cone over the chain map Ψ is homotopy equivalent to the cone over the composite of Ψ with any strong deformation retraction G. Taking G as in (4.4.15) tensored on the right with the identity on (1) we have that $[\![\ ]\!]$ is homotopy equivalent to the cone of the map $\Psi' = G\Psi$ below



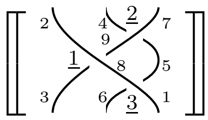
(4.4.28)

where we have left the enumeration on the bottom complex for clarity.

Similarly, by Lemma 4.4.3 the cone over the chain map

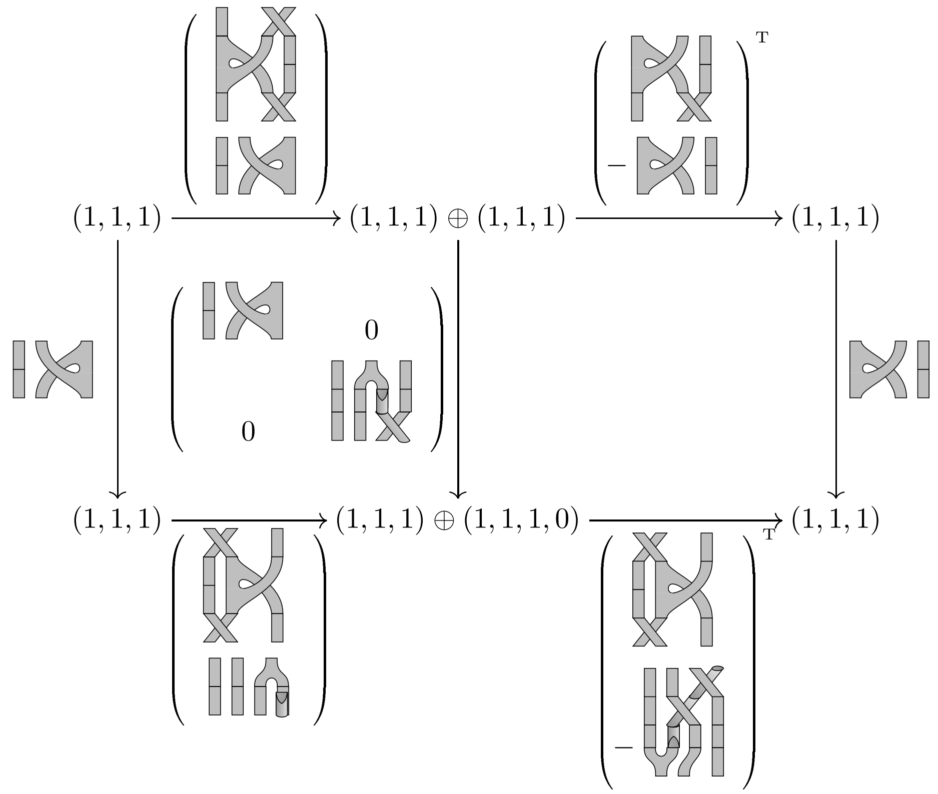
$\Phi:$  $\rightarrow$  (4.4.29)

is homotopy equivalent to the complex



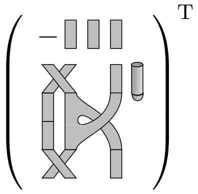
(4.4.30)

where Φ is given by

$\Phi :=$ 

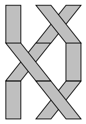
(4.4.31)

but again by Lemma 4.4.2 the cone $\Gamma(\Phi)$ is equal to the cone $\Gamma(G\Phi)$ where G is a strong deformation retraction. We take G as in (4.4.15), but with the arcs renumbered appropriately. This leads to a map G whose only nonzero component G^0 is given by

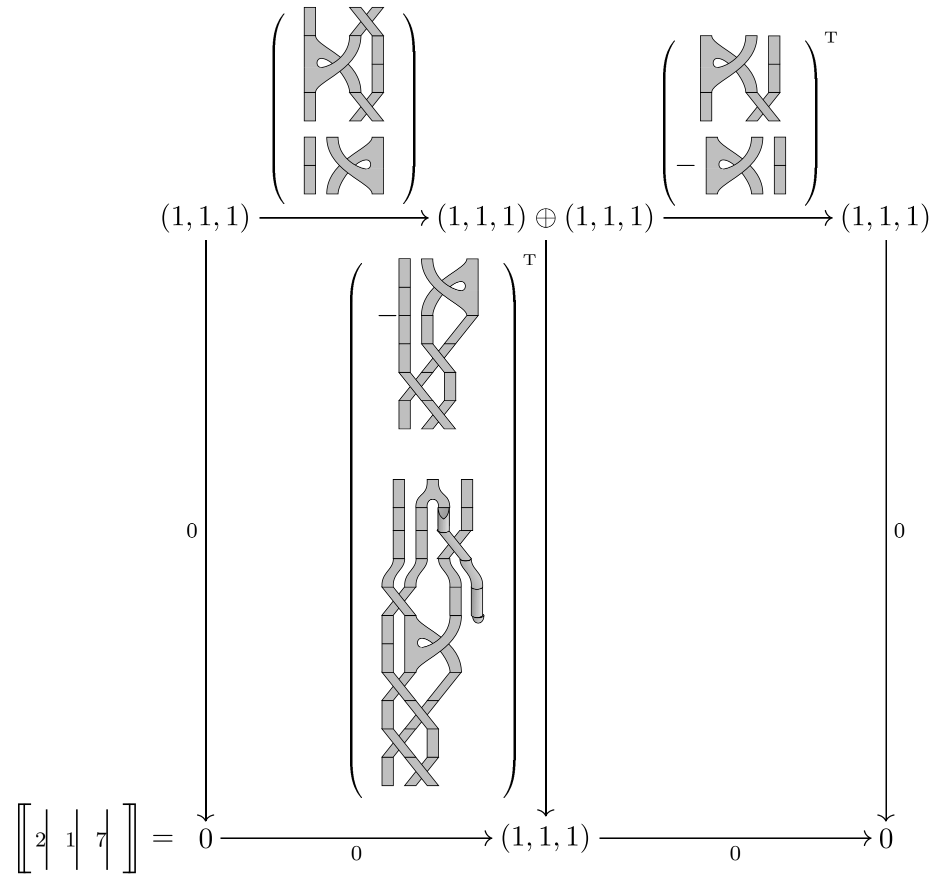
$G^0 :=$ 

. (4.4.32)

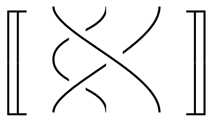
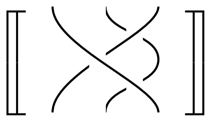
In order to make the enumeration of the complex $[\![\,|\,|\,|\,]\!]$ the same as in (4.4.28), we also postcompose the composite $G\Phi$ with the chain isomorphism f of Proposition 4.2.5 whose only nonzero component is given as follows:

$f^0 :=$  . (4.4.33)

Thus, the complex (4.4.30) is homotopy equivalent to the cone of the composite $\Phi' = fG\Phi$ given by

$\Phi' :=$  (4.4.34)

As an easy exercise in computing boundary permutations of open-closed cobordisms, one can see that the chain maps (4.4.28) and (4.4.34) are equal in the category **Kob**. Hence their cones $\Gamma(\Psi') = \Gamma(\Phi')$ are equal, making the complexes

 and  (4.4.35)

homotopy equivalent by Lemma 4.4.2 in the category $\mathbf{Kom}(\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}}))$, and therefore equal in **Kob**.

We leave it to the reader to verify the other possible colouring of the Reidemeister three move as well as the Reidemeister three move for the other crossing configurations[5].

[5]Bar-Natan's paper provides the proof for the other version of Reidemeister three not proven here.

Theorem 4.4.4. The isomorphism class of the complex $[\![T]\!]$ regarded in **Kob** is an invariant of the of the tangle T.

4.5 Applying an open-closed TQFT and obtaining a homology theory

In this section we bare the fruit of our labours. We have translated Bar-Natan's tangle homology into the language of open-closed cobordisms. All that remains is to apply an open-closed TQFT to the complex $[\![T]\!]$ associated to a tangle T. Given an open-closed TQFT $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$ where $\mathcal{C}$ is a symmetric monoidal abelian category, we immediately get a functor $Z\colon \mathbf{Kom}\big(\mathbf{Mat}(\mathbf{2Cob}^{\mathrm{ext}})\big) \to \mathbf{Kom}(\mathcal{C})$. Provided that the TQFT preserves the relations (4.4.1)–(4.4.3), then applying this functor to the complex $[\![T]\!]$ results in a complex $Z([\![T]\!])$ in $\mathbf{Kom}(\mathcal{C})$ that is an invariant of the tangle up to homotopy equivalence. Hence, the isomorphism class of the homology groups $H\big(Z([\![T]\!])\big)$ is an invariant of T.

The remainder of this chapter is devoted to providing examples of open-closed topological quantum field theories satisfying the relations (4.4.1)–(4.4.3), represented algebraically in the following:

Definition 4.5.1. A commutative Frobenius algebra $C = (C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$ is said to *satisfy Bar-Natan's conditions* if the following three conditions hold:

$$(\varepsilon_C \circ \eta_C)(1) = 0 \qquad \text{(S='sphere')}, \tag{4.5.1}$$

$$(\varepsilon_C \circ \mu_C \circ \Delta_C \circ \eta_C)(1) = 2 \qquad \text{(T='torus')}, \tag{4.5.2}$$

$$\begin{aligned} &\Delta_C \circ \eta_C \circ (\varepsilon_C \otimes \varepsilon_C) + (\eta_C \otimes \eta_C) \circ \varepsilon_C \circ \mu_C \\ &\quad -(\eta_C \circ \varepsilon_C) \otimes \mathrm{id}_C - \mathrm{id}_C \otimes (\eta_C \circ \varepsilon_C) = 0 \qquad \text{(4Tu='four tubes')}. \end{aligned} \tag{4.5.3}$$

In order for the tangle homology theory to possess the nice gluing properties, the knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ defining the TQFT $Z\colon \mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$ must arise from a state sum construction. Namely, the algebra A must be strongly separable and the other structure given by Theorem 3.2.18.

4.5.1 Examples

Recall the following:

Definition 4.5.2. Let k be a field. Khovanov's [12] commutative Frobenius algebra $(C_{\mathrm{Kh}}, \mu, \eta, \Delta, \varepsilon)$ is the algebra $C_{\mathrm{Kh}} = k[x]/(x^2)$ with the Frobenius algebra structure given in the k-basis $\{1, x\}$ by $\mu(1 \otimes 1) = 1$, $\mu(1 \otimes x) = x$, $\mu(x \otimes 1) = x$, $\mu(x \otimes x) = 0$, $\eta(1) = 1$, $\Delta(1) = 1 \otimes x + x \otimes 1$, $\Delta(x) = x \otimes x$, $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$.

Definition 4.5.3. Let k be a field. Lee's [22] commutative Frobenius algebra $(C_{\mathrm{Lee}}, \mu, \eta, \Delta, \varepsilon)$ is the algebra $C_{\mathrm{Lee}} = k[x]/(x^2-1)$ with the Frobenius algebra structure given by $\mu(1\otimes 1) = 1$, $\mu(1\otimes x) = x$, $\mu(x\otimes 1) = x$, $\mu(x\otimes x) = 1$, $\eta(1) = 1$, $\Delta(1) = 1\otimes x + x\otimes 1$, $\Delta(x) = x\otimes x + 1\otimes 1$, $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$.

Definition 4.5.4. Let k be a field. Bar-Natan's [23] commutative Frobenius algebra $(C_{\mathrm{BN}}, \mu, \eta, \Delta, \varepsilon)$ is the algebra $C_{\mathrm{BN}} = k[x]/(x^2-x)$ with the Frobenius algebra structure given by $\mu(1\otimes 1) = 1$, $\mu(1\otimes x) = x$, $\mu(x\otimes 1) = x$, $\mu(x\otimes x) = x$, $\eta(1) = 1$, $\Delta(1) = 1\otimes x + x\otimes 1 - 1\otimes 1$, $\Delta(x) = x\otimes x$, $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$.

In the following examples, we denote the algebra of $m \times m$-matrices over some commutative ring R by $M_m(R)$ and write $\{e_{pq}\}_{1\le p,q\le m}$ for a system of generators for which the multiplication reads $\mu(e_{pq} \otimes e_{rs}) = \delta_{qr} e_{ps}$ and the unit $\eta(1) = \sum_{p=1}^{m} e_{pp}$. For a direct product $M_{m_1}(R) \oplus \cdots \oplus M_{m_n}(R)$ of n such matrix algebras, we write $\{e_{pq}^{(j)}\}_{1\le j\le n, 1\le p,q\le m_j}$ for generators with $\mu(e_{pq}^{(j)} \otimes e_{rs}^{(\ell)}) = \delta_{j\ell}\delta_{qr} e_{ps}^{(j)}$, and $\eta(1) = \sum_{j=1}^{n} I^{(j)}$ with $I^{(j)} = \sum_{p=1}^{m_j} e_{pp}^{(j)}$ for the unit.

Example 4.5.5. Let k be a field and C_{Kh} be Khovanov's commutative Frobenius algebra over k (Definition 4.5.2). Consider the algebra $A := M_m(k) \otimes_k C_{\mathrm{Kh}}$ which has the k-basis $\{e_{pq}, \widetilde{e}_{pq}\}_{1\le p,q\le m}$ where we have written e_{pq} for $e_{pq} \otimes 1$ and $\widetilde{e}_{pq} := e_{pq} \otimes x$.

Then $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ forms a symmetric Frobenius algebra with $\eta_A(1) = \sum_{p=1}^{m} e_{pp}$, $\mu_A(e_{pq} \otimes e_{rs}) = \delta_{qr} e_{ps}$, $\mu_A(e_{pq} \otimes \widetilde{e}_{rs}) = \delta_{qr}\widetilde{e}_{ps}$, $\mu_A(\widetilde{e}_{pq} \otimes e_{rs}) = \delta_{qr}\widetilde{e}_{ps}$, $\mu_A(\widetilde{e}_{pq} \otimes \widetilde{e}_{rs}) = 0$, $\varepsilon_A(e_{pq}) = 0$, $\varepsilon_A(\widetilde{e}_{pq}) = \delta_{pq}$, $\Delta_A(e_{pq}) = \sum_{r=1}^{m}(e_{pr}\otimes\widetilde{e}_{rq} + \widetilde{e}_{pr}\otimes e_{rq})$ and $\Delta_A(\widetilde{e}_{pq}) = \sum_{r=1}^{m} \widetilde{e}_{pr}\otimes\widetilde{e}_{rq}$. Note that this Frobenius algebra is both a tensor product of algebra and a tensor product of coalgebra structures.

Neither C_{Kh} nor A are strongly separable. If $\operatorname{char} k = 2$, then $(A, C_{\mathrm{Kh}}, \imath, \imath^*)$ forms a knowledgeable Frobenius algebra with $\imath(1) = \eta_A(1)$, $\imath(x) = 0$, $\imath^*(e_{pq}) = 0$, and $\imath^*(\widetilde{e}_{pq}) = \delta_{pq} x$.

The assumption of $\operatorname{char} k = 2$ was made in order to satisfy the Cardy condition. Its left hand side is given by $(\mu_A \circ \tau_{A,A} \circ \Delta_A)(e_{pq}) = 2\delta_{pq} \sum_{r=1}^{m} \widetilde{e}_{rr}$ and $(\mu_A \circ \tau_{A,A} \circ \Delta_A)(\widetilde{e}_{pq}) = 0$ while its right hand side reads $(\imath \circ \imath^*)(e_{pq}) = 0$ and $(\imath \circ \imath^*)(\widetilde{e}_{pq}) = 0$.

Example 4.5.6. Let k be a field and C_{Kh} be Khovanov's commutative Frobenius algebra over k. The truncated polynomial algebra $A = k[y]/(y^p)$, $p \geq 2$, forms a commutative and therefore symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ with $\Delta_A(y^\ell) = \sum_{j=0}^{p-1-\ell} y^{j+\ell} \otimes y^{p-1-j}$ for all $\ell \in \{0, \ldots, p-1\}$, $\varepsilon_A(y^{p-1}) = 1$, and $\varepsilon_A(y^\ell) = 0$ for all $\ell \in \{0, \ldots, p-2\}$. The window element is $a = py^{p-1}$ which is a zero divisor, and so A is not strongly separable.

If char $k = p$, then $(A, C_{0,0}, \imath, \imath^*)$ forms a knowledgeable Frobenius algebra with $\imath(1) = 1$, $\imath(x) = 0$, $\imath^*(y^{p-1}) = 1$, and $\imath^*(y^\ell) = 0$ for all $\ell \in \{0, \ldots, p-2\}$.

For a field k of characteristic char $k \neq p$, the above example fails to satisfy the Cardy condition whose left hand side reads $(\mu_A \circ \tau_{A,A} \circ \Delta_A)(1) = py^{p-1}$ and $(\mu_A \circ \tau_{A,A} \circ \Delta_A)(y^\ell) = 0$ for all $\ell \in \{0, \ldots, p-2\}$ whereas its right hand side is $(\imath \circ \imath^*)(y^\ell) = 0$ for all $\ell \in \{0, \ldots, p-1\}$.

Example 4.5.7. Let k be a field and $p = 2n + 1$, $n \in \mathbb{N}$. Consider the p-dimensional vector space A with basis $\{X_{-n}, X_{-n+1}, \ldots, X_n\}$. It forms a symmetric Frobenius algebra $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ with $\mu_A(X_1 \otimes X_j) = X_j = \mu_A(X_j \otimes X_1)$, $\mu_A(X_j \otimes X_{-j}) = X_{-1}$ for all $-n \leq j \leq n$, $\eta_A(1) = X_1$, $\Delta_A(1) = \sum_{\ell=-n}^{n} X_\ell \otimes X_{-\ell}$, $\Delta_A(X_{-1}) = X_{-1} \otimes X_{-1}$, $\Delta_A(X_j) = X_{-1} \otimes X_j + X_j \otimes X_{-1}$ for all $j \notin \{1, -1\}$, and $\varepsilon_A(X_{-1}) = 1$. The operations μ_A, η_A, Δ_A, and ε_A are 0 on all other basis vectors. The window element reads $a = pX_{-1}$. It is a zero divisor, and so A is not strongly separable.

If char $k = p$, then there is a knowledgeable Frobenius algebra $(A, C_{0,0}, \imath, \imath^*)$ with $\imath(1) = 1$, $\imath(x) = 0$, $\imath^*(X_{-1}) = x$, and $\imath^*(X_j) = 0$ for all $j \neq -1$.

Again, for a field k with char $k \neq p$, the example fails to satisfy the Cardy condition whose left hand side gives $(\mu_A \circ \tau_{A,A} \circ \Delta_A)(1) = pX_{-1}$ and $(\mu_A \circ \tau_{A,A} \circ \Delta_A)(X_j) = 0$ for all $j \neq 1$ whereas its right hand side is $(\imath \circ \imath^*)(X_j) = 0$ for all j.

Remark 4.5.8. Examples (4.5.6) and (4.5.7) both supply examples of a knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ whose commutative part is not the centre $Z(A)$, and furthermore, C is not a trivial enlargement of the centre.

Example 4.5.9. Let k be a field and C_{Lee} be Lee's Frobenius algebra (Definition 4.5.3). Consider the algebra $A := M_m(k) \otimes_k C_{\mathrm{Lee}}$.

Then $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ forms a symmetric Frobenius algebra with $\eta_A(1) = \sum_{p=1}^{m} e_{pp}$, $\mu_A(e_{pq} \otimes e_{rs}) = \delta_{qr} e_{ps}$, $\mu_A(e_{pq} \otimes \widetilde{e}_{rs}) = \delta_{qr} \widetilde{e}_{ps}$, $\mu_A(\widetilde{e}_{pq} \otimes e_{rs}) = \delta_{qr} \widetilde{e}_{ps}$, $\mu_A(\widetilde{e}_{pq} \otimes \widetilde{e}_{rs}) = \delta_{qr} e_{ps}$, $\varepsilon_A(e_{pq}) = 0$, $\varepsilon(\widetilde{e}_{pq}) = \delta_{pq}$, $\Delta_A(e_{pq}) = \sum_{r=1}^{m} (e_{pr} \otimes \widetilde{e}_{rq} + \widetilde{e}_{pr} \otimes e_{rq})$, $\Delta_A(\widetilde{e}_{pq}) = \sum_{r=1}^{m} (e_{pr} \otimes e_{rq} + \widetilde{e}_{pr} \otimes \widetilde{e}_{rq})$. Again, this is a tensor product of Frobenius algebra structures.

Both A and C_{Lee} are strongly separable if and only if char $k \neq 2$. If char $k = 2$, then $(A, C_{\text{Lee}}, \imath, \imath^*)$ forms a knowledgeable Frobenius algebra with $\imath(1) = \imath(x) = \eta_A(1)$, $\imath^*(e_{pq}) = 0$, and $\imath^*(\widetilde{e}_{pq}) = \delta_{pq}(1 + x)$.

Note that the left hand side of the Cardy condition yields $(\mu_A \circ \tau_{A,A} \circ \Delta_A)(e_{pq}) = 2\delta_{pq} \sum_{r=1}^{m} \widetilde{e}_{rr}$ and $(\mu_A \circ \tau_{A,A} \circ \Delta_A)(\widetilde{e}_{pq}) = 2\delta_{pq} \sum_{r=1}^{m} e_{rr}$ while its right hand side yields $(\imath \circ \imath^*)(e_{pq}) = 0$ and $(\imath \circ \imath^*)(\widetilde{e}_{pq}) = 2\delta_{pq} \sum_{r=1}^{m} e_{rr}$. If char $k = 2$, the Cardy condition therefore holds.

Knowledgeable Frobenius algebras $(A, C, \imath, \imath^*)$ in which C is Lee's Frobenius algebra (Definition 4.5.3) for char $k \neq 2$ or Bar-Natan's (Definition 4.5.4) in any characteristic, are provided by the following proposition.

Proposition 4.5.10. Let k be a field and $A := M_{m_1}(k) \oplus \cdots \oplus M_{m_n}(k)$ be the direct product of $m_j \times m_j$-matrix algebras, $1 \leq j \leq n$, such that char k does not divide m_j for all j. In this case, (A, μ_A, η_A) is a strongly separable algebra. The elements $z_j := \sum_{p=1}^{m_j} e_{pp}^{(j)}$ form a k-basis of the centre $Z(A)$ which is strongly separable, too.

Every symmetric Frobenius algebra structure $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ is of the form $\varepsilon_A(e_{pq}^{(j)}) = \alpha_j \delta_{pq}$ and $\Delta_A(e_{pq}^{(j)}) = \alpha_j^{-1} \sum_{r=1}^{m_j} e_{pr}^{(j)} \otimes e_{rq}^{(j)}$ for some $\alpha_j \in k \backslash \{0\}$. The window element is given by $a = \sum_{j=1}^{n} \alpha_j^{-1} m_j z_j$.

The knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ of the state sum construction (Chapter 3) has a commutative Frobenius algebra C that satisfies Bar-Natan's conditions if and only if the following two conditions hold:

1. $n = 2$,

2. $\alpha_2^2 = -\alpha_1^2$.

Proof. The knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ of the state sum construction (see Theorem (3.2.18)) is given by the centre $C := Z(A)$ with the commutative Frobenius algebra structure $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$ with $\mu_C(z_j \otimes z_\ell) = \delta_{j\ell} z_j$, $\eta_C(1) = \sum_{j=1}^{n} z_j$, $\Delta_C(z_j) = \alpha_j^{-2} z_j \otimes z_j$ and $\varepsilon_C(z_j) = \alpha_j^2$ with $\imath(z_j) = \sum_{p=1}^{m_j} e_{pp}^{(j)}$ and $\imath^*(e_{pq}^{(j)}) = \alpha_j^{-1} \delta_{pq} z_j$.

The value of the torus is $(\varepsilon_C \circ \mu_C \circ \Delta_C \circ \eta_C)(1) = n$, and so the condition (4.5.2) is equivalent to $n = 2$. In this case, the value of the sphere is $(\varepsilon_C \circ \eta_C)(1) = \alpha_1^2 + \alpha_2^2$, and so the condition (4.5.1) is equivalent to $\alpha_2^2 = -\alpha_1^2$. If these first two conditions hold, the third one (4.5.3) is always satisfied. $\square$

Remark 4.5.11.

1. In Proposition 4.5.10, if char $k \neq 2$ and if there exist $\alpha_j \in k$ such that $\alpha_1^2 = 1/2$ and $\alpha_2^2 = -1/2$, then there is an isomorphism of Frobenius algebras $\varphi\colon C_{\mathrm{Lee}} \to C$ with Lee's Frobenius algebra $C_{\mathrm{Lee}} = k[x]/(x^2-1)$ (Definition 4.5.3) given by $\varphi(1) = z_1 + z_2$ and $\varphi(x) = z_1 - z_2$.

2. For k of arbitrary characteristic with $\alpha_j \in k$ such that $\alpha_1^2 = 1$ and $\alpha_2^2 = -1$, there is an isomorphism of Frobenius algebras $\psi\colon C_{\mathrm{BN}} \to C$ with Bar-Natan's Frobenius algebra $C_{\mathrm{BN}} = k[x]/(x^2-x)$ (Definition 4.5.4) given by $\psi(1) = z_1 + z_2$ and $\psi(x) = z_1$.

4.6 Concluding remarks

We have presented in this chapter one approach towards adapting Khovanov homology to a tangle homology theory that can be naturally translated into a computable algebraic theory. In Bar-Natan's picture world construction of tangle homology, he considers his surfaces to be embedded into $\mathbb{R}^3$ whereas we have chosen to consider the surfaces as abstract manifolds. As a future endeavour one might consider open-closed cobordisms that are embedded into $\mathbb{R}^3$. The corresponding TQFTs have algebraic descriptions implicit in the work of Runkel, Fjelstad, Fuchs, and Schweigert (see for example [35, 41] and the references therein) on boundary conformal field theory. Very roughly speaking, such an algebraic theory would correspond to a version of knowledgeable Frobenius algebras defined in a modular tensor category. However, one should note that recent work by Gad Naot suggest that the difference between embedded cobordisms versus abstract cobordisms may not be relevant for link homology [79].

Another possible adaptation of the work described in this chapter would be to consider unoriented open-closed cobordisms and their corresponding TQFTs. This version of open-closed topological field theory was treated by Alexeevski and Natanzon where an algebraic characterization was also supplied [31]. Perhaps, a simpler presentation could be obtained by not worrying about the orientations of the open-closed cobordisms. The true test lies in whether or not there exist interesting examples of their algebraic structures that generalize the Frobenius algebras of Bar-Natan, Lee, or Khovanov.

Finally, it is worth pointing out that although it was not considered in this thesis, S-coloured knowledgeable Frobenius algebras would provide an algebraic structure well suited for constructing tangle homology theories for tangles with coloured end points. That is, given a tangle T whose boundary points are labeled from the set S, an S-coloured knowledgeable Frobenius algebra can be used to construct algebraic tangle homology theories by following the procedure outlined in this chapter for the uncoloured case.

Appendix A

Symmetric monoidal categories

In this appendix, we collect some key definitions for easier reference.

Definition A.0.1. A *monoidal category* consists of:

- a category $\mathcal{C}$.
- a functor called the *tensor product* $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, where we write $\otimes(X, Y) = X \otimes Y$ and $\otimes(f, g) = f \otimes g$ for objects $X, Y \in |\mathcal{C}|$ and morphisms f, g in $\mathcal{C}$.
- an object called the *unit object* $\mathbb{1} \in |\mathcal{C}|$.
- natural isomorphisms called the *associator*:

$$\alpha_{X,Y,Z} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \tag{A.0.1}$$

  the *left unit constraint*:

$$\lambda_X \colon \mathbb{1} \otimes X \to X, \tag{A.0.2}$$

  and the *right unit constraint*:

$$\rho_X \colon X \otimes \mathbb{1} \to X. \tag{A.0.3}$$

such that the following diagrams commute for all objects $W, X, Y, Z \in |\mathcal{C}|$:

$$\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "a_{W\otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y}\otimes \mathrm{id}_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{W,X\otimes Y,Z}"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{id}_W \otimes a_{X,Y,Z}"']
\end{tikzcd} \tag{A.0.4}$$

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "\rho_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \lambda_Y"] \\
 & X \otimes Y &
\end{tikzcd} \tag{A.0.5}$$

Definition A.0.2. A *braided monoidal category* consists of:

- a monoidal category $\mathcal{C}$.
- a natural isomorphism called the *braiding*:

$$\tau_{X,Y} \colon X \otimes Y \to Y \otimes X. \tag{A.0.6}$$

such that these two diagrams commute, called the *hexagon equations*:

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "\tau_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "\alpha_{Y,Z,X}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[dr, "\tau_{X,Y}\otimes \mathrm{id}_Z"'] & & & Y \otimes (Z \otimes X) \\
 & (Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes \tau_{X,Z}"'] &
\end{tikzcd} \tag{A.0.7}$$

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "\tau_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[dr, "\alpha^{-1}_{Z,X,Y}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\alpha^{-1}_{X,Y,Z}"] \arrow[dr, "\mathrm{id}_X \otimes \tau_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \\
 & X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "\tau_{X,Z}\otimes \mathrm{id}_Y"'] &
\end{tikzcd} \tag{A.0.8}$$

Definition A.0.3. A *symmetric monoidal category* is a braided monoidal category $\mathcal{C}$ for which the braiding satisfies $\tau_{Y,X} \circ \tau_{X,Y} = \mathrm{id}_{X\otimes Y}$ for all objects X and Y.

Definition A.0.4. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \otimes, \mathbb{1}', \alpha', \lambda', \rho')$ be monoidal categories. A *monoidal functor* $\psi \colon \mathcal{C} \to \mathcal{C}'$ is a triple $\psi = (\psi, \psi_2, \psi_0)$ consisting of,

- a functor $\psi\colon C \to C'$,
- a natural isomorphism $\psi_2\colon \psi(X)\otimes\psi(Y) \to \psi(X\otimes Y)$, where for brevity we suppress the subscripts indicating the dependence of this isomorphism on X and Y, and
- an isomorphism $\psi_0\colon \mathbb{1}' \to \psi(\mathbb{1})$,

such that the following diagrams commute for all objects $X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
(\psi(X)\otimes\psi(Y))\otimes\psi(Z) \arrow[r, "\psi_2\otimes \mathrm{id}_{\psi(Z)}"] \arrow[d, "a_{\psi(X),\psi(Y),\psi(Z)}"'] & \psi(X\otimes Y)\otimes\psi(Z) \arrow[r, "\psi_2"] & \psi((X\otimes Y)\otimes Z) \arrow[d, "\psi(\alpha_{X,Y,Z})"] \\
\psi(X)\otimes(\psi(Y)\otimes\psi(Z)) \arrow[r, "\mathrm{id}_{\psi(X)}\otimes\psi_2"] & \psi(X)\otimes\psi(Y\otimes Z) \arrow[r, "\psi_2"] & \psi(X\otimes(Y\otimes Z))
\end{tikzcd} \tag{A.0.9}$$

$$\begin{tikzcd}
\mathbb{1}'\otimes\psi(X) \arrow[r, "\lambda'_{\psi(X)}"] \arrow[d, "\psi_0\otimes\mathrm{id}_{\psi(X)}"'] & \psi(X) \\
\psi(\mathbb{1})\otimes\psi(X) \arrow[r, "\psi_2"] & \psi(\mathbb{1}\otimes X) \arrow[u, "\psi(\lambda_X)"']
\end{tikzcd} \tag{A.0.10}$$

$$\begin{tikzcd}
\psi(X)\otimes\mathbb{1}' \arrow[r, "\rho'_{\psi(X)}"] \arrow[d, "\mathrm{id}_{\psi(X)}\otimes\psi_0"'] & \psi(X) \\
\psi(X)\otimes\psi(\mathbb{1}) \arrow[r, "\psi_2"] & \psi(X\otimes\mathbb{1}) \arrow[u, "\psi(\rho_X)"']
\end{tikzcd} \tag{A.0.11}$$

The monoidal functor is called *strict* if ψ_2 and ψ_0 are identities.

Definition A.0.5. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ and $(\mathcal{C}', \otimes, \mathbb{1}', \alpha', \lambda', \rho', \tau')$ be symmetric monoidal categories. A *symmetric monoidal functor* $\psi\colon \mathcal{C} \to \mathcal{C}'$ is a monoidal functor for which the following additional diagram commutes for all $X, Y \in \mathcal{C}$:

$$\begin{tikzcd}
\psi(X)\otimes\psi(Y) \arrow[r, "\tau'_{X,Y}"] \arrow[d, "\psi_2"'] & \psi(Y)\otimes\psi(X) \arrow[d, "\psi_2"] \\
\psi(X\otimes Y) \arrow[r, "\psi(\tau)"] & \psi(Y\otimes X)
\end{tikzcd} \tag{A.0.12}$$

Definition A.0.6. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \otimes, \mathbb{1}', \alpha', \lambda', \rho')$ be monoidal categories and $\psi\colon \mathcal{C} \to \mathcal{C}'$ and $\psi'\colon \mathcal{C} \to \mathcal{C}'$ be monoidal functors. A *monoidal natural transformation* $\varphi\colon \psi \Rightarrow \psi'$ is a natural transformation such that for all objects X, Y of $\mathcal{C}$, the following diagrams commute,

$$\begin{tikzcd}
\psi(X)\otimes\psi(Y) \arrow[r, "\varphi_X\otimes\varphi_Y"] \arrow[d, "\psi_2"'] & \psi'(X)\otimes\psi'(Y) \arrow[d, "\psi'_2"] \\
\psi(X\otimes Y) \arrow[r, "\varphi_{X\otimes Y}"] & \psi'(X\otimes Y)
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathbb{1}' \arrow[d, "\psi_0"'] \arrow[dr, "\psi'_0"] & \\
\psi(\mathbb{1}) \arrow[r, "\varphi(\mathbb{1})"'] & \psi'(\mathbb{1})
\end{tikzcd} \tag{A.0.13}$$

Definition A.0.7. Let $\mathcal{C}$ be a small symmetric monoidal category and let $\mathcal{C}'$ be an arbitrary symmetric monoidal category. We denote by $\mathbf{Symm\text{-}Mon}(\mathcal{C}, \mathcal{C}')$ the category of symmetric monoidal functors $\mathcal{C} \to \mathcal{C}'$ and monoidal natural transformations between them. It is clear that the tensor product of symmetric monoidal functors and monoidal natural transformations defines a symmetric monoidal structure on the category $\mathbf{Symm\text{-}Mon}(\mathcal{C}, \mathcal{C}')$.

Definition A.0.8. Let $\mathcal{C}$ and $\mathcal{C}'$ be monoidal categories. We say that $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent as monoidal categories* if there is an equivalence of categories $\mathcal{C} \simeq \mathcal{C}'$ given by functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}$ and natural isomorphisms $\eta\colon 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathcal{C}'}$ such that both F and G are monoidal functors and η and ε are monoidal natural transformations.

If $\mathcal{C}$ and $\mathcal{C}'$ are symmetric monoidal categories, we say that they are *equivalent as symmetric monoidal categories* if in addition F and G are symmetric monoidal functors.

Appendix B

Abelian categories

In this appendix we review the basics of Abelian categories. For more details see [70].

Definition B.0.9. A *zero object* in a category $\mathcal{C}$ is an object $\mathbf{0}$ that is both initial and terminal. A morphism $f\colon X \to Y$ is called the *zero morphism* when it factors through the zero object $\mathbf{0}$.

Recall that there is exactly one zero morphism between each object X and Y of $\mathcal{C}$ and the composite of a zero morphism with any other morphism is again the zero morphism.

Definition B.0.10. Let $f\colon X \to Y$ in the category $\mathcal{C}$. Then when they exist, the *kernel* of f is the equalizer of f and the zero morphism $\mathbf{0}\colon X \to Y$ and the *cokernel* of f is defined dually as the coequalizer of f and the zero morphism. The kernel and cokernel of f are denoted as $\ker f$ and $\operatorname{coker} f$, respectively.

Definition B.0.11. An **Ab**-category $\mathcal{C}$, or pre-additive category, is a category enriched in abelian groups. This means that each Hom set $\mathcal{C}(X,Y)$ is an abelian group and composition is bilinear relative to this addition.

Definition B.0.12. Given objects X and Y in the **Ab**-category $\mathcal{C}$ a *biproduct* of X and Y is an object $X \oplus Y$ together with morphisms

$$X \underset{p_X}{\overset{s_X}{\rightleftarrows}} X \oplus Y \underset{s_Y}{\overset{p_Y}{\rightleftarrows}} B\,, \tag{B.0.1}$$

such that

$$p_X \circ s_X = 1_X, \quad p_Y \circ s_Y = 1_Y, \quad p_X \circ s_Y = 0, \quad p_Y \circ s_X = 0, \quad s_X \circ p_X + s_Y \circ p_Y = 1_{X\oplus Y}. \tag{B.0.2}$$

Definition B.0.13. An *additive category* $\mathcal{C}$ is an **Ab**-category that is equipped with a zero object and a biproduct for every pair of objects.

Definition B.0.14. An *abelian category* $\mathcal{C}$ is an **Ab**-category satisfying the following conditions:

i) $\mathcal{C}$ has a zero object,

ii) $\mathcal{C}$ has binary biproducts,

iii) Every morphism in $\mathcal{C}$ has a kernel and cokernel,

iv) Every monic morphism is a kernel, and every epi a cokernel.

Conditions (i) and (ii) ensure that $\mathcal{C}$ is an additive category.

Proposition B.0.15. In an abelian category $\mathcal{C}$, every morphism f can be factored uniquely (up to isomorphism) as $f = \text{im } f \circ \text{coim } f$, with $\text{coim } f$ a monomorphism and $\text{im } f$ an epimorphism. In particular, $\text{im } f = \ker(\text{coker } f)$ and $\text{coim } f = \text{coker } (\ker f)$.

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Index