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The Metalanguage of Category Theory

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Thank you to my parents, Catherine, John, Mike, Sarah, and Nathanael.

Abstract of the Dissertation

The Metalanguage of Category Theory

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Doctor of Philosophy, Mathematics

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Category theory is known as a language of mathematics. The fundamental concepts of the language are systematized in a *fibrant double category*, a two-dimensional structure also known as a *bicategory equipped with proarrows*.

We give a new definition of the structure: a *bifibrant double category* is a “two-sided bifibration” from a category to itself, with a weak composition and identity. This way of thinking gives a way to construct the fully three-dimensional category of bifibrant double categories, as follows.

A category forms a bifibrant double category, by forming the union of the arrow double category with its opposite; we call this the *weave double category*. Then a *two-sided bifibration* or *matrix category* is a span of categories forming a bimodule of weave double categories. We construct a three-dimensional category of categories, functors, profunctors, and matrix categories; squares are transformations, *matrix functors*, and *matrix profunctors*, and cubes are *matrix transformations*. This structure is a “bifibrant triple category without interchange”, which we call a *metalogic*.

A bifibrant double category is a pseudomonad in the metalogic of matrix categories. This defines the objects of a three-dimensional construction: a *double functor* is a morphism of pseudomonads, a *vertical profunctor* is a “vertical monad” between pseudomonads, and a *horizontal profunctor* is a bimodule of pseudomonads; a *vertical transformation* is a morphism of vertical monads, a *horizontal transformation* is a morphism of bimodules, and a *double profunctor* is a bimodule of vertical monads. A *double transformation* is a transformation of vertical bimodules. These form the metalogic of bifibrant double categories.

Contents

Introduction

Category theory is known as a unifying language of mathematics [13]. In recent years, the community of Applied Category Theory has begun to explore its potential as a unifying language of all kinds of science [6]. Here, we propose that category theory is the *language of thinking*.

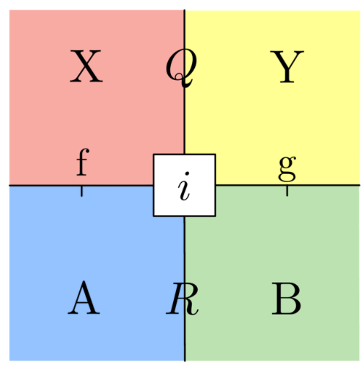
A "world" is a collection of *types* of things, and *processes* between types; these form a category.

A "thought" of the world is a *relation* of types, and a "process of thinking" is an *inference* between relations; these form a category of "thoughts" which depends on pairs in the category of the "world".

A *logic* is a two-dimensional structure of relations and inferences, over pairs of types and processes: the structure known as an equipment, or framed bicategory [17]. We give a new definition, via the notion of *two-sided bifibration* [2.2], and this motivates the name *bifibrant double category*.

The language of *string diagrams* [15] is dual to conventional diagrams: types and processes are colored areas and vertical-pointing "bars", relations and inferences are "strings" and "beads".

bif. dbl. cat.	dim.	**logic**
object	0	type
tight morphism	V	process
loose morphism	H	relation
square	2	inference



In this thesis, we construct the metalanguage of logics. The language of "metalogic" is both visual and formal, expressed in both three-dimensional string diagrams and the co/descent calculus of matrix categories. Imagery and syntax are complementary, so intuition and computation can strengthen each other.

The simplest kind of logic is *binary logic*: types and processes are sets and functions; relations and inferences are binary relations and entailments. This is known as the predicate logic of sets.

How do we make logics? This is summarized in a motto:

"a category is a matrix with composition and identity".

A category is a type of objects, indexing a matrix of morphisms, with the structure of composition and identity. In [17], Shulman presented the two main ways that we construct universes of categories:

1. A *bifibered monoidal category* $\mathcal{R} \to \mathbb{A}$ forms a logic, in which a relation $R : \mathrm{A} \mid \mathrm{B}$ is an object R over $\mathrm{A} \times \mathrm{B}$; this is a matrix, i.e. two-variable dependent type $\mathrm{a} : \mathrm{A}, \mathrm{b} : \mathrm{B} \vdash R(\mathrm{a}, \mathrm{b}) : \mathbb{V}$.

2. *Monads* in a logic, self-relations equipped with composition and identity, form a richer logic. A monad in a logic of matrices is a category, "enriched in" or "internal to" that logic.

These two constructions define the language of the *co/end calculus* [14]. Bimodules of monads are matrices-with-composition; they compose by *coend*, a coequalizer of a coproduct, and divide by *end*, an equalizer of a product. Below are the formulae for composition and transformation.

$$
\begin{aligned}
R \circ S &= \quad \Sigma \mathrm{b} \quad R(-, \mathrm{b}) \times S(\mathrm{b}, -) \\
[P, Q] &= \quad \Pi \mathrm{x}, \mathrm{y} \quad P(\mathrm{x}, \mathrm{y}) \to Q(\mathrm{x}, \mathrm{y})
\end{aligned}
$$

This is generalized logic [12]: coend is the "bilinear existential", and end is the "natural universal". As a language of categories is the co/end calculus of a logic, we propose that *category theory is logic*.

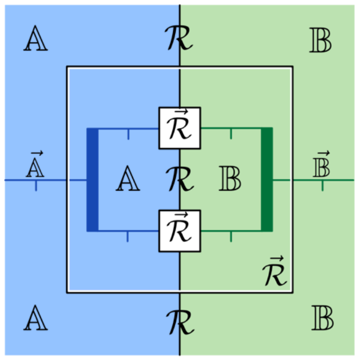
Now, the central insight of our thesis: a logic is a *matrix category* with composition and identity. For each pair of types there is a *category* of relations, and each pair of processes gives a *profunctor* of inferences. So relations form a *matrix of categories*, and inferences form a *matrix of profunctors*.

We develop the notion of a "matrix of categories", and its three-dimensional language, as follows.

Chapter 1: Spans of categories.

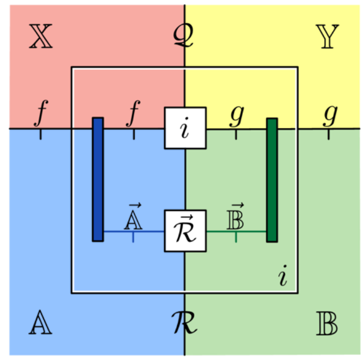
A *span of categories* $\mathbb{A} \leftarrow \mathcal{R} \to \mathbb{B}$ is a equivalent to a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$ and profunctors $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})$, with *sequential composition* and identity. In the same way, a *span of profunctors* $i \leftarrow f \to g$ is equivalent to a matrix of profunctors $i(\mathrm{f}, \mathrm{g}) : \mathcal{Q}(\mathrm{X}, \mathrm{Y}) \mid \mathcal{R}(\mathrm{A}, \mathrm{B})$ with composition and identity.

We introduce *three-dimensional* string diagrams: spans of categories are horizontal strings, profunctors are vertical bars, and functors are drawn as a closed loop or "bead within a bead", interpreted as a transformation from inner to outer.



span category

$\vec{\mathcal{R}}(\mathrm{a}_1, \mathrm{b}_1) \circ \vec{\mathcal{R}}(\mathrm{a}_2, \mathrm{b}_2) \Rightarrow \vec{\mathcal{R}}(\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2)$



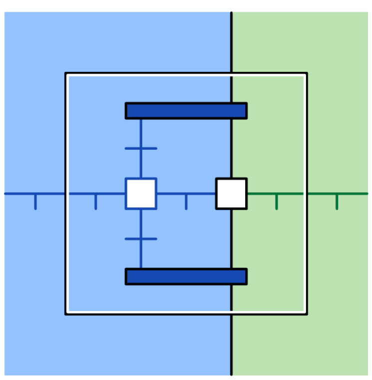
span profunctor

$i(\mathrm{f}, \mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \Rightarrow i(\mathrm{fa}, \mathrm{gb})$

We introduce the concept of *displayed profunctor* 1.2, and show the double category of span categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ to be equivalent to that of displayed categories $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$. The matrices $\mathcal{R}(\mathrm{A}, \mathrm{B})$ are the basic data of the co/descent calculus.

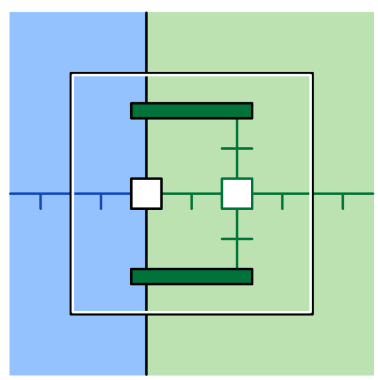
Chapter 2: Matrix categories.

A *matrix category* $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is a span of categories, with actions by both arrows and "op-arrows" in $\mathbb{A}$ and $\mathbb{B}$: the *weave double category* $\langle\mathbb{A}\rangle$ is the union of the arrow double category and its opposite $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$, forming a logic, and $\mathcal{R}$ is a bimodule from $\langle\mathbb{A}\rangle$ to $\langle\mathbb{B}\rangle$. In the terminology of [20], a matrix category would be a "two-sided bifibration".



matrix category

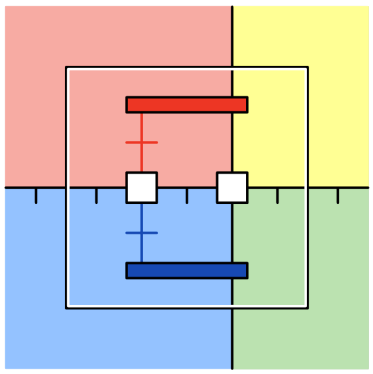
$\odot_{\mathbb{A}} : \langle\mathbb{A}\rangle(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_1, \mathrm{B}) \rightarrow \mathcal{R}(\mathrm{A}_0, \mathrm{B})$



matrix category

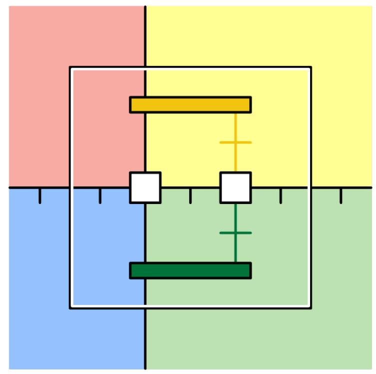
$\odot_{\mathbb{B}} : \mathcal{R}(\mathrm{A}, \mathrm{B}_0) \times \langle\mathbb{B}\rangle(\mathrm{B}_0, \mathrm{B}_1) \rightarrow \mathcal{R}(\mathrm{A}, \mathrm{B}_1)$

This generalizes from categories to profunctors: the *arrow profunctor* $\vec{f} : \vec{\mathbb{X}} \mid \vec{\mathbb{A}}$ consists of commutative squares $\mathrm{f}_0 \cdot \mathrm{a} = \mathrm{x} \cdot \mathrm{f}_1$, and parallel composition . The *weave vertical profunctor* $\langle f \rangle : \langle \mathbb{A} \rangle \mid \langle \mathbb{B} \rangle$ is the union of $\vec{f}$ and its opposite. A *matrix profunctor* $i(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \mid \mathcal{R}(\mathbb{A}, \mathbb{B})$ is a span of profunctors $f \leftarrow i \rightarrow g$, which is a bimodule from $\langle f \rangle$ to $\langle g \rangle$.



matrix profunctor

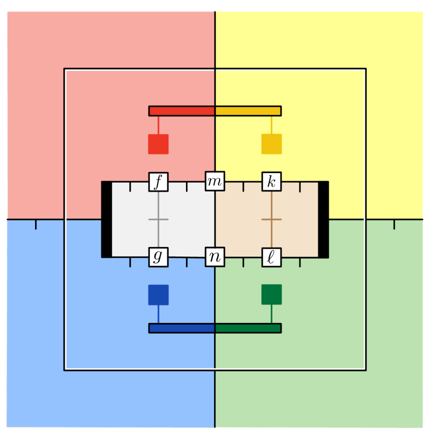
$\odot_f : \langle f \rangle(\mathrm{f}_0, \mathrm{f}_1) \times i(\mathrm{f}_1, \mathrm{g}) \Rightarrow i(\mathrm{f}_0, \mathrm{g})$



matrix profunctor

$\odot_g : i(\mathrm{f}, \mathrm{g}_0) \times \langle g \rangle(\mathrm{g}_0, \mathrm{g}_1) \Rightarrow i(\mathrm{f}, \mathrm{g}_1)$

Morphisms of matrix categories and matrix profunctors are *matrix functors* and *matrix transformations*. These form a double category $\mathrm{Mat}\mathbb{C}\mathrm{at}$ over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$. Sequential composition of matrix profunctors over that of profunctors is defined by a coequalizer, which nullifies the parallel action of zig-zags reassociating $[(\mathrm{f}_0, \mathrm{g}_0)] = [(\mathrm{f}_1, \mathrm{g}_1)] : \langle f \circ g \rangle$ and $[(\mathrm{k}_0, \mathrm{l}_0)] = [(\mathrm{k}_1, \mathrm{l}_1)] : \langle k \circ \ell \rangle$. (Definition 43)

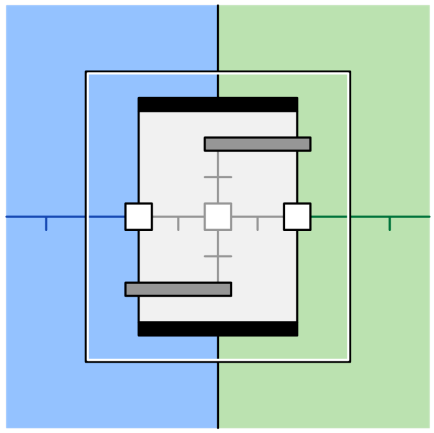


sequential composite

$$(m, n) \;\equiv\; (\mathrm{f} \odot m \odot \mathrm{k}, \mathrm{g} \odot n \odot \mathrm{k}) \;:\; m \diamond n$$

Moreover, $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a logic, and $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ is a *double fibration* [4]: sequential composition of matrix profunctors preserves substitution of transformations (starting at Prop. 46). Hence we call the structure $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ a *fibered logic*.

We then define *parallel composition* of matrix categories in Section 2.5. While profunctors compose by quotient, matrix categories compose by *codescent object* [20], which adjoins an associator isomorphism for the action by arrows and oparrows of the middle category.



parallel composition

$\alpha : (R, \bar{\mathrm{b}} \odot S) \cong (R \odot \bar{\mathrm{b}}, S)$

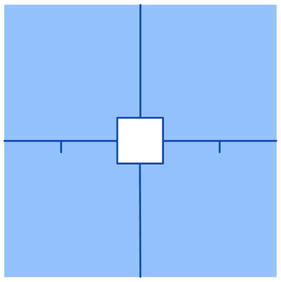
Dually, the category of matrix functors is constructed as a *descent object* [19]. So composition and transformation of matrix categories are dual, just as in the co/end calculus (Theorem 55).

$$\begin{array}{rcl} \mathcal{R} \otimes \mathcal{S} & = & \vec{\Sigma}\mathrm{B}. \quad \mathcal{R}(-, \mathrm{B}) \times \mathcal{S}(\mathrm{B}, -) \\ {[\mathcal{P}, \mathcal{Q}]} & = & \vec{\Pi}\mathrm{X}, \mathrm{Y}. \quad \mathcal{P}(\mathrm{X}, \mathrm{Y}) \to \mathcal{Q}(\mathrm{X}, \mathrm{Y}) \end{array}$$

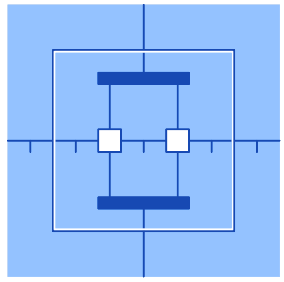
However, parallel composition does not preserve sequential composition of matrix profunctors: because both dimensions are bimodules, both compositions involve colimits which the other cannot represent. So $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at}$ is like a triple category without interchange, a structure on span categories: we define a *metalogic* to be a fibered logic $\mathbb{M} \to \mathbb{C} \times \mathbb{C}$, which forms a 2-weak category in $\mathrm{Span}\mathbb{C}\mathrm{at}$. [Definition 54]

Chapter 3: The metalogic of logics.

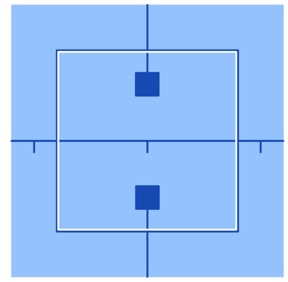
A bifibrant double category, i.e. a logic, is a pseudomonad in $\mathbb{M}\mathrm{at}\mathbb{C}\mathrm{at}$.



logic

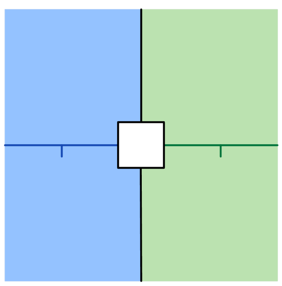


composition

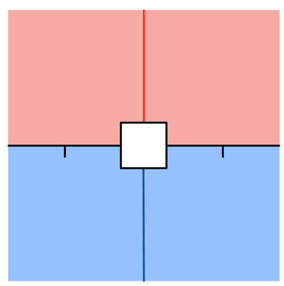


unit

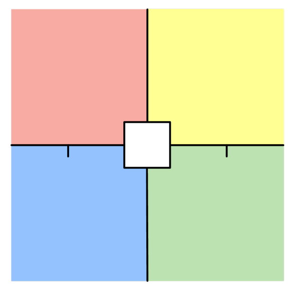
Because a logic is two-dimensional, there are *two* kinds of relations between logics: a *vertical profunctor* consists of processes between logics, and a *horizontal profunctor* consists of relations between logics. Two pairs are connected by a *double profunctor*, which consists of inferences between relations, along processes.



meta relation
[horiz. profunctor]

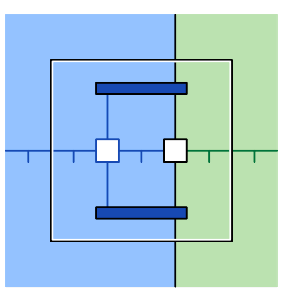


meta process
[vert. profunctor]

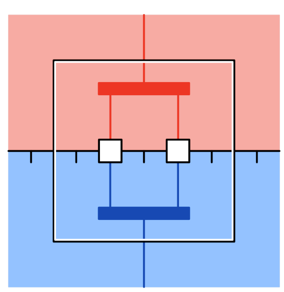


meta inference
[dbl. profunctor]

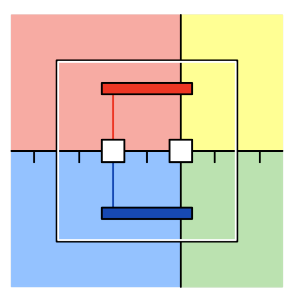
For horizontal profunctors, parallel composition is a familiar *bimodule* action. Yet because vertical profunctors are orthogonal, parallel composition defines a *monad* structure, and so double profunctors are bimodules thereof.



H-prof. composition

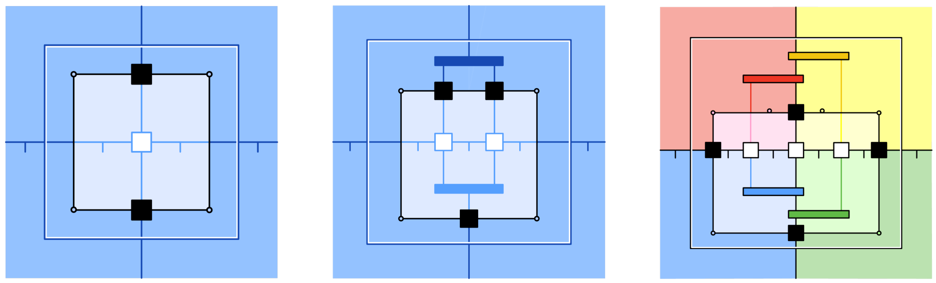


V-prof. composition



D-prof. composition

So logics have two kinds of "relations", and one kind of "function": a *double functor* $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ maps squares of $\mathbb{A}_0$ to squares of $\mathbb{A}_1$, preserving relation composition and unit up to coherent isomorphism. This generalizes to transformations of vertical, horizontal, and double profunctors; all four are defined by mapping squares in a way that coheres with parallel composition.



double functor — preserves composition; — double transformation

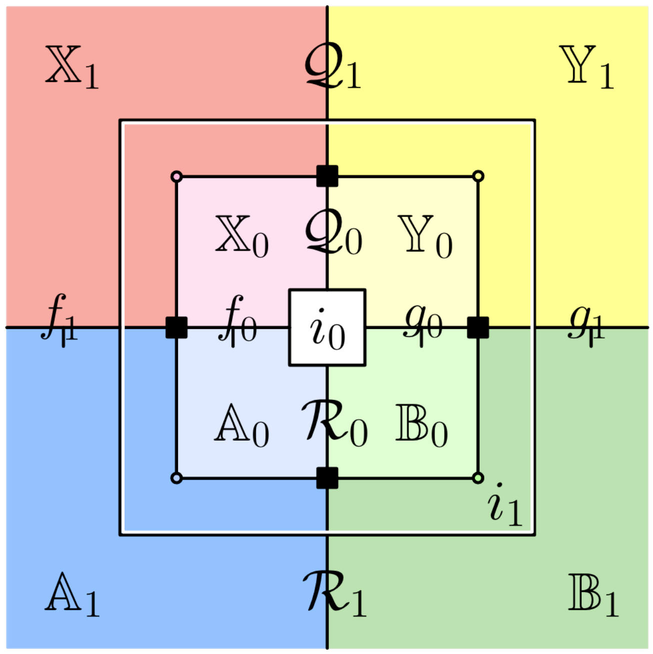
All together, logics form a metalogic: morphisms are functors, profunctors, and matrix categories; squares are vertical transformations, horizontal transformations, and double profunctors; and cubes are double transformations.

Below, the outline: we construct the metalogic of matrix categories, then apply the "horizontal pseudomonad" construction to form the metalogic of bifibrant double categories; and we give a metalogical interpretation of this structure.

	$\mathrm{Mat}\mathbb{C}\mathrm{at}$	$\mathrm{H.PsMnd}(-)$	$bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$	Logic
0	category	(H)-pseudomonad	bifibrant double category	logic
V	profunctor	(H)-vertical monad	vertical profunctor	meta process
H	matrix category	(H)-pseudobimodule	horizontal profunctor	meta relation
VH	matrix profunctor	(H)-vertical bimodule	double profunctor	meta inference
T	functor	ps. mnd. morphism	double functor	flow type
TV	transformation	v. mnd. morphism	vertical transformation	flow process
TH	matrix functor	ps. bim. morphism	horizontal transformation	flow relation
TVH	matrix transformation	v. bim. morphism	double transformation	flow inference

As a double profunctor consists of inferences between logics, a double transformation is a "flow" of *meta*-reasoning, a way to transform one system of reasoning into another.

In this sense, the language of $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ is the language of metalogic.



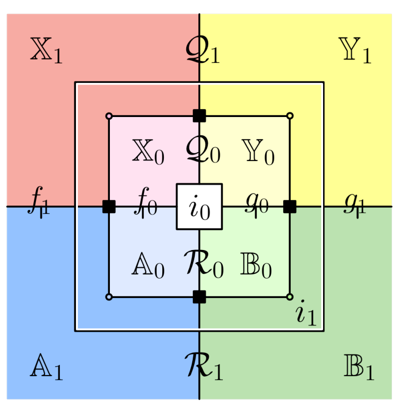
Chapter 1

Spans of categories

Our aim is to define the setting in which we can construct and explore logics. The most basic infrastructure we need first is *spans of categories*: a pair of categories of “types” which index a category of “relations”.

The morphisms of $\mathbb{C}\mathrm{at}$ are functors and profunctors; now spans of categories constitute a *third* dimension. Some work has considered $\mathrm{Span}(\mathbb{C}\mathrm{at})$ as a tricategory [20], but $\mathbb{C}\mathrm{at}$ is a double category, and so $\mathrm{Span}(\mathbb{C}\mathrm{at})$ is really a *lax triple category*, a.k.a. “intercategory” [7]. Profunctors and spans of profunctors are essential to metalogic, as these give processes and inferences between logics.

In this section, we introduce the three-dimensional visual language of spans of categories. Because one represents a category of relations connecting a pair of categories of types, we draw a span of categories as a string in the horizontal dimension.



span transformation $[\![i]\!] : i_0 \Rightarrow i_1$

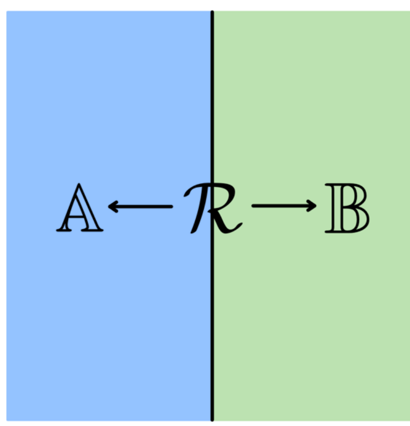
These spans form the new horizontal dimension, while profunctors are vertical, and functors are *transversal*, i.e. "out of the page". Hence the string diagrams of $\mathbb{C}\mathrm{at}$ are rotated to form the left and right faces of a cube, while the middle "horizontal slice" is span categories and span functors (top and bottom), span profunctors and span transformations (inner to outer).

To understand a span of categories as a dependent type, we show that inverse image along $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ determines a *displayed category*, a diagram $\mathcal{R}^* : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$ of categories and profunctors, with a monad structure for composition of $\mathcal{R}$. This extends to an equivalence of double categories $\mathrm{Span}\mathbb{C}\mathrm{at} \simeq \mathrm{Dis}\mathbb{C}\mathrm{at}$.

1.1 Span Category

Let $\mathbb{A}$ and $\mathbb{B}$ be categories. A **span category** from $\mathbb{A}$ to $\mathbb{B}$ is a category $\mathcal{R}$ with functors $\pi^{\mathcal{R}}_{\mathbb{A}} : \mathcal{R} \rightarrow \mathbb{A}$ and $\pi^{\mathcal{R}}_{\mathbb{B}} : \mathcal{R} \rightarrow \mathbb{B}$; we can denote the span by $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$, or $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$. Note this data is equivalent to a functor $(\pi^{\mathcal{R}}_{\mathbb{A}}, \pi^{\mathcal{R}}_{\mathbb{B}}) : \mathcal{R} \rightarrow \mathbb{A} \times \mathbb{B}$. The pair $\mathbb{A}, \mathbb{B}$ are the **base categories**, and $\mathcal{R}$ is the **total category**; we may refer to the span simply as $\mathcal{R}$.

We can draw a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ simply as a string.



span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$

We can see a span category as a *matrix of categories*, by inverse image along $\mathcal{R} \rightarrow \mathbb{A} \times \mathbb{B}$. The notion of inverse image along a functor $\mathcal{R} \rightarrow \mathbb{C}$ has been given by Street in [18]; the resulting map $\mathcal{R} : \mathbb{C} \rightarrow \mathbb{C}\mathrm{at}$ is called a *normal lax functor*. The notion was later developed for use in type theory, and rebranded as "displayed category" [1].

Definition 1. A **displayed category** $\mathcal{R} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}\mathrm{at}$ gives, for each pair:

objects $\mathrm{A} : \mathbb{A}, \mathrm{B} : \mathbb{B}$	a category $\mathcal{R}(\mathrm{A}, \mathrm{B})$
morphisms $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1), \mathrm{b} : \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$	a profunctor $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) : \mathcal{R}(\mathrm{A}_0, \mathrm{B}_0) \mid \mathcal{R}(\mathrm{A}_1, \mathrm{B}_1)$
composable pairs $(\mathrm{a}_1, \mathrm{b}_1), (\mathrm{a}_2, \mathrm{b}_2)$	a transformation $r \cdot r : \vec{\mathcal{R}}(\mathrm{a}_1, \mathrm{b}_1) \circ \vec{\mathcal{R}}(\mathrm{a}_2, \mathrm{b}_2) \Rightarrow \vec{\mathcal{R}}(\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2)$
objects $\mathrm{A} : \mathbb{A}, \mathrm{B} : \mathbb{B}$	an equality $\mathcal{R}(\mathrm{A}, \mathrm{B})(-, -) = \vec{\mathcal{R}}(\mathrm{id}_{\mathrm{A}}, \mathrm{id}_{\mathrm{B}})$

so that composition is associative and unital, i.e. $(r \cdot r) \cdot r = r \cdot (r \cdot r)$ and $\mathrm{id}_{\mathcal{R}} \cdot r = r = r \cdot \mathrm{id}_{\mathcal{R}}$.

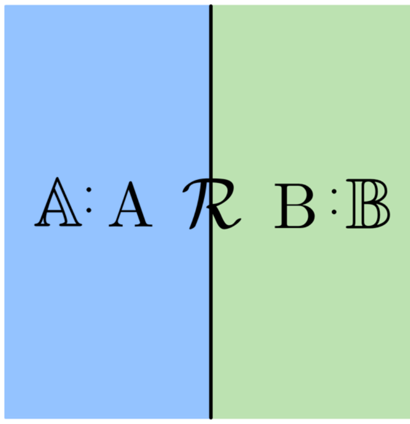
We give the main proposition, and then expound through the visual language of span categories.

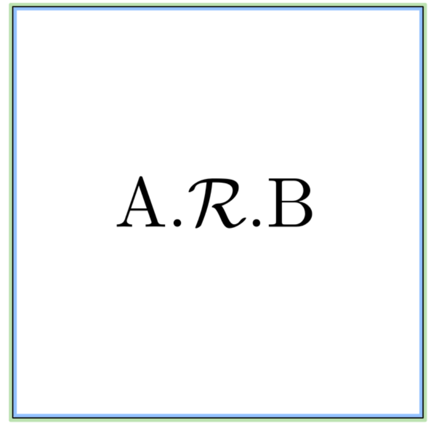
Proposition 2. Let $\mathbb{A}, \mathbb{B}$ be categories, and let $\mathbb{A} \leftarrow \mathcal{R} \to \mathbb{B}$ be a span of categories. Inverse image along $\mathcal{R} \to \mathbb{A} \times \mathbb{B}$ determines a displayed category $\mathcal{R} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}\mathrm{at}$. [18]

For each pair of objects $\mathrm{A} : \mathbb{A}, \mathrm{B} : \mathbb{B}$ there is a category $\mathcal{R}(\mathrm{A}, \mathrm{B})$ of objects $R : \mathcal{R}$ which map to (A, B), also known as the "fiber over" (A, B); this may also be denoted $\mathcal{R}^{\mathrm{A}}_{\mathrm{B}}$. This is given by pullback in Cat, of $\mathcal{R}$ along the functor which selects the pair (A, B).

$$\begin{tikzcd}
\mathcal{R}(\mathrm{A}, \mathrm{B}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{R} \arrow[d] \\
* \arrow[r, "(\mathrm{A},\mathrm{B})"] & \mathbb{A} \times \mathbb{B}
\end{tikzcd}$$

Color syntax now expands to a dependent type system of categories. The above pullback is depicted by substituting objects A, B in the color of each category $\mathbb{A}, \mathbb{B}$. In this way, substitution determines a matrix of categories. An entry is drawn on the right as a type in $\mathbb{C}\mathrm{at}$, which we color white as the "ambient" logic, outlined in blue and green to indicate that it is a diagram indexed by categories $\mathbb{A}$ and $\mathbb{B}$.

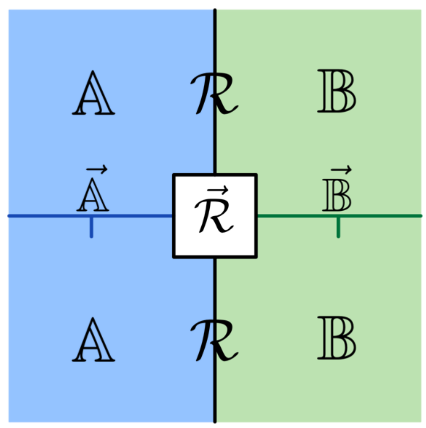




category $\mathcal{R}(\mathrm{A},\mathrm{B})$

Now, to consider the *morphisms* of a span category $\mathbb{A} \leftarrow \mathcal{R} \to \mathbb{B}$, we consider the induced span of *profunctors*, which we denote $\vec{\mathbb{A}} \leftarrow \vec{\mathcal{R}} \to \vec{\mathbb{B}}$. Profunctors are drawn as "bars" pointing downward, and the hom of $\mathcal{R}$ is drawn as a *bead* from the $\mathcal{R}$ string to itself along the homs of $\mathbb{A}$ and $\mathbb{B}$.

$$\begin{tikzcd}
\mathbb{A} \arrow[d, "\vec{\mathbb{A}}"', "|"] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "\vec{\mathcal{R}}"] \arrow[l, phantom, "\Longleftarrow"] \arrow[r, phantom, "\Longrightarrow"] & \mathbb{B} \arrow[d, "\vec{\mathbb{B}}", "|"'] \\
\mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{B}
\end{tikzcd}$$

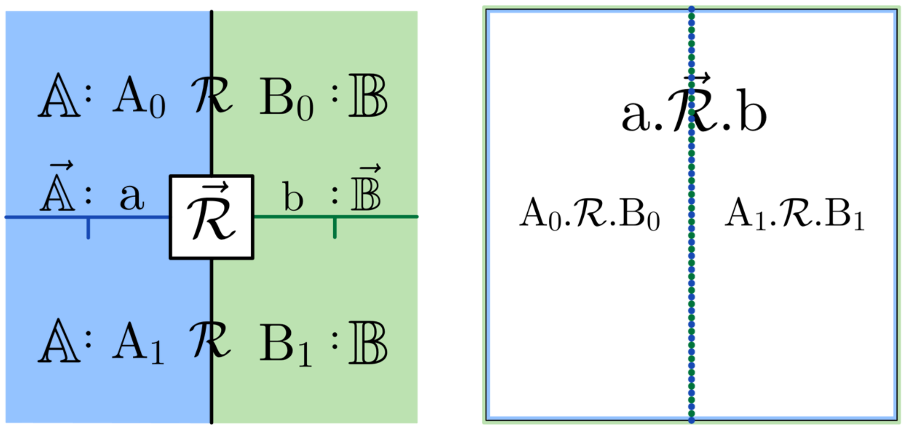


span hom-profunctor $\vec{\mathbb{A}} \leftarrow \vec{R} \to \vec{\mathbb{B}}$

Just as for objects the functor $\mathcal{R} \to \mathbb{A}\times\mathbb{B}$ gives a matrix of categories, for morphisms the transformation $\vec{\mathcal{R}} \Rightarrow \vec{\mathbb{A}}\times\vec{\mathbb{B}}$ determines a *matrix of profunctors*: for each pair of morphisms $\mathrm{a} : \mathbb{A}(\mathrm{A}_0,\mathrm{A}_1), \mathrm{b} : \mathbb{B}(\mathrm{B}_0,\mathrm{B}_1)$ there is a profunctor $\vec{\mathcal{R}}(\mathrm{a},\mathrm{b})$ from the category $\mathcal{R}(\mathrm{A}_0,\mathrm{B}_0)$ to $\mathcal{R}(\mathrm{A}_1,\mathrm{B}_1)$, also denoted $\vec{R}^{\mathrm{a}}_{\mathrm{b}}$. This is given by pullback in Prof of the hom of $\mathcal{R}$ along the functor which maps the walking arrow to (a,b).

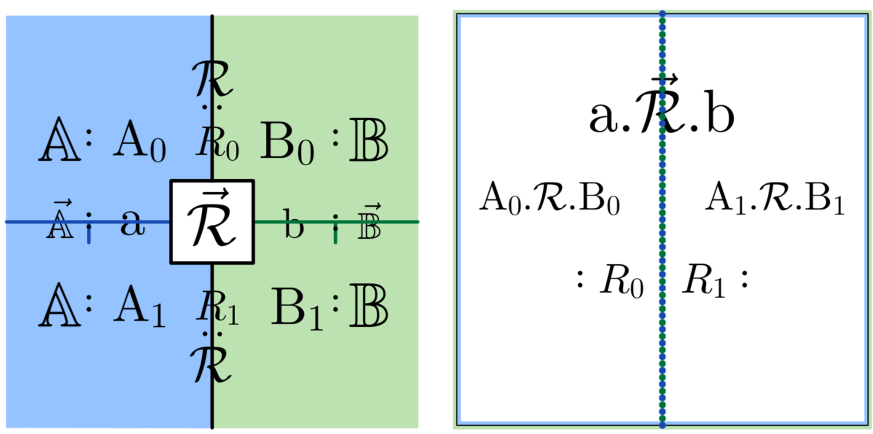
$$\begin{tikzcd}
\vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \vec{\mathcal{R}} \arrow[d] \\
{[0 \to 1]} \arrow[r, "(\mathrm{a},\mathrm{b})"] & \vec{\mathbb{A}} \times \vec{\mathbb{B}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R}(\mathrm{A}_0,\mathrm{B}_0) \arrow[r, "\vec{\mathcal{R}}(\mathrm{a},\mathrm{b})", "|"'] \arrow[d, dotted] \arrow[r, phantom, ""{name=U}] & \mathcal{R}(\mathrm{A}_1,\mathrm{B}_1) \arrow[d, dotted] \\
(\mathrm{A}_0,\mathrm{B}_0) \arrow[r, "(\mathrm{a},\mathrm{b})"] & (\mathrm{A}_1,\mathrm{B}_1)
\end{tikzcd}$$

The profunctor $\vec{\mathcal{R}}(\mathrm{a},\mathrm{b})$ is represented in color syntax by substituting a pair of morphisms (a,b) into the hom-profunctors $\vec{\mathbb{A}},\vec{\mathbb{B}}$. This determines a diagram of categories and profunctors $\vec{\mathcal{R}}:\vec{\mathbb{A}}\times\vec{\mathbb{B}}\to\mathrm{Prof}$, depicted on the right. Each profunctor is drawn as a blue and green "string of beads", as its elements can be understood as two-dimensional morphisms.



profunctor $\vec{\mathcal{R}}(\mathrm{a},\mathrm{b}):\mathcal{R}(\mathrm{A}_0,\mathrm{B}_0)\,|\,\mathcal{R}(\mathrm{A}_1,\mathrm{B}_1)$

We can now go one level further, to see the morphisms of the span category. Given $R_0:\mathcal{R}(\mathrm{A}_0,\mathrm{B}_0)$ and $R_1:\mathcal{R}(\mathrm{A}_1,\mathrm{B}_1)$ we have $\vec{\mathcal{R}}^{\mathrm{a}}_{\mathrm{b}}(R_0,R_1)$ is the set of morphisms $r:\mathcal{R}(R_0,R_1)$ over (a,b).



set $\vec{\mathcal{R}}^{\mathrm{a}}_{\mathrm{b}}(R_0,R_1)$

As the string diagram suggests, we can think of the objects of $\mathcal{R}$ as *relations*, i.e. horizontal morphisms, and morphisms of $\mathcal{R}$ as *inferences*, i.e. squares in a double category. Once we define matrix categories, by adding horizontal composition, this interpretation will be literal.

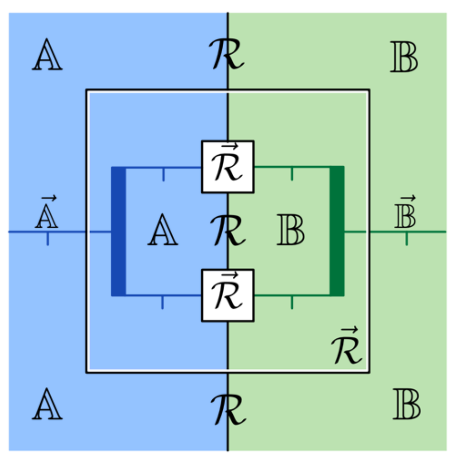
This completes the data of a span category, which as we see is two-dimensional; we now consider its structure of composition and unit, which is *three-dimensional*. We can draw a cube "head on" to see the inner source 2-cell and the four side faces; then we can "slice down the middle" to see the 3-cell which connects the source 2-cell to the target 2-cell outside.

A span category has a **composition** transformation $r \cdot r : \vec{\mathcal{R}} \circ \vec{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$ over composition of $\mathbb{A}$ and $\mathbb{B}$. We draw equalities as dotted lines.

$$\begin{tikzcd}
\mathbb{A} \arrow[dddd, "\shortmid" marking] \arrow[dr, dotted, no head] & & \mathcal{R} \arrow[ll] \arrow[rr] \arrow[d, dotted, no head] & & \mathbb{B} \arrow[dddd, "\shortmid" marking] \arrow[dl, dotted, no head] \\
& \mathbb{A} \arrow[d, "\shortmid" marking] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "\shortmid" marking] \arrow[l, phantom, shift right=4ex, "\Leftarrow"] \arrow[r, phantom, shift right=4ex, "\Rightarrow"] & \mathbb{B} \arrow[d, "\shortmid" marking] & \\
{} \arrow[r, phantom, "\Longleftarrow"] & \mathbb{A} \arrow[d, "\shortmid" marking] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "\shortmid" marking] \arrow[l, phantom, shift right=4ex, "\Leftarrow"] \arrow[r, phantom, shift right=4ex, "\Rightarrow"] & \mathbb{B} \arrow[d, "\shortmid" marking] \arrow[r, phantom, "\Longrightarrow"] & {} \\
& \mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \mathbb{B} & \\
\mathbb{A} \arrow[ur, dotted, no head] & & \mathcal{R} \arrow[ll] \arrow[rr] & & \mathbb{B} \arrow[ul, dotted, no head]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \arrow[r, "\vec{\mathcal{R}}", "\shortmid" marking] \arrow[d, equal] & \mathcal{R} \arrow[r, "\vec{\mathcal{R}}", "\shortmid" marking] \arrow[d, Rightarrow] & \mathcal{R} \arrow[d, equal] \\
\mathcal{R} \arrow[rr, "\vec{\mathcal{R}}", "\shortmid" marking] & & \mathcal{R}
\end{tikzcd}$$

We can draw a three-dimensional string diagram in the same way, "head on", but now we can see more: because the source and target 2-cells are drawn as "beads", the target can be depicted as a large "hollow" bead. We're looking at the front of a box and "poking a hole" to look inside.

$$\begin{tikzcd}
\mathbb{A} \arrow[dddd, "\shortmid" marking] \arrow[dr, dotted, no head] & & \mathcal{R} \arrow[ll] \arrow[rr] \arrow[d, dotted, no head] & & \mathbb{B} \arrow[dddd, "\shortmid" marking] \arrow[dl, dotted, no head] \\
& \mathbb{A} \arrow[d, "\shortmid" marking] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "\shortmid" marking] \arrow[l, phantom, shift right=4ex, "\Leftarrow"] \arrow[r, phantom, shift right=4ex, "\Rightarrow"] & \mathbb{B} \arrow[d, "\shortmid" marking] & \\
{} \arrow[r, phantom, "\Longleftarrow"] & \mathbb{A} \arrow[d, "\shortmid" marking] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "\shortmid" marking] \arrow[l, phantom, shift right=4ex, "\Leftarrow"] \arrow[r, phantom, shift right=4ex, "\Rightarrow"] & \mathbb{B} \arrow[d, "\shortmid" marking] \arrow[r, phantom, "\Longrightarrow"] & {} \\
& \mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \mathbb{B} & \\
\mathbb{A} \arrow[ur, dotted, no head] & & \mathcal{R} \arrow[ll] \arrow[rr] & & \mathbb{B} \arrow[ul, dotted, no head]
\end{tikzcd}$$



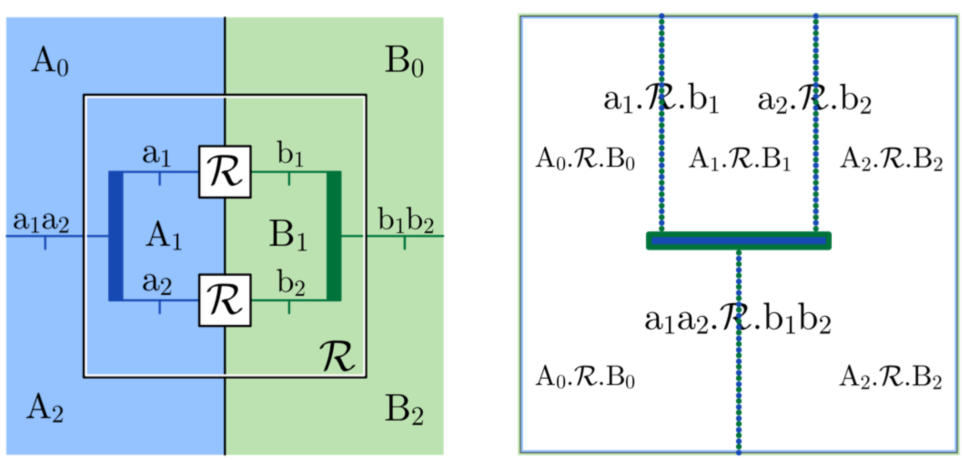
composition span transformation
$r \cdot r : \vec{\mathcal{R}} \circ \vec{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$

Yet to see the actual 3-morphism, we still need to "slice down the middle". As we do so, we draw the middle slice as its "displayed category" equivalent.

The span transformation $r\cdot r : \vec{\mathcal{R}}\circ\vec{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$ determines a matrix of transformations: for each composable pair of pairs $(a_1,b_1) : \mathbb{A}(A_0,A_1)\times\mathbb{B}(B_0,B_1)$ and $(a_2,b_2) : \mathbb{A}(A_1,A_2)\times\mathbb{B}(B_1,B_2)$, there is a transformation $r_1\cdot r_2 : \vec{\mathcal{R}}(a_1,b_1)\circ\vec{\mathcal{R}}(a_2,b_2) \Rightarrow \vec{\mathcal{R}}(a_1a_2,b_1b_2)$. This is given by functoriality of pullback in Prof.

$$\begin{tikzcd}
\vec{\mathcal{R}}\circ\vec{\mathcal{R}} \arrow[rrr, "r\cdot r"] \arrow[ddd] & & & \vec{\mathcal{R}} \arrow[ddd] \\
& \vec{\mathcal{R}}(a_1,b_1)\circ\vec{\mathcal{R}}(a_2,b_2) \arrow[ul] \arrow[r, dashed, "r_1\cdot r_2"] \arrow[d] & \vec{\mathcal{R}}(a_1a_2,b_1b_2) \arrow[ur] \arrow[d] & \\
& {[0\to 1]} \arrow[r, equal] \arrow[dl, "{((a_1,b_1),(a_2,b_2))}"'] & {[0\to 1]} \arrow[dr, "{(a_1a_2,b_1b_2)}"] & \\
(\vec{\mathbb{A}}\times\vec{\mathbb{B}})\circ(\vec{\mathbb{A}}\times\vec{\mathbb{B}}) \arrow[rrr] & & & \mathbb{A}\times\mathbb{B}
\end{tikzcd}$$

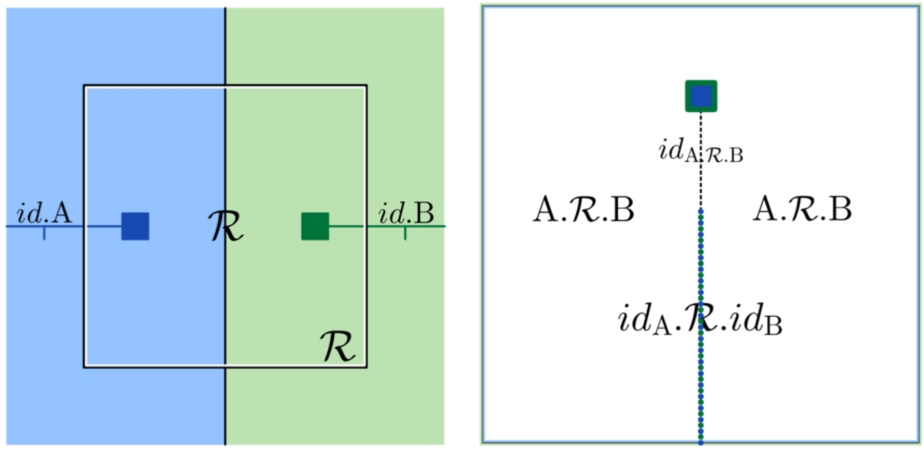
Again, this is given in color syntax by substituting morphisms (a_1,b_1) and (a_2,b_2) into the homs of $\mathbb{A}$ and $\mathbb{B}$. As diagrams become more complex, we may leave types implicit when they can be inferred in context. We may also use $\mathcal{R}(a,b)$ or $\mathcal{R}^{a}_{b}$ for the hom-profunctors, rather than $\vec{\mathcal{R}}(a,b)$.



$r_1\cdot r_2 : \mathcal{R}(a_1,b_1)\circ\mathcal{R}(a_2,b_2) \Rightarrow \mathcal{R}(a_1a_2,b_1b_2)$

On the left, we see the “hollow shell” of the cube; then to see the 3-morphism we slice down the middle: on the right is the span transformation $\vec{\mathcal{R}}\circ\vec{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$, as a matrix of transformations.

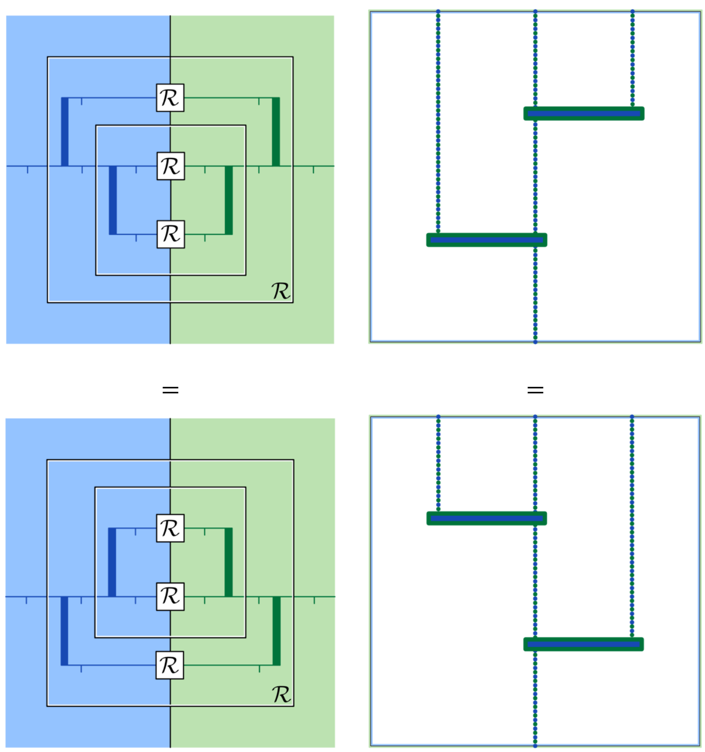
The second structure of a span category $\mathcal{R}$ is a **unit** transformation $\mathrm{id}_{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$. For each pair of objects, there is an *equality* unit transformation which identifies the profunctor $\vec{\mathcal{R}}(\mathrm{id}_A,\mathrm{id}_B)$ with the hom of $\mathcal{R}(A,B)$. So, the identities in $\mathcal{R}(A,B)$ become identities in $\vec{\mathcal{R}}(\mathrm{id}_A,\mathrm{id}_B)$.



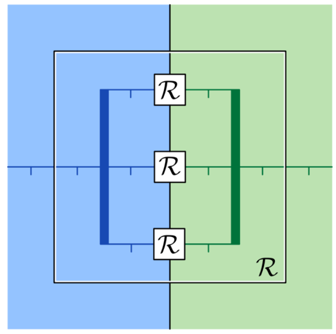
$$\mathrm{id}_{\mathcal{R}} : \mathcal{R}^{\mathrm{A}}_{\mathrm{B}} \Rightarrow \mathcal{R}^{\mathrm{A}}_{\mathrm{B}}(-,-) = \mathcal{R}^{\mathrm{id.A}}_{\mathrm{id.B}}$$

Finally, this structure satisfies two properties: composition is associative and unital.

For any composable triple r_1, r_2, r_3 we have $r_1 \cdot (r_2 \cdot r_3) = (r_1 \cdot r_2) \cdot r_3$.

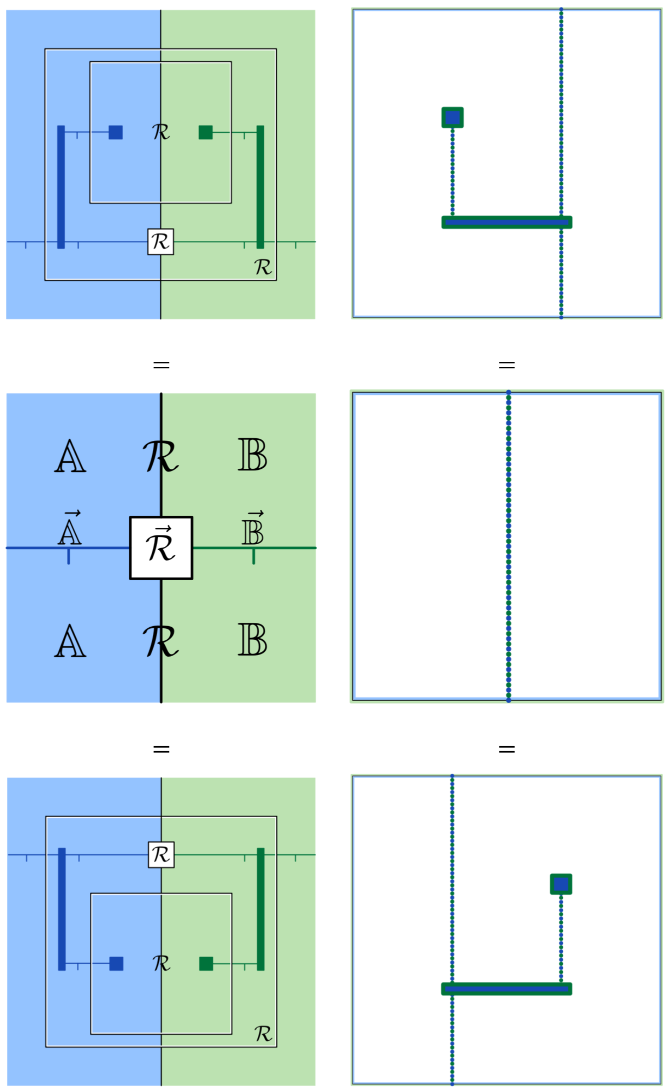


We introduce the **coherence principle** for three-dimensional string diagrams: in definitions, if we draw a cube which can be constructed in multiple ways, it means that these constructions are equal. Hence the above equation of associativity can be drawn as a single cube.



associativity of span category composition

For any morphism $r : \vec{R}^{\mathrm{a}}_{\mathrm{b}}(R_0, R_1)$ we have $\mathrm{id}_{R_0} \cdot r = r = r \cdot \mathrm{id}_{R_1}$.



In summary, a span of categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ is equivalent to a displayed category $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$: a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$ and profunctors $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})$, with composition $\vec{\mathcal{R}}(\mathrm{a}_1, \mathrm{b}_1) \circ \vec{\mathcal{R}}(\mathrm{a}_2, \mathrm{b}_2) \Rightarrow \vec{\mathcal{R}}(\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2)$ which is associative and unital.

The relations of a logic form such a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$; this is why span categories provide essential infrastructure for metalogic. Once we add the structure of parallel composition, a span category will be a "metarelation", i.e. horizontal profunctor, between logics.

1.1.1 Span Functor

Let $\mathbb{A}_0 \leftarrow \mathcal{R}_0 \rightarrow \mathbb{B}_0$ and $\mathbb{A}_1 \leftarrow \mathcal{R}_1 \rightarrow \mathbb{B}_1$ be span categories. A **span functor** from $\mathcal{R}_0$ to $\mathcal{R}_1$ is a pair of functors $[\![\mathbb{A}]\!] : \mathbb{A}_0 \rightarrow \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \rightarrow \mathbb{B}_1$, and a functor $[\![R]\!] : \mathcal{R}_0 \rightarrow \mathcal{R}_1$ such that the two squares commute, i.e. for any $R : \mathcal{R}_0$ over (A, B) we have that $[\![R]\!] : \mathcal{R}_1$ lies over $([\![\mathrm{A}]\!], [\![\mathrm{B}]\!])$.

This is equivalent to one commutative square, $[\![R]\!] : \mathcal{R}_0 \rightarrow \mathcal{R}_1$ over $[\![\mathrm{A}]\!] \times [\![\mathrm{B}]\!] : \mathbb{A}_0 \times \mathbb{B}_0 \rightarrow \mathbb{A}_1 \times \mathbb{B}_1$.

$$\begin{tikzcd}
\mathbb{A}_0 \arrow[d, "{[\![\mathbb{A}]\!]}"] & \mathcal{R}_0 \arrow[l] \arrow[r] \arrow[d, "{[\![\mathcal{R}]\!]}"] & \mathbb{B}_0 \arrow[d, "{[\![\mathbb{B}]\!]}"] \\
\mathbb{A}_1 & \mathcal{R}_1 \arrow[l] \arrow[r] & \mathbb{B}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R}_0 \arrow[r, "{[\![\mathcal{R}]\!]}"] \arrow[d] & \mathcal{R}_1 \arrow[d] \\
\mathbb{A}_0 \times \mathbb{B}_0 \arrow[r, "{[\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]}"'] & \mathbb{A}_1 \times \mathbb{B}_1
\end{tikzcd}$$

Just as a span category forms a matrix of categories, a span functor forms a matrix of functors.

Proposition 3. Let $\mathcal{R}_0 : \mathbb{A}_0 \times \mathbb{B}_0 \rightarrow \mathbb{C}\mathrm{at}$ and $\mathcal{R}_1 : \mathbb{A}_1 \times \mathbb{B}_1 \rightarrow \mathbb{C}\mathrm{at}$ be displayed categories, and let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \rightarrow \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \rightarrow \mathbb{B}_1$ be functors. A **displayed functor** $[\![R]\!] : \mathbb{A}_0 \times \mathbb{B}_0 \rightarrow \vec{\mathbb{C}\mathrm{at}}_0$ over $[\![\mathbb{A}]\!], [\![\mathbb{B}]\!]$ from $\mathcal{R}_0$ to $\mathcal{R}_1$ gives for each pair:

objects $\mathrm{A} : \mathbb{A}, \mathrm{B} : \mathbb{B}$	a functor $[\![R]\!](\mathrm{A}, \mathrm{B}) : \mathcal{R}_0(\mathrm{A}, \mathrm{B}) \rightarrow \mathcal{R}_1([\![\mathrm{A}]\!], [\![\mathrm{B}]\!])$
morphisms $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1), \mathrm{b} : \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$	a transformation $[\![r]\!](\mathrm{a}, \mathrm{b}) : \vec{\mathcal{R}}_0(\mathrm{a}, \mathrm{b}) \Rightarrow \vec{\mathcal{R}}_1([\![\mathrm{a}]\!], [\![\mathrm{b}]\!])$
composable pairs $(\mathrm{a}_1, \mathrm{b}_1), (\mathrm{a}_2, \mathrm{b}_2)$	an equality $[\![r]\!](\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2) = [\![r]\!](\mathrm{a}_1, \mathrm{b}_1) \cdot [\![r]\!](\mathrm{a}_2, \mathrm{b}_2)$
objects $R : \mathcal{R}_0(\mathrm{A}, \mathrm{B})$	an equality $[\![\mathrm{id}_R]\!] = \mathrm{id}_{[\![R]\!]}$

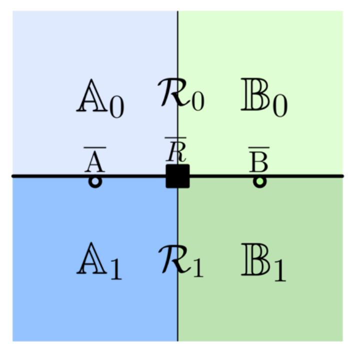
Proposition 4. Let $\mathbb{A}_0 \leftarrow \mathcal{R}_0 \rightarrow \mathbb{B}_0$ and $\mathbb{A}_1 \leftarrow \mathcal{R}_1 \rightarrow \mathbb{B}_1$ be span categories, let $[\![\mathrm{A}]\!] : \mathbb{A}_0 \rightarrow \mathbb{A}_1$ and $[\![\mathrm{B}]\!] : \mathbb{B}_0 \rightarrow \mathbb{B}_1$ be functors, and let $[\![R]\!] : \mathcal{R}_0 \rightarrow \mathcal{R}_1$ be a span functor over $[\![\mathrm{A}]\!], [\![\mathrm{B}]\!]$.

Inverse image along $[\![R]\!]$ determines a displayed functor $[\![R]\!] : \mathbb{A}_0 \times \mathbb{B}_0 \rightarrow \vec{\mathbb{C}\mathrm{at}}_0$.

Just as a displayed category is a lax functor or "poly-monad", a displayed functor is also known as a *transformation* of lax functors, i.e. a homomorphism of such monads. We now expound the idea.

A functor is a *transversal* morphism in $\mathrm{Span}\mathbb{C}\mathrm{at}$, and it is drawn as a string with a small "bubble" pointer, filled with the color of its source. A span functor, like a transformation, is drawn as a solid black bead, to distinguish from the "open" bead of a span profunctor.

$$\begin{tikzcd}
\mathbb{A}_0 \arrow[d, "{[\![\mathrm{A}]\!]}"] & \mathcal{R}_0 \arrow[l] \arrow[r] \arrow[d, "{[\![R]\!]}"] & \mathbb{B}_0 \arrow[d, "{[\![\mathrm{B}]\!]}"] \\
\mathbb{A}_1 & \mathcal{R}_1 \arrow[l] \arrow[r] & \mathbb{B}_1
\end{tikzcd}$$

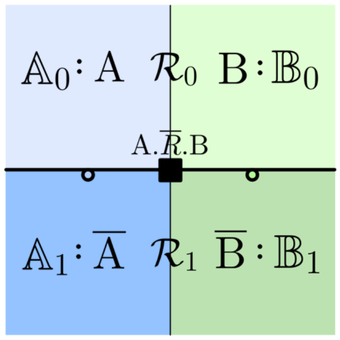


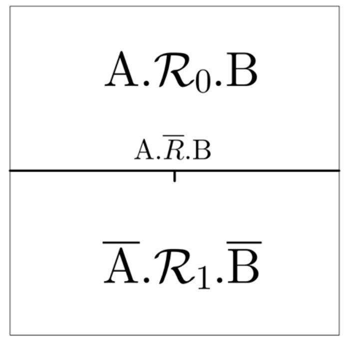
span functor $[\![R]\!] : \mathcal{R}_0 \rightarrow \mathcal{R}_1$

Inverse image defines a matrix of functors $[\![R]\!](\mathrm{A},\mathrm{B}) : \mathcal{R}_0(\mathrm{A},\mathrm{B}) \rightarrow \mathcal{R}_1([\![\mathrm{A}]\!],[\![\mathrm{B}]\!])$, by functoriality of pullback.

$$\begin{tikzcd}
\mathcal{R}_0 \arrow[rrr, "{[\![R]\!]}"] \arrow[ddd] & & & \mathcal{R}_1 \arrow[ddd] \\
& \mathcal{R}_0(\mathrm{A},\mathrm{B}) \arrow[ul] \arrow[r, dashed, "{[\![R]\!](\mathrm{A},\mathrm{B})}"] \arrow[d] \arrow[dr, phantom, "\llcorner", very near start] & \mathcal{R}_1([\![\mathrm{A}]\!],[\![\mathrm{B}]\!]) \arrow[ur] \arrow[d] \arrow[dl, phantom, "\lrcorner", very near start] & \\
& * \arrow[r, equal] \arrow[dl, "{(\mathrm{A},\mathrm{B})}"'] & * \arrow[dr, "{([\![\mathrm{A}]\!],[\![\mathrm{B}]\!])}"] & \\
\mathbb{A}_0 \times \mathbb{B}_0 \arrow[rrr, "{[\![\mathrm{A}]\!] \times [\![\mathrm{B}]\!]}"] & & & \mathbb{A}_1 \times \mathbb{B}_1
\end{tikzcd}$$

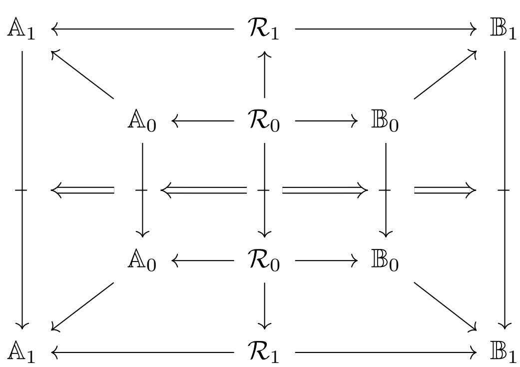
Each functor is determined in color syntax by substituting a pair of objects A,B into the base categories $\mathbb{A}_0, \mathbb{B}_0$ of the source span category $\mathcal{R}_0$.

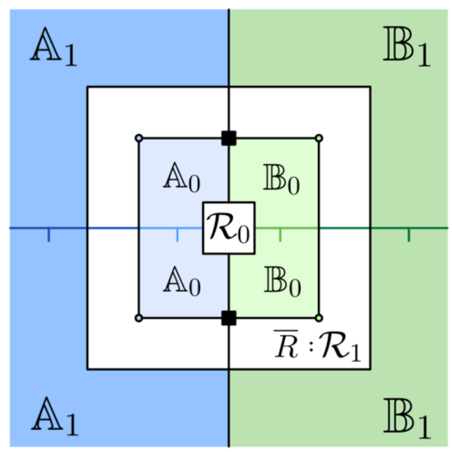




functor $[\![R]\!](\mathrm{A},\mathrm{B}) : \mathcal{R}_0(\mathrm{A},\mathrm{B}) \to \mathcal{R}_1([\![\mathrm{A}]\!],[\![\mathrm{B}]\!])$

In the same way for morphisms, the span functor induces a transformation of span profunctors. As span profunctors are two-dimensional, this transformation is *three-dimensional*, depicted below on the right. To distinguish this transformation in the diagram, we may designate it in white space between the span functor and the hom of the target span category.

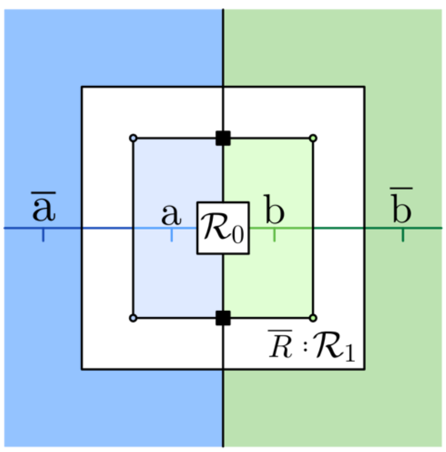


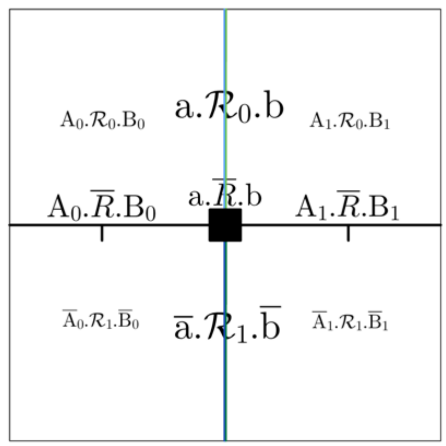


Inverse image determines a matrix of transformations $[\![r]\!](\mathrm{a},\mathrm{b}) : \vec{\mathcal{R}}_0(\mathrm{a},\mathrm{b}) \Rightarrow \vec{\mathcal{R}}_1([\![\mathrm{a}]\!],[\![\mathrm{b}]\!])$, by functoriality of pullback in Prof.

$$\begin{tikzcd}
\vec{\mathcal{R}}_0 \arrow[rrr, "{[\![r]\!]}"] \arrow[ddd] & & & \vec{\mathcal{R}}_1 \arrow[ddd] \\
& \vec{\mathcal{R}}_0(\mathrm{a},\mathrm{b}) \arrow[ul] \arrow[r, dashed, "{[\![r]\!](\mathrm{a},\mathrm{b})}"] \arrow[d] \arrow[dr, phantom, very near start, "\llcorner"] & \vec{\mathcal{R}}_1([\![\mathrm{a}]\!],[\![\mathrm{b}]\!]) \arrow[ur] \arrow[d] \arrow[dl, phantom, very near start, "\lrcorner"] & \\
& {[0 \to 1]} \arrow[r, equal] \arrow[dl, "{(\mathrm{a},\mathrm{b})}"'] & {[0 \to 1]} \arrow[dr, "{([\![\mathrm{a}]\!],[\![\mathrm{b}]\!])}"] & \\
\vec{\mathbb{A}}_0 \times \vec{\mathbb{B}}_0 \arrow[rrr, "{([\![\mathrm{a}]\!],[\![\mathrm{b}]\!])}" description] & & & \vec{\mathbb{A}}_1 \times \vec{\mathbb{B}}_1
\end{tikzcd}$$

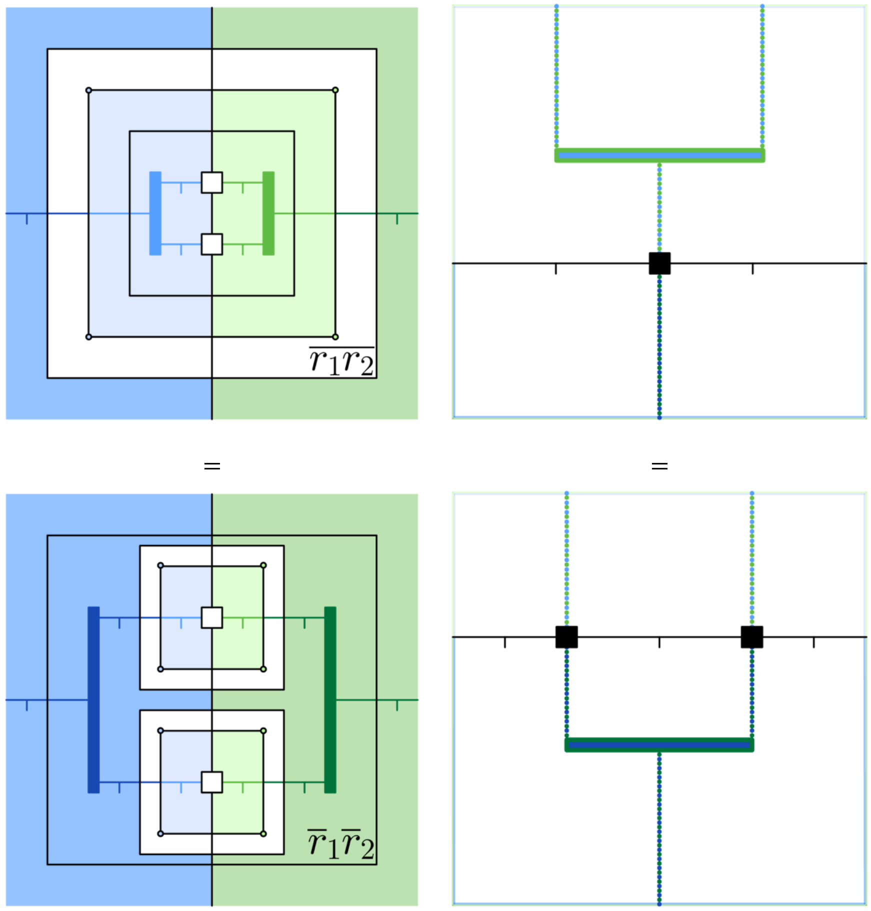
Again, this is represented in color syntax by substitution.



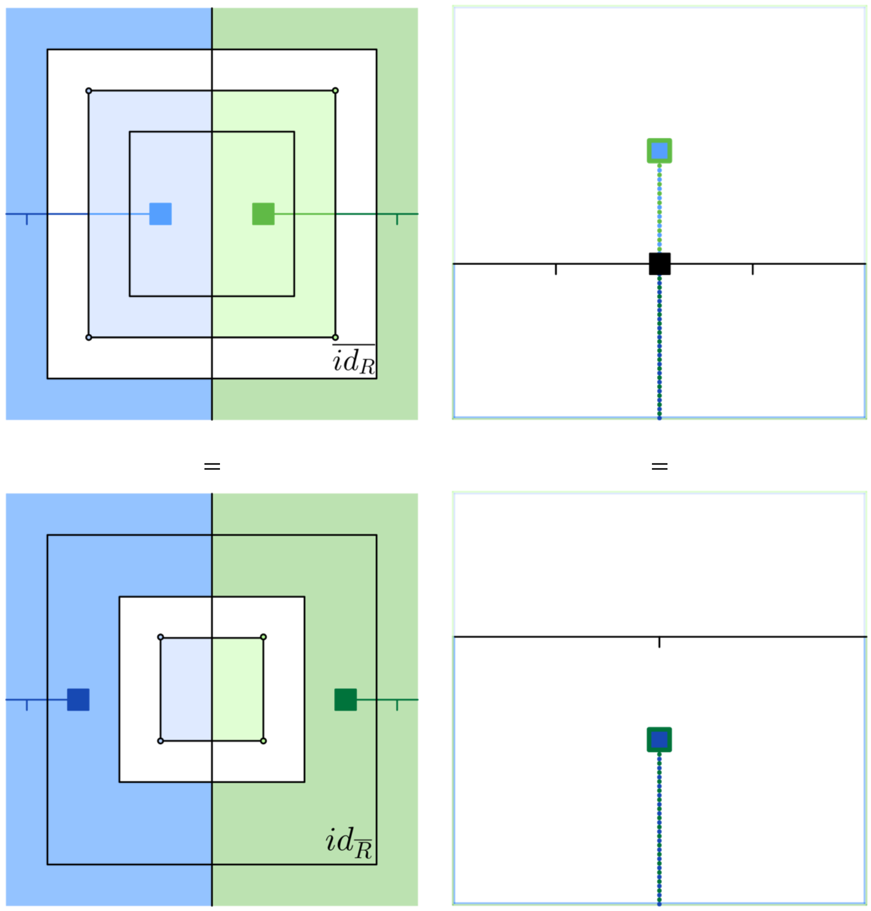


transformation $[\![R]\!](\mathrm{a},\mathrm{b}) : \mathcal{R}_0(\mathrm{a},\mathrm{b}) \Rightarrow \mathcal{R}_1([\![\mathrm{a}]\!],[\![\mathrm{b}]\!])$

This completes the structure of a span functor $[\![R]\!] : \mathcal{R}_0 \to \mathcal{R}_1$; lastly, this structure has the property that it preserves the composition and unit transformations of the span categories $\mathcal{R}_0, \mathcal{R}_1$.



$\overline{r_1 r_2}$
=
=
$\overline{r}_1\overline{r}_2$



So, a span functor $[\![R]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ over functors $[\![\mathrm{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathrm{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ is equivalent to a matrix of functors $[\![R]\!](\mathrm{A}, \mathrm{B}) : \mathcal{R}_0(\mathrm{A}, \mathrm{B}) \to \mathcal{R}_1([\![\mathrm{A}]\!], [\![\mathrm{B}]\!])$ and transformations $[\![r]\!](\mathrm{a}, \mathrm{b}) : \vec{\mathcal{R}}_0(\mathrm{a}, \mathrm{b}) \Rightarrow \vec{\mathcal{R}}_1([\![\mathrm{a}]\!], [\![\mathrm{b}]\!])$, which preserves the composition and unit of $\mathcal{R}_0$ and $\mathcal{R}_1$.

1.2 Span Profunctor

Recall that the collage of a profunctor forms a category, simply by making elements into morphisms. This justifies the interpretation of these elements simply as *morphisms between categories*. So now, a span of profunctors can be understood to consist of “morphisms between span categories”.

We introduce a new concept, *displayed profuntor*, given by inverse image along a transformation; a displayed profunctor is a relation of displayed categories, completing the equivalence between the logic of span categories and that of displayed categories.

Definition 5. Let $\mathbb{X} \leftarrow \mathcal{Q} \rightarrow \mathbb{Y}$ and $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ be spans of categories. A **span profunctor** from $\mathcal{Q}$ to $\mathcal{R}$ is a pair of profunctors $f : \mathbb{X} \mid \mathbb{A}$ and $g : \mathbb{Y} \mid \mathbb{B}$, and a profunctor $i : \mathcal{Q} \mid \mathcal{R}$ with transformations $\pi^i_f : i \Rightarrow f(\pi^{\mathcal{Q}}_{\mathbb{X}}, \pi^{\mathcal{R}}_{\mathbb{A}})$ and $\pi^i_g : i \Rightarrow g(\pi^{\mathcal{Q}}_{\mathbb{Y}}, \pi^{\mathcal{R}}_{\mathbb{B}})$, denoted $i(f,g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \mid \mathcal{R}(\mathbb{A}, \mathbb{B})$, with elements $i : i(\mathrm{f}, \mathrm{g})(Q, R) \equiv i^{\mathrm{f}}_{\mathrm{g}}(Q, R)$.

Note this data is equivalent to a transformation $(\pi^i_f, \pi^i_g) : i \Rightarrow (f \times g)(\pi^{\mathcal{Q}}_{\mathbb{X}} \times \pi^{\mathcal{Q}}_{\mathbb{Y}}, \pi^{\mathcal{R}}_{\mathbb{A}} \times \pi^{\mathcal{R}}_{\mathbb{B}})$.

$$\begin{tikzcd}
\mathbb{X} \arrow[d, "f"', "\shortmid" marking] & \mathcal{Q} \arrow[l] \arrow[r] \arrow[d, "i", "\shortmid" marking] \arrow[l, phantom, shift right=6, "\Leftarrow"] \arrow[r, phantom, shift right=6, "\Rightarrow"] & \mathbb{Y} \arrow[d, "g", "\shortmid" marking] \\
\mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Q} \arrow[r, "i", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathcal{R} \arrow[d] \\
\mathbb{X} \times \mathbb{Y} \arrow[r, "f \times g"', "\shortmid" marking] & \mathbb{A} \times \mathbb{B}
\end{tikzcd}$$

Above, we found that inverse image along a functor $\mathcal{R} \rightarrow \mathbb{A} \times \mathbb{B}$ determines a displayed category, a map $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$ with composition and unit. Now, we show that inverse image along a *transformation* $i \Rightarrow f \times g$ determines a *displayed profunctor*: a bimodule of displayed categories.

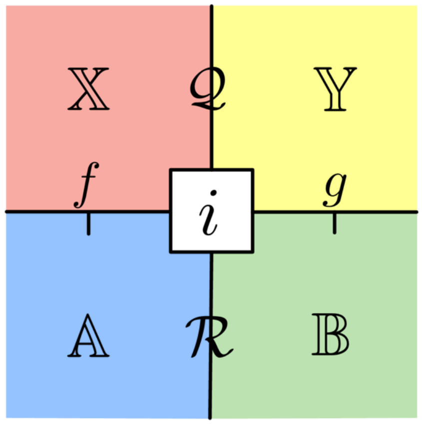
Proposition 6. Let $\mathbb{X} \leftarrow \mathcal{Q} \rightarrow \mathbb{Y}$ and $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ be span categories, giving displayed categories $\mathcal{Q} : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{C}\mathrm{at}$ and $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$. Let $i(f,g)$ be a span profunctor from $\mathcal{Q}$ to $\mathcal{R}$.

Inverse image along the transformation $i \Rightarrow f \times g$ determines a **displayed profunctor** $i : f \times g \rightarrow \mathrm{Prof}$ from displayed category $\mathcal{Q}$ to displayed category $\mathcal{R}$, which gives for each pair:

elements $\mathrm{f} : f(\mathrm{X}, \mathrm{A}), \mathrm{g} : g(\mathrm{Y}, \mathrm{B})$	a profunctor $i(\mathrm{f}, \mathrm{g}) : \mathcal{Q}(\mathrm{X}, \mathrm{A}) \mid \mathcal{R}(\mathrm{Y}, \mathrm{B})$
composable pairs $(\mathrm{x}, \mathrm{f}), (\mathrm{y}, \mathrm{g})$	a transformation $q \cdot i : \vec{\mathcal{Q}}(\mathrm{x}, \mathrm{y}) \circ i(\mathrm{f}, \mathrm{g}) \Rightarrow i(\mathrm{xf}, \mathrm{yg})$
composable pairs $(\mathrm{f}, \mathrm{a}), (\mathrm{g}, \mathrm{b})$	a transformation $i \cdot r : i(\mathrm{f}, \mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \Rightarrow i(\mathrm{fa}, \mathrm{gb})$
with associativity	$(q \cdot i) \cdot r = q \cdot (i \cdot r)$
and unitality	$\mathrm{id}.Q \cdot i = i = i \cdot \mathrm{id}.R$.

Just as a displayed category is a map $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$ with a "monad" structure for composition, i.e. a "lax functor", a displayed profunctor is a *bimodule* of such monads. One of the few references for this concept is given by Paré [16]. We now expound the concept, continuing to expand the visual language of $\mathrm{Span}\mathbb{C}\mathrm{at}$.

Generalizing the hom of a span category, a span profunctor can be drawn as a bead which connects the string of one span category to another, along the profunctors f and g drawn as bars pointing downward.

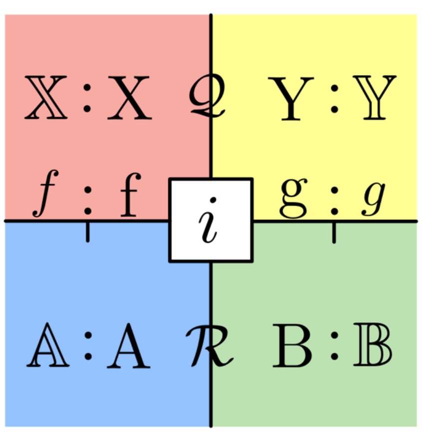


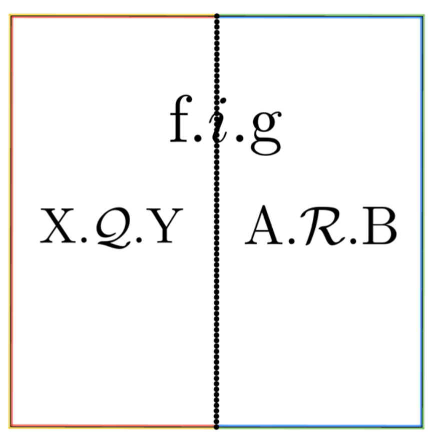
span profunctor $i(f,g) : \mathcal{Q} \mid \mathcal{R}$

Inverse image along the transformation $i \Rightarrow f \times g$ determines a *matrix of profunctors*: for each $\mathrm{f} : f(\mathrm{X}, \mathrm{A})$ and $\mathrm{g} : g(\mathrm{Y}, \mathrm{B})$ there is a profunctor $i(\mathrm{f}, \mathrm{g})$ from category $\mathcal{Q}(\mathrm{X}, \mathrm{Y})$ to category $\mathcal{R}(\mathrm{A}, \mathrm{B})$. This is given by pullback in Prof of $i \to f \times g$ along the transformation which maps the walking arrow to the pair $(\mathrm{f}, \mathrm{g}) : f \times g$.

$$\begin{tikzcd} i(\mathrm{f},\mathrm{g}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & i \arrow[d] \\ {[0 \to 1]} \arrow[r, "(\mathrm{f},\mathrm{g})"] & f \times g \end{tikzcd}$$

This pullback is represented in color syntax by substitution of a pair f, g onto the "bars" of the profunctors f, g. The resulting profunctor $i(\mathrm{f}, \mathrm{g}) : \mathcal{Q}(\mathrm{X}, \mathrm{Y}) \mid \mathcal{R}(\mathrm{A}, \mathrm{B})$ is a relation in the logic of $\mathbb{C}\mathrm{at}$, drawn on the right.



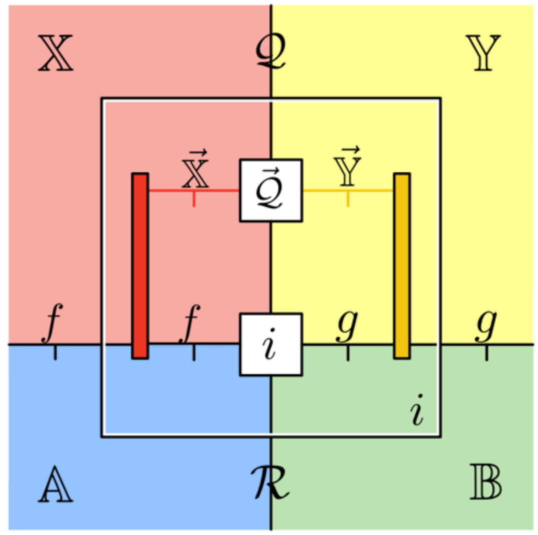


profunctor $i(\mathrm{f}, \mathrm{g}) : \mathcal{Q}(\mathrm{X}, \mathrm{Y}) \mid \mathcal{R}(\mathrm{A}, \mathrm{B})$

So the above is the data of a span profunctor, which is two-dimensional. Now we explicate its *structure*, sequential composition, which is *three-dimensional*.

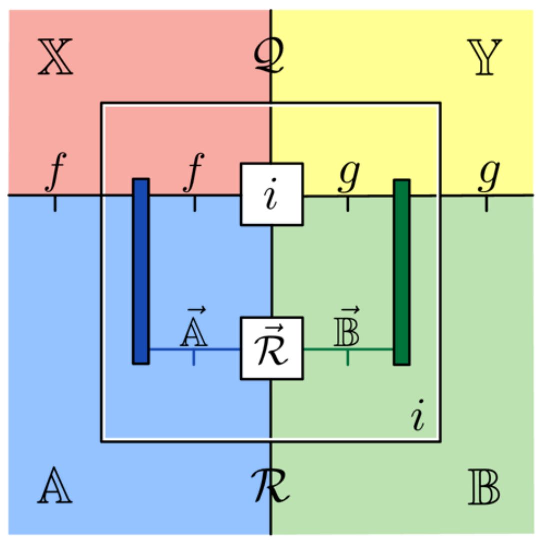
A span profunctor $i : \mathcal{Q} \mid \mathcal{R}$ has a *precompose* action $\vec{\mathcal{Q}} \circ i \to i$, and a *postcompose* action by $i \circ \mathcal{R} \to i$. Below, these are given in conventional diagrams, and then string diagrams.

$$
\begin{tikzcd}
\mathbb{X} \arrow[dddddd, "f"'] & & \mathcal{Q} \arrow[ll] \arrow[rr] \arrow[d, dotted, no head] & & \mathbb{Y} \arrow[dddddd, "g"] \\
& \mathbb{X} \arrow[dd, "\vec{\mathbb{X}}"'] \arrow[ul, dotted, no head] & \mathcal{Q} \arrow[l] \arrow[r] \arrow[dd, dashed, "\vec{\mathcal{Q}}"] & \mathbb{Y} \arrow[dd, "\vec{\mathbb{Y}}"] \arrow[ur, dotted, no head] & \\
& \arrow[r, phantom, "\Longleftarrow"] & \arrow[r, phantom, "\Longrightarrow"] & & \\
\arrow[r, phantom, "\Longleftarrow"] & \mathbb{X} \arrow[dd, "f"'] & \mathcal{Q} \arrow[l] \arrow[r] \arrow[dd, dashed, "i"] & \mathbb{Y} \arrow[dd, "g"] \arrow[r, phantom, "\Longrightarrow"] & \\
& \arrow[r, phantom, "\Longleftarrow"] & \arrow[r, phantom, "\Longrightarrow"] & & \\
& \mathbb{A} \arrow[dl, dotted, no head] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \mathbb{B} \arrow[dr, dotted, no head] & \\
\mathbb{A} & & \mathcal{R} \arrow[ll] \arrow[rr] & & \mathbb{B}
\end{tikzcd}
$$



precompose action $\vec{\mathcal{Q}} \circ i \Rightarrow i$

$$
\begin{tikzcd}
\mathbb{X} \arrow[dddddd, "f"'] & & \mathcal{Q} \arrow[ll] \arrow[rr] \arrow[d, dotted, no head] & & \mathbb{Y} \arrow[dddddd, "g"] \\
& \mathbb{X} \arrow[dd, "f"'] \arrow[ul, dotted, no head] & \mathcal{Q} \arrow[l] \arrow[r] \arrow[dd, dashed, "i"] & \mathbb{Y} \arrow[dd, "g"] \arrow[ur, dotted, no head] & \\
& \arrow[r, phantom, "\Longleftarrow"] & \arrow[r, phantom, "\Longrightarrow"] & & \\
\arrow[r, phantom, "\Longleftarrow"] & \mathbb{A} \arrow[dd, "\vec{\mathbb{A}}"'] & \mathcal{R} \arrow[l] \arrow[r] \arrow[dd, dashed, "\vec{\mathcal{R}}"] & \mathbb{B} \arrow[dd, "\vec{\mathbb{B}}"] \arrow[r, phantom, "\Longrightarrow"] & \\
& \arrow[r, phantom, "\Longleftarrow"] & \arrow[r, phantom, "\Longrightarrow"] & & \\
& \mathbb{A} \arrow[dl, dotted, no head] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \mathbb{B} \arrow[dr, dotted, no head] & \\
\mathbb{A} & & \mathcal{R} \arrow[ll] \arrow[rr] & & \mathbb{B}
\end{tikzcd}
$$

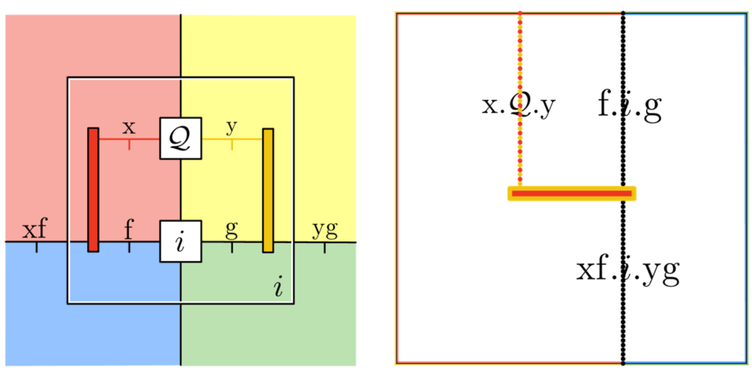


postcompose action $i \circ \vec{\mathcal{R}} \Rightarrow i$

Precomposition by $\mathcal{Q}$ is a matrix of transformations (indexed by composable pairs) $\mathrm{comp}_{\mathcal{Q}} : \vec{\mathcal{Q}}(\mathrm{x}, \mathrm{y}) \circ i(\mathrm{f}, \mathrm{g}) \Rightarrow i(\mathrm{xf}, \mathrm{yg})$. This is given by the functoriality of pullback in $\mathbb{P}\mathrm{rof}$.

$$
\begin{tikzcd}
\vec{\mathcal{Q}} \circ i \arrow[rrr] \arrow[ddd] & & & i \arrow[ddd] \\
& \vec{\mathcal{Q}}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \arrow[ul] \arrow[r, dashed] \arrow[d] \arrow[d, phantom, "\llcorner", shift right=30pt] & i(\mathrm{xf},\mathrm{yg}) \arrow[ur] \arrow[d] \arrow[d, phantom, "\lrcorner", shift left=30pt] & \\
& {[0 \to 1]} \arrow[r, equal] \arrow[dl, "{((\mathrm{x},\mathrm{y}),(\mathrm{f},\mathrm{g}))}"'] & {[0 \to 1]} \arrow[dr, "{(\mathrm{xf},\mathrm{yg})}"] & \\
(\vec{\mathbb{X}} \times \vec{\mathbb{Y}}) \circ (f \times g) \arrow[rrr] & & & f \times g
\end{tikzcd}
$$

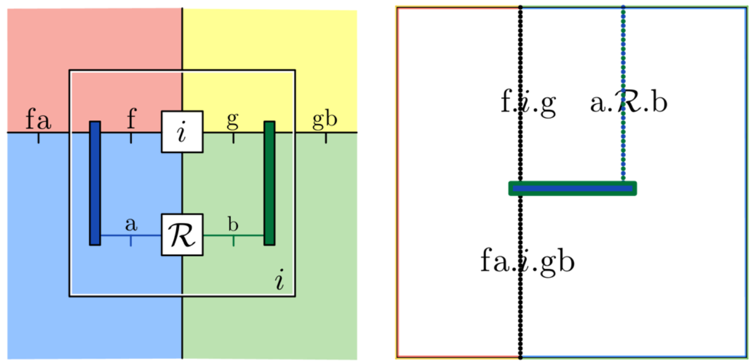
So, substitution in the string diagram for composition determines a transformation in $\mathbb{C}\mathrm{at}$.



$q \cdot i : \vec{\mathcal{Q}}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg})$

Postcomposition by $\vec{\mathcal{R}}$ is a matrix of transformations $\mathrm{comp}_{\mathcal{R}} : i(\mathrm{f},\mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb})$.

$$
\begin{tikzcd}
i \circ \vec{\mathcal{R}} \arrow[rrr] \arrow[ddd] & & & i \arrow[ddd] \\
& i(\mathrm{f},\mathrm{g}) \circ \vec{R}(\mathrm{a},\mathrm{b}) \arrow[ul] \arrow[r, dashed] \arrow[d] \arrow[d, phantom, "\llcorner", shift right=30pt] & i(\mathrm{fa},\mathrm{gb}) \arrow[ur] \arrow[d] \arrow[d, phantom, "\lrcorner", shift left=20pt] & \\
& {[0 \to 1]} \arrow[r, equal] \arrow[dl, "{((\mathrm{f},\mathrm{g}),(\mathrm{a},\mathrm{b}))}"'] & {[0 \to 1]} \arrow[dr, "{(\mathrm{fa},\mathrm{gb})}"] & \\
(f \times g) \circ (\vec{\mathbb{A}} \times \vec{\mathbb{B}}) \arrow[rrr] & & & f \times g
\end{tikzcd}
$$

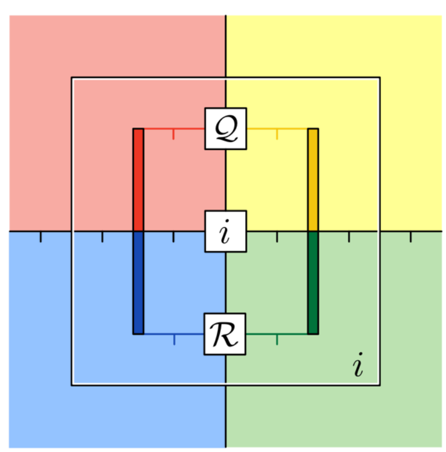


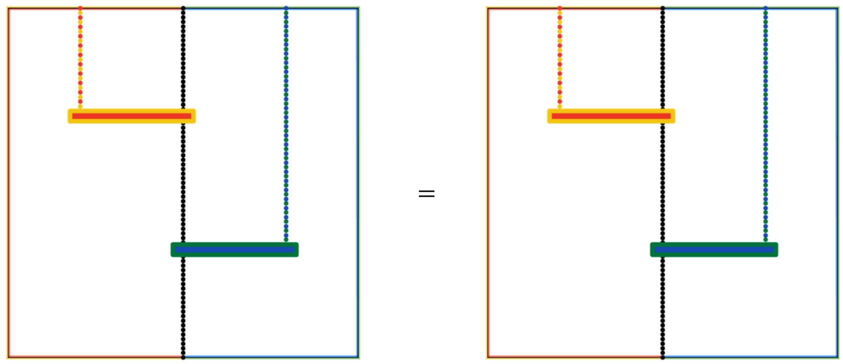
$$i \cdot r : i(\mathrm{f}, \mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \Rightarrow i(\mathrm{fa}, \mathrm{gb})$$

Hence the structure of a span profunctor $i : \mathcal{Q} \mid \mathcal{R}$, precomposition by $\vec{\mathcal{Q}}$ and postcomposition by $\vec{\mathcal{R}}$, is given by matrices of transformations $\vec{\mathcal{Q}}(\mathrm{x}, \mathrm{y}) \circ i(\mathrm{f}, \mathrm{g}) \Rightarrow i(\mathrm{xf}, \mathrm{yg})$ and $i(\mathrm{f}, \mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \Rightarrow i(\mathrm{fa}, \mathrm{gb})$. To complete the exposition, this structure satisfies the *property* of associativity.

$$\begin{tikzcd}
\vec{\mathcal{Q}}(\mathrm{x}, \mathrm{y}) \circ i(\mathrm{f}, \mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \arrow[r] \arrow[d] & \vec{\mathcal{Q}}(\mathrm{x}, \mathrm{y}) \circ i(\mathrm{fa}, \mathrm{gb}) \arrow[d] \\
i(\mathrm{xf}, \mathrm{yg}) \circ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \arrow[r] & i(\mathrm{xfa}, \mathrm{ygb})
\end{tikzcd}$$

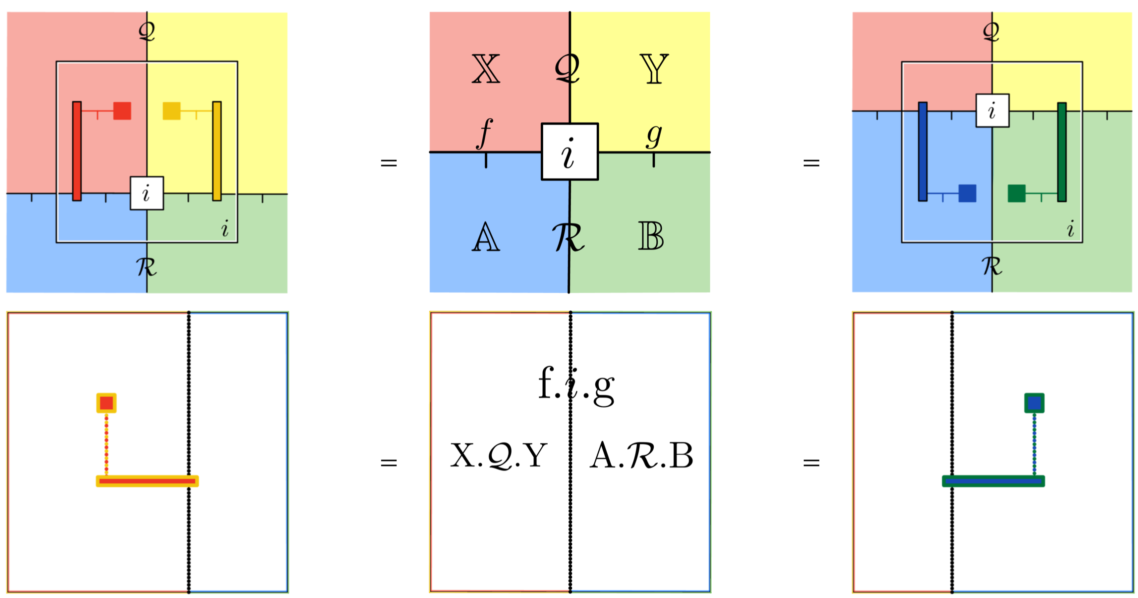
By the "coherence principle" of string diagrams, introduced for span categories, associativity can be depicted simply by drawing the cube $\mathcal{Q} \circ i \circ \mathcal{R} \to i$. This expresses that the cube is "coherent" or well-defined, i.e. the two transformations $\vec{\mathcal{Q}} \circ i \circ \vec{\mathcal{R}} \to i$ are equal.





$\mathcal{Q}(\mathrm{x},\mathrm{y}) \cdot (i(\mathrm{f},\mathrm{g}) \cdot \mathcal{R}(\mathrm{a},\mathrm{b}))$ $\quad$ $(\mathcal{Q}(\mathrm{x},\mathrm{y}) \cdot i(\mathrm{f},\mathrm{g})) \cdot \mathcal{R}(\mathrm{a},\mathrm{b})$

Finally, composition is unital.



$\mathrm{id}_Q \cdot i$ $\quad$ i $\quad$ $i \cdot \mathrm{id}_R$

In summary, just as a span category can be understood as a matrix of categories, a span profunctor can be understood as a matrix of profunctors $i(\mathrm{f},\mathrm{g}) : \mathcal{Q}(\mathrm{X},\mathrm{Y}) \mid \mathcal{R}(\mathrm{A},\mathrm{B})$, with actions for sequential composition $q \cdot i : \vec{\mathcal{Q}}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg})$ and $i \cdot r : i(\mathrm{f},\mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb})$, which are associative.

This concept is precisely what was needed to complete the framework for metalogic: the *inferences* of a logic, form a matrix of profunctors. Once we add parallel composition, span profunctors will form "metainferences", i.e. double profunctors, between logics. Metalogic is the language of metainferences and their transformations.

1.2.1 Span Transformation

To complete the double category of span categories, we define *transformations* of span profunctors. Just as a span profunctor can be understood as giving morphisms between span categories, a span transformation is simply a functor of such morphisms.

Definition 7. Let $\mathcal{Q}_0 : \mathbb{X}_0 \,\|\, \mathbb{Y}_0, \mathcal{R}_0 : \mathbb{A}_0 \,\|\, \mathbb{B}_0, \mathcal{Q}_1 : \mathbb{X}_1 \,\|\, \mathbb{Y}_1, \mathcal{R}_1 : \mathbb{A}_1 \,\|\, \mathbb{B}_1$ be span categories.

Let $\llbracket Q \rrbracket : \mathcal{Q}_0 \to \mathcal{Q}_1$ and $\llbracket R \rrbracket : \mathcal{R}_0 \to \mathcal{R}_1$ be span functors over $(\llbracket \mathbb{X} \rrbracket, \llbracket \mathbb{Y} \rrbracket)$ and $(\llbracket \mathbb{A} \rrbracket, \llbracket \mathbb{B} \rrbracket)$.

Let $i_0(f_1, g_1) : \mathcal{Q}_0 \mid \mathcal{R}_0, i_1(f_1, g_1) : \mathcal{Q}_1 \mid \mathcal{R}_1$ be span profunctors.

A **span transformation** $\llbracket i \rrbracket : i_0 \Rightarrow i_1$ is a pair of transformations $\llbracket f \rrbracket : f_0 \Rightarrow f_1$ over $(\llbracket \mathbb{X} \rrbracket, \llbracket \mathbb{A} \rrbracket)$ and $\llbracket g \rrbracket : g_0 \Rightarrow g_1$ over $(\llbracket \mathbb{Y} \rrbracket, \llbracket \mathbb{B} \rrbracket)$, and a transformation $\llbracket i \rrbracket : i_0 \Rightarrow i_1$ over $(\llbracket \mathcal{Q} \rrbracket, \llbracket \mathcal{R} \rrbracket)$, such that the two squares commute.

$$
\begin{tikzcd}
\mathbb{X}_1 \arrow[dddd, "f_1"', "|" marking] & & \mathcal{Q}_1 \arrow[ll] \arrow[rr] \arrow[dddd, "i_1", "|" marking] & & \mathbb{Y}_1 \arrow[dddd, "g_1", "|" marking] \\
& \mathbb{X}_0 \arrow[ul, "\llbracket \mathrm{X} \rrbracket"] \arrow[dd, "f_0"', "|" marking] & \mathcal{Q}_0 \arrow[l] \arrow[r] \arrow[u, "\llbracket Q \rrbracket"] \arrow[dd, "i_0", "|" marking] & \mathbb{Y}_0 \arrow[ur, "\llbracket \mathrm{Y} \rrbracket"'] \arrow[dd, "g_0", "|" marking] & \\
\arrow[r, phantom, "\Swarrow \llbracket \mathrm{f} \rrbracket"] & & \arrow[phantom, "\Downarrow \llbracket i \rrbracket"] & & \arrow[l, phantom, "\Searrow \llbracket \mathrm{g} \rrbracket"] \\
& \mathbb{A}_0 \arrow[dl, "\llbracket \mathrm{A} \rrbracket"'] & \mathcal{R}_0 \arrow[l] \arrow[r] \arrow[d, "\llbracket R \rrbracket"] & \mathbb{B}_0 \arrow[dr, "\llbracket \mathrm{B} \rrbracket"] & \\
\mathbb{A}_1 & & \mathcal{R}_1 \arrow[ll] \arrow[rr] & & \mathbb{B}_1
\end{tikzcd}
$$

Note this is equivalent to one commutative square of transformations, $[\![i]\!] : i_0 \to i_1$ over $[\![f]\!] \times [\![g]\!]$.

$$
\begin{tikzcd}
\mathcal{Q}_0 \arrow[rrrr, "i_0"] \arrow[dr] \arrow[ddd, "{[\![Q]\!]}"'] & & & & \mathcal{R}_0 \arrow[dl] \arrow[ddd, "{[\![R]\!]}"] \\
& \mathbb{X}_0 \times \mathbb{Y}_0 \arrow[rr, "f_0 \times g_0"'] \arrow[d, "{[\![\mathrm{X}]\!]\times[\![\mathrm{Y}]\!]}"] & & \mathbb{A}_0 \times \mathbb{B}_0 \arrow[d, "{[\![\mathrm{A}]\!]\times[\![\mathrm{B}]\!]}"] & \\
& \mathbb{X}_1 \times \mathbb{Y}_1 \arrow[rr, "f_1 \times g_1"] & & \mathbb{A}_1 \times \mathbb{B}_1 & \\
\mathcal{Q}_1 \arrow[rrrr, "i_1"'] \arrow[ur] & & & & \mathcal{R}_1 \arrow[ul]
\arrow[from=1-3, to=2-3, phantom, "\Downarrow"]
\arrow[from=2-3, to=3-3, phantom, "\Downarrow {[\![\mathrm{f}]\!]\times[\![\mathrm{g}]\!]}"]
\arrow[from=4-3, to=3-3, phantom, "\Uparrow"]
\arrow[from=1-4, to=4-4, phantom, "\Downarrow {[\![i]\!]}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
i_0 \arrow[r, "{[\![i]\!]}"] \arrow[d] & i_1 \arrow[d] \\
f_0 \times g_0 \arrow[r, "{[\![\mathrm{f}]\!]\times[\![\mathrm{g}]\!]}"] & f_1 \times g_1
\end{tikzcd}
$$

Just as a span profunctor is equivalent to a matrix of profunctors, a span transformation is a matrix of transformations.

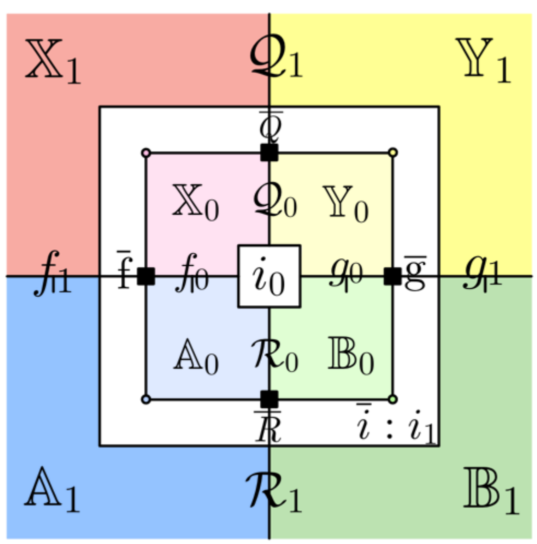
Definition 8. A **displayed transformation** $[\![i]\!] : f \times g \to \vec{\mathbb{C}\mathrm{at}}_1$ gives for each pair

morphisms $\mathrm{f} : f(\mathrm{X}, \mathrm{A}), \mathrm{g} : g(\mathrm{Y}, \mathrm{B})$ a transformation $[\![i]\!](\mathrm{f}, \mathrm{g}) : i_0(\mathrm{f}, \mathrm{g}) \Rightarrow i_1([\![\mathrm{f}]\!], [\![\mathrm{g}]\!])$

preserving composition.

We now expound the idea, completing the visual language of $\mathrm{Span}\mathbb{C}\mathrm{at}$.

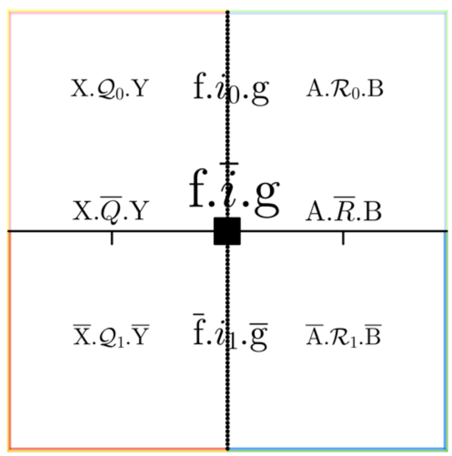
A span transformation is a cube: the inner face is the source span profunctor i_0, and the outer face is the target span profunctor i_1. The left and right faces are transformations $[\![\mathrm{f}]\!] : f_0 \Rightarrow f_1$ and $[\![\mathrm{g}]\!] : g_0 \to g_1$, and the top and bottom faces are span functors $[\![Q]\!] : \mathcal{Q}_0 \to \mathcal{Q}_1$ and $[\![R]\!] : \mathcal{R}_0 \to \mathcal{R}_1$.



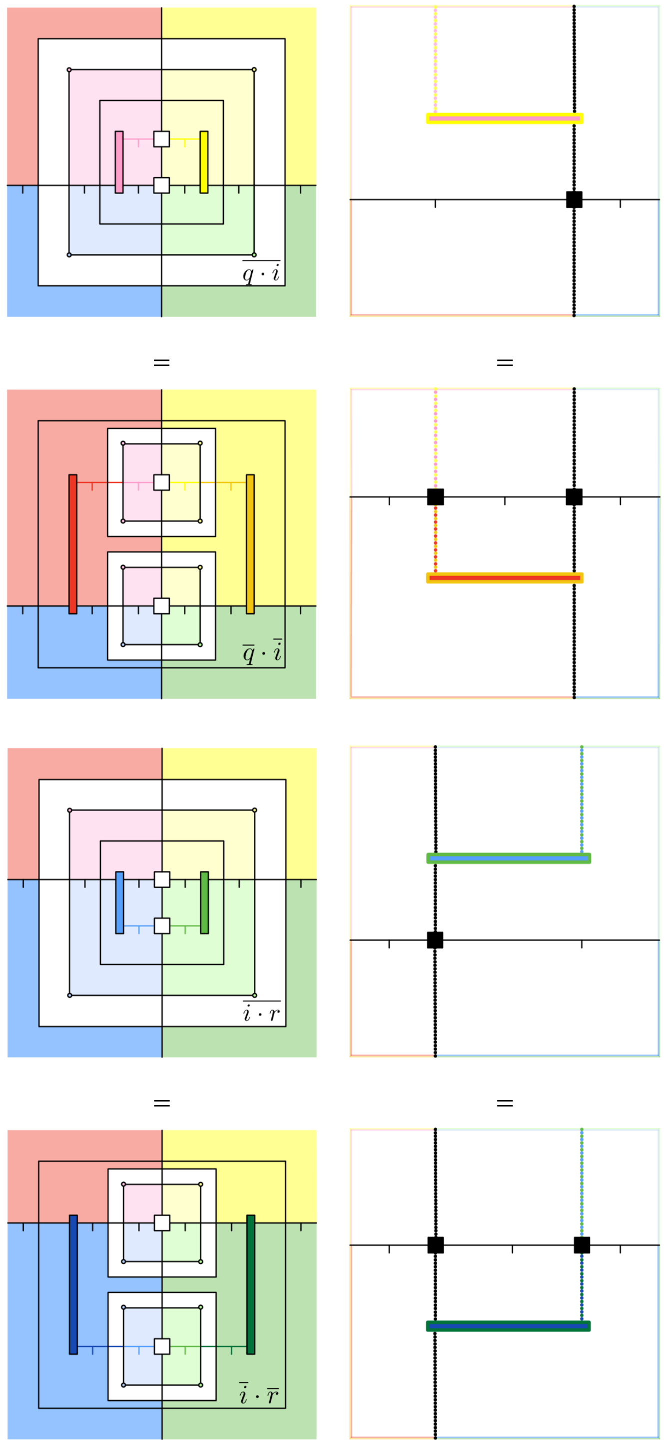
Substitution determines a matrix of transformations, again by functoriality of pullback.

$$
\begin{tikzcd}
i_0 \arrow[rrr, "{[\![i]\!]}"] \arrow[ddd] & & & i_1 \arrow[ddd] \\
& i_0(\mathrm{f},\mathrm{g}) \arrow[ul] \arrow[r, dashed, "{\bar{i}(\mathrm{f},\mathrm{g})}"] \arrow[d] \arrow[d, phantom, "\llcorner", very near start, shift right=3ex] & i_1([\![\mathrm{f}]\!],[\![\mathrm{g}]\!]) \arrow[ur] \arrow[d] \arrow[d, phantom, "\lrcorner", very near start, shift left=3ex] & \\
& {[0 \to 1]} \arrow[r, equal] \arrow[dl, "{(\mathrm{f},\mathrm{g})}"'] & {[0 \to 1]} \arrow[dr, "{([\![\mathrm{f}]\!],[\![\mathrm{g}]\!])}"] & \\
f_0 \times g_0 \arrow[rrr, "{[\![\mathrm{f}]\!]\times[\![\mathrm{g}]\!]}"] & & & f_1 \times g_1
\end{tikzcd}
$$

This is represented in color syntax by substituting elements $\mathrm{f} : f_0, \mathrm{g} : g_0$ into the profunctors of the source span profunctor i_0.



So, the data of a span transformation is three-dimensional. Then it just has one property: the transformation is *natural* with respect to the actions of i_0 and i_1.


$\overline{q \cdot i}$
=
=
$\bar{q} \cdot \bar{i}$
$\overline{i \cdot r}$
=
=
$\bar{i} \cdot \bar{r}$

1.3 The double category of span categories

Span categories are the objects of a double category $\mathrm{Span}\mathbb{C}\mathrm{at}$; its relations are span profunctors, whose composition is spans of profunctor composition. Understanding span profunctors to contain inferences, this composition is *sequential* composition of inference.

Definition 9. Let $m(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \,|\, \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,l) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \,|\, \mathcal{T}(\mathbb{Z},\mathbb{C})$ be span profunctors. The **sequential composite** $(m \circ n)(f \circ g, k \circ \ell) : \mathcal{R}(\mathbb{X},\mathbb{A}) \,|\, \mathcal{T}(\mathbb{Z},\mathbb{C})$ is the span of profunctor composites.

$$\begin{tikzcd}
\mathbb{X} & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "m" description, "|" marking] & \mathbb{A} \\
\mathbb{Y} \arrow[from=u, "f"', "|" marking] \arrow[ur, phantom, "\Leftarrow"] & \mathcal{S} \arrow[l] \arrow[r] \arrow[d, "n" description, "|" marking] \arrow[ur, phantom, "\Rightarrow"] & \mathbb{B} \arrow[from=u, "k", "|" marking] \\
\mathbb{Z} \arrow[from=u, "g"', "|" marking] \arrow[ur, phantom, "\Leftarrow"] & \mathcal{T} \arrow[l] \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & \mathbb{C} \arrow[from=u, "l", "|" marking]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[d, "f \circ g"', "|" marking] \arrow[dr, phantom, "\Leftarrow"] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "m \circ n" description, "|" marking] \arrow[dr, phantom, "\Rightarrow"] & \mathbb{A} \arrow[d, "k \circ \ell", "|" marking] \\
\mathbb{Z} & \mathcal{T} \arrow[l] \arrow[r] & \mathbb{C}
\end{tikzcd}$$

An element of $f \circ g$ is an indexed pair $\mathrm{Y}.(\mathrm{f},\mathrm{g}) : f(\mathrm{X},\mathrm{Y}) \times g(\mathrm{Y},\mathrm{Z})$, and of $k \circ \ell$ is $\mathrm{B}.(\mathrm{k},\mathrm{l}) : k(\mathrm{A},\mathrm{B}) \times \ell(\mathrm{B},\mathrm{C})$. Then an element of $m \circ n$ over $((\mathrm{f},\mathrm{g}),(\mathrm{k},\mathrm{l}))$ is a pair $S.(m,n) : m^{\mathrm{f}}_{\mathrm{k}}(R,S) \times n^{\mathrm{g}}_{\mathrm{l}}(S,T)$, quotiented by the relation of associativity: for any $s : \mathcal{S}(S_0,S_1)$ we have $S_0.(m, s \cdot n) = S_1.(m \cdot s, n)$.

Composition of span profunctors is functorial, i.e. the composite of span transformations $\overline{m} : m_0 \Rightarrow m_1$ and $\bar{n} : n_0 \Rightarrow n_1$ maps $(m,n) : m_0(\mathrm{f},\mathrm{k})(R,S) \times n_0(\mathrm{g},\mathrm{l})(S,T)$ to $(\overline{m},\bar{n}) : m_1(\bar{\mathrm{f}},\bar{\mathrm{k}})(\overline{R},\overline{S}) \times n_1(\bar{\mathrm{g}},\bar{\mathrm{l}})(\overline{S},\overline{T})$. This defines horizontal composition of the double category of span categories.

Proposition 10. Span categories and span functors, span profunctors and span transformations form a double category $\mathrm{Span}\mathbb{C}\mathrm{at}$.

In the same way, displayed categories form a double category.

Proposition 11. Displayed categories and displayed functors, displayed profunctors and displayed transformations form a double category $\mathrm{Dis}\mathbb{C}\mathrm{at}$.

Proof. Sequential composition of displayed profunctors is defined: given $m : f \times k \to \mathrm{Prof}$ and $n : g \times l \to \mathrm{Prof}$, the composite $(m \circ n) : (f \circ g) \times (k \circ l) \to \mathrm{Prof}$ is $(m \circ n)((\mathrm{f},\mathrm{g}),(\mathrm{k},\mathrm{l})) = m(\mathrm{f},\mathrm{k}) \times n(\mathrm{g},\mathrm{l})$. This is functorial, defining parallel composition of the double category $\mathrm{Dis}\mathbb{C}\mathrm{at}$. □

Hence to summarize the exposition of the section, we have an equivalence of double categories.

Theorem 12. The double category of span categories is equivalent to that of displayed categories.

$$\mathrm{Span}\mathbb{C}\mathrm{at} \simeq \mathrm{Dis}\mathbb{C}\mathrm{at}$$

1.4 Parallel composition

Span categories have *sequential* composition. In the next section, we define "matrix category" as a span category with *parallel* composition actions; and similarly for "matrix profunctors". So, we first need to define parallel composition of span categories, and span profunctors.

Definition 13. Let $\mathcal{R} : \mathbb{A} \parallel \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \parallel \mathbb{C}$ be span categories. The **parallel composite** $\mathcal{R} * \mathcal{S} : \mathbb{A} \parallel \mathbb{C}$ is a span category defined by composition of spans in $\mathbb{C}\mathrm{at}$. This means that an object of $\mathcal{R} * \mathcal{S}$ over $\mathrm{A} : \mathbb{A}, \mathrm{C} : \mathbb{C}$ is a pair $R : \mathcal{R}(\mathrm{A}, \mathrm{B}), S : \mathcal{S}(\mathrm{B}, \mathrm{C})$ for some $\mathrm{B} : \mathbb{B}$. Hence the composite is equivalent to the matrix of categories

$$(\mathcal{R} * \mathcal{S})(\mathrm{A}, \mathrm{C}) = \Sigma \mathrm{B} : \mathbb{B}.\ \mathcal{R}(\mathrm{A}, \mathrm{B}) \times \mathcal{S}(\mathrm{B}, \mathrm{C})$$

and similarly for morphisms.

$$(\vec{\mathcal{R}} * \vec{\mathcal{S}})(\mathrm{a}, \mathrm{c}) = \Sigma \mathrm{b} : \mathbb{B}.\ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \times \vec{\mathcal{S}}(\mathrm{b}, \mathrm{c})$$

Composition and unit of $\mathcal{R} \circ \mathcal{S}$ are given by that of $\mathcal{R}$ and $\mathcal{S}$; this structure is associative and unital.

In the same way, we define parallel composition of span profunctors.

Definition 14. Let $i(d, f) : \mathcal{O}(\mathbb{U}, \mathbb{X}) \mid \mathcal{P}(\mathbb{V}, \mathbb{Y})$ and $m(f, k) : \mathcal{R}(\mathbb{X}, \mathbb{A}) \mid \mathcal{S}(\mathbb{Y}, \mathbb{B})$ be span profunctors. The **parallel composite** $(i * m)(d, k) : (\mathcal{Q} * \mathcal{S})(\mathbb{U}, \mathbb{A}) \mid (\mathcal{R} * \mathcal{T})(\mathbb{V}, \mathbb{B})$ is the span composite in Prof.

$$\begin{tikzcd}
\mathbb{U} & \mathcal{O} \arrow[l] \arrow[r] & \mathbb{X} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{A} \\
d \arrow[u, phantom] & i \arrow[l, Rightarrow] \arrow[r, Rightarrow] & f & m \arrow[l, Rightarrow] \arrow[r, Rightarrow] & k \\
\mathbb{V} & \mathcal{P} \arrow[l] \arrow[r] & \mathbb{Y} & \mathcal{S} \arrow[l] \arrow[r] & \mathbb{B}
\arrow[from=1-1, to=3-1, "|"] \arrow[from=1-2, to=3-2, "|"] \arrow[from=1-3, to=3-3, "|"] \arrow[from=1-4, to=3-4, "|"] \arrow[from=1-5, to=3-5, "|"]
\end{tikzcd}$$

This means that an element of $i * m$ over $\mathrm{d} : d(\mathrm{U}, \mathrm{V}), \mathrm{k} : k(\mathrm{A}, \mathrm{B})$ is a pair $i : i(\mathrm{d}, \mathrm{f})$ and $m : m(\mathrm{f}, \mathrm{k})$ for some $\mathrm{f} : f(\mathrm{X}, \mathrm{Y})$. This can be understood as pairs of horizontally composable squares.

$$
\begin{tikzcd}
\mathrm{U} \arrow[r, "O", "|" marking] \arrow[d, "\mathrm{d}"'] \arrow[dr, phantom, "\Downarrow i"] & \mathrm{X} \arrow[r, "R", "|" marking] \arrow[d, "\mathrm{f}"] \arrow[dr, phantom, "\Downarrow m"] & \mathrm{A} \arrow[d, "\mathrm{k}"] \\
\mathrm{V} \arrow[r, "P"', "|" marking] & \mathrm{Y} \arrow[r, "S"', "|" marking] & \mathrm{B}
\end{tikzcd}
$$

Hence the composite is equivalent to the following matrix of profunctors.

$$(i * m)(\mathrm{d}, \mathrm{k}) = \Sigma \mathrm{f} : f(\mathrm{X}, \mathrm{Y}).\ i(\mathrm{d}, \mathrm{f}) \times m(\mathrm{f}, \mathrm{k})$$

Parallel composition is functorial with respect to span functors and span transformations.

Because we only need to define composition in order to define matrix categories, we do not continue to define the whole three-dimensional structure of $\mathrm{Span}\mathbb{C}\mathrm{at}$. It is an *intercategory* [7], meaning that parallel composition is only *lax* functorial with respect to sequential composition, i.e. for the diagram

$$
\begin{tikzcd}
\mathbb{U} \arrow[d, "d", "|" marking] & \mathcal{O} \arrow[l] \arrow[r] \arrow[d, "i"] \arrow[l, phantom, shift right=4, "\Longleftarrow"] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \mathbb{X} \arrow[d, "f"] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "m"] \arrow[l, phantom, shift right=4, "\Longleftarrow"] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \mathbb{A} \arrow[d, "k"', "|" marking] \\
\mathbb{V} \arrow[d, "e", "|" marking] & \mathcal{P} \arrow[l] \arrow[r] \arrow[d, "j"] \arrow[l, phantom, shift right=4, "\Longleftarrow"] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \mathbb{Y} \arrow[d, "g"] & \mathcal{S} \arrow[l] \arrow[r] \arrow[d, "n"] \arrow[l, phantom, shift right=4, "\Longleftarrow"] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \mathbb{B} \arrow[d, "\ell"', "|" marking] \\
\mathbb{W} & \mathcal{Q} \arrow[l] \arrow[r] & \mathbb{Z} & \mathcal{T} \arrow[l] \arrow[r] & \mathbb{C}
\end{tikzcd}
$$

there is a noninvertible transformation $(i * m) \circ (j * n) \Rightarrow (i \circ j) * (m \circ n)$.

Chapter 2

Matrix categories

A logic is a category of types and processes, indexing a *matrix category* of relations and inferences.

Section 2.1. To define matrix category, we first determine how a category forms a logic. Existing literature has defined two-sided fibrations as bimodules of arrow double categories [20]; yet these are not logics, because they lack conjoints. So in the first section, we define the logic of the *weave double category* to be the coproduct of the arrow double category with its opposite.

Section 2.2 Then, we define matrix categories to be bimodules of weave double categories. The "weave construction" extends to profunctors, giving the notion of *matrix profunctor* [2.3]. These form a double category $\mathrm{Mat}\mathbb{C}\mathrm{at}$, which is fibered over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ [2.4].

Section 2.5. Last, we define *parallel composition* of matrix categories, making $\mathrm{Mat}\mathbb{C}\mathrm{at}$ a kind of three-dimensional category which we call a "metalogic".

2.1 Fibrations and bifibrations

A category is seen as a 1-dimensional structure of objects and morphisms; yet *reasoning* in a category consists of 2-dimensional equalities between composites of morphisms. Every category forms a double category, in fact three double categories, whose squares are commutative squares.

Two are known: the *arrow double category* $\overrightarrow{\mathbb{A}}$ and its opposite $\overleftarrow{\mathbb{A}}$; modules are *fibrations* and *opfibrations*. Yet $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$ are not logics; so we define the *weave double category* $\langle\mathbb{A}\rangle$ to be the union $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$. It is a logic, and its modules are *bifibrations*.

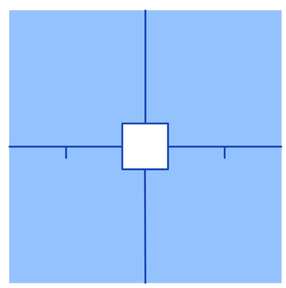
2.1.1 Arrow double category

Definition 15. Let $\mathbb{A}$ be a category. The **arrow double category** $\overrightarrow{\mathbb{A}}$ is as follows: the base category is $\mathbb{A}$; a loose morphism is a morphism of $\mathbb{A}$, and a square is a commutative square. Composition is vertical composition of squares, and for each morphism there is an identity square.

We denote (vertical) processes by a, and (horizontal) relations by $\hat{\mathrm{a}}$.

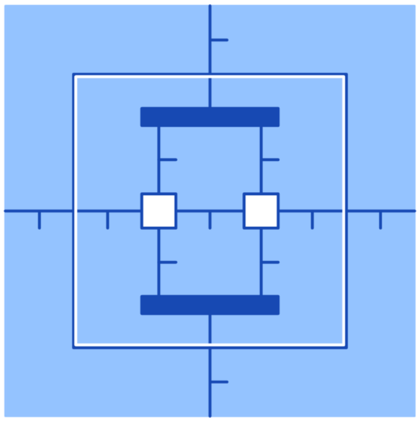
$$\begin{tikzcd}
A^0_0 \arrow[r, "\hat{a}^1_0"] \arrow[d, "a_0"'] & A^1_0 \arrow[d, "a_1"] \\
A^0_1 \arrow[r, "\hat{a}^1_1"] & A^1_1
\end{tikzcd}$$

$$(a_0, a_1) : \overrightarrow{\mathbb{A}}(\hat{a}^1_0 \to \hat{a}^1_1)$$

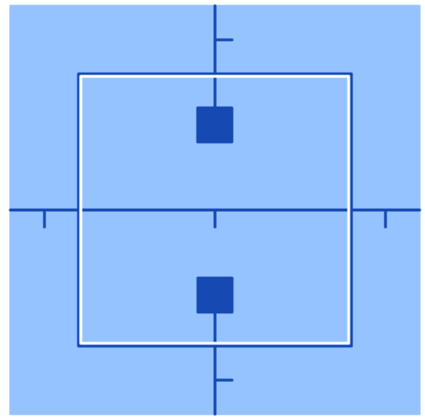


Horizontal composition is that of morphisms and squares, and horizontal units are identities.

$$\begin{tikzcd}
A^0_0 \arrow[r, "\hat{a}^1_0"] \arrow[d, "a_0"'] & A^1_0 \arrow[r, "\hat{a}^2_0"] \arrow[d, "a_1"] & A^2_0 \arrow[d, "a_2"] \\
A^0_1 \arrow[r, "\hat{a}^1_1"] & A^1_1 \arrow[r, "\hat{a}^2_1"] & A^2_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[d, "a"'] & A_0 \arrow[d, "a"] \\
A_1 \arrow[r, equal] & A_1
\end{tikzcd}$$



arrow category composition



arrow category unit

By inducing an arrow double category, a category can *act* on span categories. If an object of $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is to be a relation from an $\mathbb{A}$-type to a $\mathbb{B}$-type, then such relations should *vary* over processes of $\mathbb{A}$ and $\mathbb{B}$ — this is a *module* of arrow double categories.

Definition 16. Let $\mathbb{A}$ and $\mathbb{B}$ be categories.

A **fibered category** over $\mathbb{A}$ is a left module of the arrow double category $\overrightarrow{\mathbb{A}}$. This is a span category $\mathcal{R} : \mathbb{A} \,\|\, 1$, with a span functor $\odot : \overrightarrow{\mathbb{A}} * \mathcal{R} \to \mathcal{R}$, and coherent isomorphisms for associativity and unitality. The action, called **substitution**, is a matrix of functors

$$\hat{\mathrm{a}} \odot R : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_1) \to \mathcal{R}(\mathrm{A}_0)$$

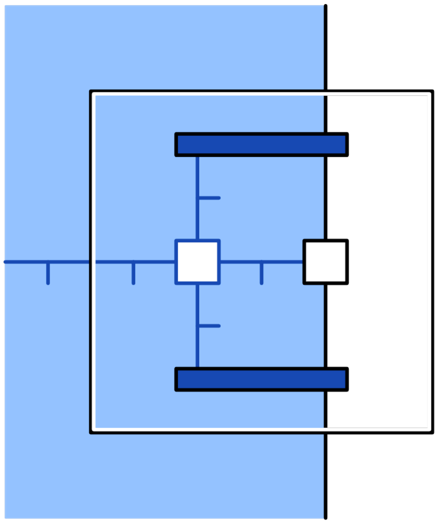
which is contravariant in A. It is also known as "pullback", and often denoted by $\mathrm{a}^*(R_1)$.

An **opfibered category** over $\mathbb{B}$ is a right module of the arrow double category $\overrightarrow{\mathbb{B}}$. This is a span category $\mathcal{R} : 1 \,\|\, \mathbb{B}$, with a span functor $\odot : \mathcal{R} * \overrightarrow{\mathbb{B}} \to \mathcal{R}$, and coherent isomorphisms for associativity and unitality. The action, called **image**, is a matrix of functors

$$R \odot \hat{\mathrm{b}} : \mathcal{R}(\mathrm{B}_0) \times \overrightarrow{\mathbb{B}}(\mathrm{B}_0, \mathrm{B}_1) \to \mathcal{R}(\mathrm{B}_1)$$

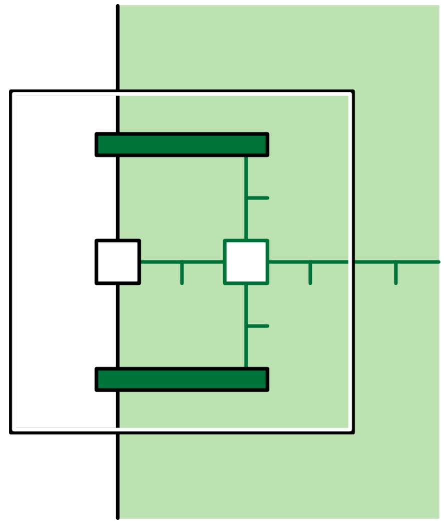
which is covariant in B. It is also known as "pushforward", and often denoted by $\mathrm{b}_!(R_0)$.

In string diagrams, with terminal category 1 as white space, the actions are drawn as follows.



substitution

$\overrightarrow{\mathbb{A}}(A_0, A_1) \times \mathcal{R}(A_1) \to \mathcal{R}(A_0)$



image

$\mathcal{R}(B_0) \times \overrightarrow{\mathbb{B}}(B_0, B_1) \to \mathcal{R}(B_1)$

Arrow double categories are special, because every process has a *companion*: there are two squares which "bend" the process up or down into a relation.

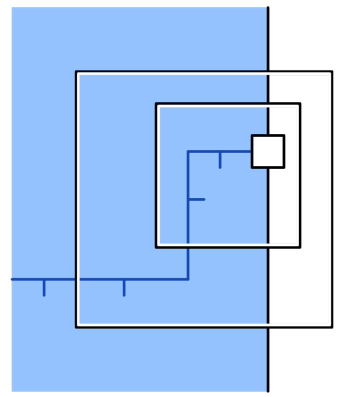
Definition 17. Let $\mathbb{A}$ be a category, with $\overrightarrow{\mathbb{A}}$ the arrow double category. Each morphism $a : \mathbb{A}(A_0, A_1)$ induces two squares: the **cartesian** square $\varepsilon.a$ and the **opcartesian** square $\eta.a$, drawn below.

$$\begin{tikzcd} A_0 \arrow[r, "a"] \arrow[d, "a"'] \arrow[dr, phantom, "\varepsilon.a"] & A_1 \arrow[d, equal] \\ A_1 \arrow[r, equal] & A_1 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta.a"] & A_0 \arrow[d, "a"] \\ A_0 \arrow[r, "a"'] & A_1 \end{tikzcd}$$

Fibered and opfibered categories are usually defined in terms of *cartesian* and *opcartesian* morphisms [9, Ch. 1,9]. These morphisms are given by the actions of squares in the arrow double category, as follows.

Proposition 18. In a fibered category $\mathcal{R}$ over $\mathbb{A}$, a morphism $r : R_0 \to R_1$ over $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1)$ is equivalent to $\eta.\mathrm{a} \circ r : R_0 \to \mathrm{a} \odot R_1$ over $\mathrm{id}.\mathrm{A}_0$, by factoring through the **cartesian** morphism $\varepsilon.\mathrm{a} \circ \mathrm{id}.R_1 : \mathrm{a} \odot R_1 \to R_1$.

$$
\begin{tikzcd}
\mathrm{A}_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta.\mathrm{a}"] & \mathrm{A}_0 \arrow[r, "R_0"] \arrow[d, "\mathrm{a}"] \arrow[dr, phantom, "\Downarrow r"] & 1 \arrow[d, equal] \\
\mathrm{A}_0 \arrow[r, "\mathrm{a}"] \arrow[d, "\mathrm{a}"] \arrow[dr, phantom, "\varepsilon.\mathrm{a}"] & \mathrm{A}_1 \arrow[r, "R_1"'] \arrow[d, equal] & 1 \arrow[d, equal] \\
\mathrm{A}_1 \arrow[r, equal] & \mathrm{A}_1 \arrow[r, "R_1"'] & 1
\end{tikzcd}
$$

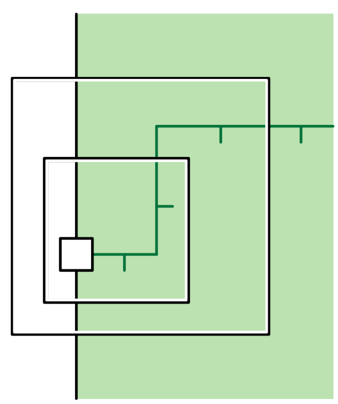


This gives a contravariant representation of morphisms over a.

$$\vec{\mathcal{R}}(\mathrm{a})(R_0, R_1) \cong \mathcal{R}(R_0, \mathrm{a} \odot R_1)$$

In an opfibered category $\mathcal{R}$ over $\mathbb{B}$, a morphism $r : R_0 \to R_1$ over $\mathrm{b} : \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$ is equivalent to a morphism $r \circ \varepsilon.\mathrm{b} : R_0 \odot \mathrm{b} \to R_1$ over $\mathrm{id}.\mathrm{B}_1$, by factoring through the **opcartesian** morphism $\mathrm{id}.R_0 \circ \eta.\mathrm{b} : R_0 \to R_0 \odot \mathrm{b}$.

$$
\begin{tikzcd}
1 \arrow[r, "R_0"] \arrow[d, equal] & \mathrm{B}_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta.\mathrm{b}"] & \mathrm{B}_0 \arrow[d, "\mathrm{b}"] \\
1 \arrow[r, "R_0"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow r"] & \mathrm{B}_0 \arrow[r, "\mathrm{b}"] \arrow[d, "\mathrm{b}"] \arrow[dr, phantom, "\varepsilon.\mathrm{b}"] & \mathrm{B}_1 \arrow[d, equal] \\
1 \arrow[r, "R_1"'] & \mathrm{B}_1 \arrow[r, equal] & \mathrm{B}_1
\end{tikzcd}
$$



This gives a covariant representation of morphisms over b.

$$\vec{\mathcal{R}}(\mathrm{b})(R_0, R_1) \cong \mathcal{R}(R_0 \odot \mathrm{b}, \mathcal{R}_1)$$

However, there is a limitation to the arrow double category: it is not a *logic*, because there are no backwards-pointing arrows to be conjoints. This may at first seem like a technicality — surely all equational reasoning of $\mathbb{A}$ can be expressed in $\overrightarrow{\mathbb{A}}$, right? Actually, no.

By introducing a second dimension, we distinguish between morphisms as *processes* and as *relations*. Based on how processes act on relations, there are four basic kinds of equations.

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{a}^0"] \arrow[d, "\mathrm{a}_0"'] & \bullet \arrow[d, "\mathrm{a}_1"] \\ \bullet \arrow[r, "\mathrm{a}^1"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "\mathrm{a}^0"] \arrow[d, "\mathrm{a}_0"'] & \bullet \arrow[d, "\mathrm{a}_1"] \\ \bullet & \bullet \arrow[l, "\mathrm{a}^1"] \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[d, "\mathrm{a}_0"'] & \bullet \arrow[l, "\mathrm{a}^0"'] \arrow[d, "\mathrm{a}_1"] \\ \bullet \arrow[r, "\mathrm{a}^1"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[d, "\mathrm{a}_0"'] & \bullet \arrow[l, "\mathrm{a}^0"'] \arrow[d, "\mathrm{a}_1"] \\ \bullet & \bullet \arrow[l, "\mathrm{a}^1"] \end{tikzcd}$$

natural	**factorization**	**composition**	**conatural**
$\mathrm{a}^0 \cdot \mathrm{a}_1 = \mathrm{a}_0 \cdot \mathrm{a}^1$	$\mathrm{a}_0 = \mathrm{a}^0 \cdot \mathrm{a}_1 \cdot \mathrm{a}^1$	$\mathrm{a}^0 \cdot \mathrm{a}_0 \cdot \mathrm{a}^1 = \mathrm{a}_1$	$\mathrm{a}^0 \cdot \mathrm{a}_0 = \mathrm{a}_1 \cdot \mathrm{a}^1$

Of course, each of the above equations can be expressed as a "natural" commutative square in an arrow double category. However, there is an obstruction to reasoning about *sequential composition*.

Associativity has two forms, "forward" and "backward": suppose that two pairs $(\mathrm{a}_1^0, \mathrm{a}_2^0)$ and $(\mathrm{a}_1^1, \mathrm{a}_2^1)$ are equal in the composite profunctor $\mathbb{A} \circ \mathbb{A}$; then there is a "zig-zag" connecting the pair: a sequence of morphisms $\hat{\mathrm{a}} : \mathbb{A}(\mathrm{A}_i, \mathrm{A}_{i+1})$ *or* $\check{\mathrm{a}} : \mathbb{A}(\mathrm{A}_{i+1}, \mathrm{A}_i)$, so that the squares commute. The two unary cases are below.

$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "\mathrm{a}_1^0"'] & \bullet \arrow[d, "\mathrm{a}_1^1"] \\ \mathrm{A}_0 \arrow[r, "\hat{\mathrm{a}}"] \arrow[d, "\mathrm{a}_2^0"'] & \mathrm{A}_1 \arrow[d, "\mathrm{a}_2^1"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "\mathrm{a}_1^0"'] & \bullet \arrow[d, "\mathrm{a}_1^1"] \\ \mathrm{A}_0 \arrow[d, "\mathrm{a}_2^0"'] & \mathrm{A}_1 \arrow[l, "\check{\mathrm{a}}"'] \arrow[d, "\mathrm{a}_2^1"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

forward associativity	**backward associativity**
$\mathrm{a}_1^0 \cdot \hat{\mathrm{a}} = \mathrm{a}_1^1$	$\mathrm{a}_1^0 = \mathrm{a}_1^1 \cdot \check{\mathrm{a}}$
$\mathrm{a}_2^0 = \hat{\mathrm{a}} \cdot \mathrm{a}_2^1$	$\check{\mathrm{a}} \cdot \mathrm{a}_2^0 = \mathrm{a}_2^1$

Forward associativity, on the left, is the composite of two "natural" squares, which can be expressed in the arrow double category. Backwards associativity, on the right, is the composite of a "factorization" and a "composition" — this *cannot be expressed* in the arrow double category.

Hence we identify the following limitation.

Proposition 19. Let $\mathbb{A}$ be a category. In the arrow double category $\overrightarrow{\mathbb{A}}$, factorization and composition squares do not compose in sequence; so backward associativity cannot be expressed.

This leads to an obstruction, when defining sequential composition of profunctors between "two-sided fibrations", i.e. bimodules of arrow double categories — the author learned this the hard way.

We could accept this limitation and still use these concepts to construct logics, but it would be more complex than necessary. Rather, we understand the problem to be that *arrow double categories are not logics*, and we instead determine the logic which a category *does* form.

2.1.2 Weave double category

Every category $\mathbb{A}$ defines a logic, called the *weave double category* $\langle\mathbb{A}\rangle$. It is the union of $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$, the arrow double category and its opposite.

In the logic of $\langle\mathbb{A}\rangle$, a relation is a *zig-zag* in $\mathbb{A}$: an alternating sequence of arrows in $\overrightarrow{\mathbb{A}}$ and oparrows in $\overleftarrow{\mathbb{A}}$; and an inference is a *weave*: a composite of squares in $\overrightarrow{\mathbb{A}}$, opsquares in $\overleftarrow{\mathbb{A}}$, and *unit isomorphisms* — the units of $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$ are "united" by adjoining isomorphisms between each identity arrow and oparrow.

Definition 20. Let $\mathbb{A}$ be a category, with arrow double category $\overrightarrow{\mathbb{A}}$.

The **op-arrow double category** $\overleftarrow{\mathbb{A}}$ is the horizontal opposite: $\overleftarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \equiv \overrightarrow{\mathbb{A}}(\mathrm{A}_1, \mathrm{A}_0)$.

We denote an **arrow** by $\hat{\mathrm{a}} : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1)$, and an **op-arrow** by $\check{\mathrm{a}} : \overleftarrow{\mathbb{A}}(\mathrm{A}_1, \mathrm{A}_0)$. We use $\bar{\mathrm{a}}$ for objects of $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$. A square of $\overrightarrow{\mathbb{A}}$ is a **square**, and a square of $\overleftarrow{\mathbb{A}}$ is an **opsquare**.

Definition 21. Define $\mathrm{Dbl}_{\mathbb{A}}$ to be the 2-category of double categories on $\mathbb{A}$, double functors over $\mathrm{id}.\mathbb{A}$, and identity-component transformations, a.k.a. icons [11].

Given double categories $\mathcal{A}_0$ and $\mathcal{A}_1$ on $\mathbb{A}$, and double functors $f, g : \mathcal{A}_0 \to \mathcal{A}_1$ over $\mathrm{id}.\mathbb{A}$, an icon $\gamma : f \Rightarrow g$ gives for each $a_0 : \mathcal{A}_0$ a 2-morphism $\gamma(a_0) : f(a_0) \Rightarrow g(a_0)$, subject to naturality.

$$
\begin{tikzcd}
\mathbb{A} \arrow[d, equal] & \mathcal{A}_0 \arrow[l] \arrow[r] \arrow[d, shift right=4, "f"'] \arrow[d, shift left=4, "g"] \arrow[d, phantom, "=\gamma\Rrightarrow"] & \mathbb{A} \arrow[d, equal] \\
\mathbb{A} & \mathcal{A}_1 \arrow[l] \arrow[r] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}_0 \arrow[r, "f(a_0)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\overset{\|}{\gamma(a_0)}\Downarrow"] & \mathrm{A}_1 \arrow[d, equal] \\
\mathrm{A}_0 \arrow[r, "g(a_0)"', "\shortmid"{marking}] & \mathrm{A}_1
\end{tikzcd}
$$

Definition 22. Let $\mathbb{A}$ be a category. Define the **weave double category** $\langle\mathbb{A}\rangle$ to be the 2-coproduct of the arrow and oparrow double categories in $\mathrm{Dbl}_{\mathbb{A}}$.

$$\langle\mathbb{A}\rangle \equiv \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$$

So for every double category $\mathbb{A} \leftarrow \mathcal{A} \to \mathbb{A}$ there is the following natural equivalence.

$$\mathrm{Dbl}_{\mathbb{A}}(\langle\mathbb{A}\rangle, \mathcal{A}) \simeq \mathrm{Dbl}_{\mathbb{A}}(\overrightarrow{\mathbb{A}}, \mathcal{A}) \times \mathrm{Dbl}_{\mathbb{A}}(\overleftarrow{\mathbb{A}}, \mathcal{A})$$

We show the weave double category consists of the following loose morphisms and squares.

A **zig-zag** of $\mathbb{A}$-morphisms is a nonempty sequence of morphisms $(\mathrm{A}_0, \bar{\mathrm{a}}_1, \ldots, \bar{\mathrm{a}}_k, \mathrm{A}_k)$ alternating with each $\bar{\mathrm{a}}_i$ either an arrow $\hat{\mathrm{a}}_i : \overrightarrow{\mathbb{A}}(\mathrm{A}_{i-1}, \mathrm{A}_i)$ or an op-arrow $\check{\mathrm{a}}_i : \overleftarrow{\mathbb{A}}(\mathrm{A}_{i-1}, \mathrm{A}_i)$.

$$\mathrm{A}_0 \xrightarrow{\hat{\mathrm{a}}_1} \mathrm{A}_1 \xleftarrow{\check{\mathrm{a}}_2} \mathrm{A}_2 \longrightarrow \cdots \longleftarrow \mathrm{A}_{k-2} \xrightarrow{\hat{\mathrm{a}}_{k-1}} \mathrm{A}_{k-1} \xleftarrow{\check{\mathrm{a}}_k} \mathrm{A}_k$$

We may abbreviate a zig-zag by $\langle\bar{\mathrm{a}}_1, \ldots, \bar{\mathrm{a}}_k\rangle$ or simply by $\langle\bar{\mathrm{a}}_k\rangle$.

A **weave** of zig-zags $w : \langle\bar{\mathrm{a}}_k\rangle \to \langle\bar{\mathrm{a}}_\ell\rangle$ is a composite of squares, opsquares, and *unit isomorphisms*.

Proposition 23. The weave double category $\langle\mathbb{A}\rangle$ is equivalent to the free strict semi-double category, i.e. associative double category without introducing a unit, on the following presentation.

Generators. Squares of $\overrightarrow{\mathbb{A}}$, opsquares of $\overleftarrow{\mathbb{A}}$, and for each object $\mathrm{A} : \mathbb{A}$ a **unit isomorphism**

$$\hat{\mathrm{id}}.\mathrm{A} \cong \check{\mathrm{id}}.\mathrm{A}\,.$$

Equations. Interchange, square and opsquare composition, vertical and horizontal associativity, and mixed-unitor naturality: the details are given in the proof.

Proof. Let $w(\mathbb{A})$ be given by the following presentation.

Generators. Squares of $\overrightarrow{\mathbb{A}}$ and opsquares of $\overleftarrow{\mathbb{A}}$

$$
\begin{tikzcd}
A_0^0 \arrow[r, "\hat{a}^0"] \arrow[d, "a_0"'] & A_1^0 \arrow[d, "a_1"] \\
A_0^1 \arrow[r, "\hat{a}^1"'] & A_1^1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0^0 & A_1^0 \arrow[l, "\check{a}^0"'] \arrow[d, "a_1"] \\
A_0^1 \arrow[from=u, "a_0"'] & A_1^1 \arrow[l, "\check{a}^1"]
\end{tikzcd}
$$

and for each object $A : \mathbb{A}$ an isomorphism of the identity arrow and the identity op-arrow.

$$
\begin{tikzcd}
A \arrow[r, dotted, "\hat{\mathrm{id}}.A"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & A \arrow[d, equal] \\
A & A \arrow[l, dotted, "\check{\mathrm{id}}.A"]
\end{tikzcd}
$$

Generated from these squares, $w(\mathbb{A})$ consists of all vertical and horizontal composites thereof, subject to interchange and the following equations.

Equations.

- For each vertical-composable pair of $\overrightarrow{\mathbb{A}}$, the vertical composite in $w(\mathbb{A})$ equals that of $\overrightarrow{\mathbb{A}}$.
- For each vertical-composable pair of $\overleftarrow{\mathbb{A}}$, the vertical composite in $w(\mathbb{A})$ equals that of $\overleftarrow{\mathbb{A}}$.

$$
\begin{tikzcd}
A_0^0 \arrow[r, "\hat{a}_0"] \arrow[d, "a_0^1"] & A_1^0 \arrow[d, "a_1^1"] \\
A_0^1 \arrow[r, "\hat{a}_1"] \arrow[d, "a_0^2"] & A_1^1 \arrow[d, "a_1^2"] \\
A_0^2 \arrow[r, "\hat{a}_2"] & A_1^2
\end{tikzcd}
\equiv
\begin{tikzcd}
A_0^0 \arrow[r, "\hat{a}_0"] \arrow[dd, "a_0^1 a_0^2"] & A_1^0 \arrow[dd, "a_1^1 a_1^2"] \\
 & \\
A_0^2 \arrow[r, "\hat{a}_2"] & A_1^2
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0^0 \arrow[d, "a_0^1"] & A_1^0 \arrow[l, "\check{a}_0"'] \arrow[d, "a_1^1"] \\
A_0^1 \arrow[d, "a_0^2"] & A_1^1 \arrow[l, "\check{a}_1"'] \arrow[d, "a_1^2"] \\
A_0^2 & A_1^2 \arrow[l, "\check{a}_2"']
\end{tikzcd}
\equiv
\begin{tikzcd}
A_0^0 \arrow[dd, "a_0^1 a_0^2"] & A_1^0 \arrow[l, "\check{a}_0"'] \arrow[dd, "a_1^1 a_1^2"] \\
 & \\
A_0^2 & A_1^2 \arrow[l, "\check{a}_2"']
\end{tikzcd}
$$

So vertical composition of $w(\mathbb{A})$ is unital, by inheriting the vertical units of $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$.

- For each vertical-composable triple of $w(\mathbb{A})$, vertical composition is associative.
- For each horizontal-composable pair of $\overrightarrow{\mathbb{A}}$, the horizontal composite in $w(\mathbb{A})$ equals that of $\overrightarrow{\mathbb{A}}$.
- For each horizontal-composable pair of $\overleftarrow{\mathbb{A}}$, the horizontal composite in $w(\mathbb{A})$ equals that of $\overleftarrow{\mathbb{A}}$.

$$
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"] \arrow[d, "\mathrm{a}_0"] & \mathrm{A}_1^0 \arrow[r, "\hat{\mathrm{a}}_2^0"] \arrow[d, "\mathrm{a}_1"] & \mathrm{A}_2^0 \arrow[d, "\mathrm{a}_2"] \\
\mathrm{A}_0^1 \arrow[r, "\hat{\mathrm{a}}_1^1"'] & \mathrm{A}_1^1 \arrow[r, "\hat{\mathrm{a}}_2^1"'] & \mathrm{A}_2^1
\end{tikzcd}
\equiv
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[rr, "\hat{\mathrm{a}}_1^0\hat{\mathrm{a}}_2^0"] \arrow[d, "\mathrm{a}_0"] & & \mathrm{A}_2^0 \arrow[d, "\mathrm{a}_2"] \\
\mathrm{A}_0^1 \arrow[rr, "\hat{\mathrm{a}}_1^1\hat{\mathrm{a}}_2^1"'] & & \mathrm{A}_2^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[d, "\mathrm{a}_0"] & \mathrm{A}_1^0 \arrow[l, "\check{\mathrm{a}}_1^0"'] \arrow[d, "\mathrm{a}_1"] & \mathrm{A}_2^0 \arrow[l, "\check{\mathrm{a}}_2^0"'] \arrow[d, "\mathrm{a}_2"] \\
\mathrm{A}_0^1 & \mathrm{A}_1^1 \arrow[l, "\check{\mathrm{a}}_1^1"] & \mathrm{A}_2^1 \arrow[l, "\check{\mathrm{a}}_2^1"]
\end{tikzcd}
\equiv
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[d, "\mathrm{a}_0"] & & \mathrm{A}_2^0 \arrow[ll, "\check{\mathrm{a}}_1^0\check{\mathrm{a}}_2^0"'] \arrow[d, "\mathrm{a}_2"] \\
\mathrm{A}_0^1 & & \mathrm{A}_2^1 \arrow[ll, "\check{\mathrm{a}}_1^1\check{\mathrm{a}}_2^1"]
\end{tikzcd}
$$

- For each horizontal-composable triple in $w(\mathbb{A})$, horizontal composition is strictly associative.

- For each arrow and each op-arrow, naturality equations for the "mixed unitor" isomorphisms $\check{\eta}(\hat{\mathrm{a}}) : \hat{\mathrm{a}} = \hat{\mathrm{a}} \circ \mathrm{i\hat{d}}.\mathrm{A}_1 \cong \hat{\mathrm{a}} \circ \mathrm{i\check{d}}.\mathrm{A}_1$ and $\hat{\eta}(\check{\mathrm{a}}) : \check{\mathrm{a}} = \check{\mathrm{a}} \circ \mathrm{i\check{d}}.\mathrm{A}_1 \cong \check{\mathrm{a}} \circ \mathrm{i\hat{d}}.\mathrm{A}_1$: the following equations hold for right unitor naturality; and similarly for left unitor naturality.

$$
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[rr, "\hat{\mathrm{a}}_1^0"] \arrow[d, equal] & & \mathrm{A}_1^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"] \arrow[d, equal] & \mathrm{A}_1^0 \arrow[r, dotted, "\mathrm{i\hat{d}}.\mathrm{A}_1^0"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"'] \arrow[d, "\mathrm{a}_0^1"] & \mathrm{A}_1^0 \arrow[d, "\mathrm{a}_1^1"] & \mathrm{A}_1^0 \arrow[l, dotted, "\mathrm{i\check{d}}.\mathrm{A}_1^0"] \arrow[d, "\mathrm{a}_1^1"] \\
\mathrm{A}_0^1 \arrow[r, "\hat{\mathrm{a}}_1^1"] \arrow[d, equal] & \mathrm{A}_1^1 \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^1 \arrow[l, dotted, "\mathrm{i\check{d}}.\mathrm{A}_1^1"'] \arrow[d, equal] \\
\mathrm{A}_0^1 \arrow[r, "\hat{\mathrm{a}}_1^1"'] \arrow[d, equal] & \mathrm{A}_1^1 \arrow[r, dotted, "\mathrm{i\hat{d}}.\mathrm{A}_1^0"'] & \mathrm{A}_1^1 \arrow[d, equal] \\
\mathrm{A}_0^1 \arrow[rr, "\hat{\mathrm{a}}_1^1"'] & & \mathrm{A}_1^1
\end{tikzcd}
\equiv
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"] \arrow[d, "\mathrm{a}_0^1"] & \mathrm{A}_1^0 \arrow[d, "\mathrm{a}_1^1"] \\
\mathrm{A}_0^1 \arrow[r, "\hat{\mathrm{a}}_1^1"'] & \mathrm{A}_1^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[d, equal] & & \mathrm{A}_1^0 \arrow[ll, "\check{\mathrm{a}}_1^0"'] \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[d, equal] & \mathrm{A}_1^0 \arrow[l, "\check{\mathrm{a}}_1^0"'] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^0 \arrow[l, dotted, "\mathrm{i\check{d}}.\mathrm{A}_1^0"'] \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[d, "\mathrm{a}_0^1"] & \mathrm{A}_1^0 \arrow[l, "\check{\mathrm{a}}_1^0"] \arrow[r, dotted, "\mathrm{i\hat{d}}.\mathrm{A}_1^0"'] \arrow[d, "\mathrm{a}_1^1"] & \mathrm{A}_1^0 \arrow[d, "\mathrm{a}_1^1"] \\
\mathrm{A}_0^1 \arrow[d, equal] & \mathrm{A}_1^1 \arrow[l, "\check{\mathrm{a}}_1^1"'] \arrow[r, dotted, "\mathrm{i\hat{d}}.\mathrm{A}_1^1"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^1 \arrow[d, equal] \\
\mathrm{A}_0^1 \arrow[d, equal] & \mathrm{A}_1^1 \arrow[l, "\check{\mathrm{a}}_1^1"] & \mathrm{A}_1^1 \arrow[l, dotted, "\mathrm{i\check{d}}.\mathrm{A}_1^0"] \arrow[d, equal] \\
\mathrm{A}_0^1 & & \mathrm{A}_1^1 \arrow[ll, "\check{\mathrm{a}}_1^1"]
\end{tikzcd}
\equiv
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[d, "\mathrm{a}_0^1"] & \mathrm{A}_1^0 \arrow[l, "\check{\mathrm{a}}_1^0"'] \arrow[d, "\mathrm{a}_1^1"] \\
\mathrm{A}_0^1 & \mathrm{A}_1^1 \arrow[l, "\check{\mathrm{a}}_1^1"]
\end{tikzcd}
$$

We show that $w(\mathbb{A})$ is a coproduct of $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$ in $\mathrm{Dbl}_{\mathbb{A}}$.

First, $w(\mathbb{A})$ is a double category: the horizontal unit of A can be chosen to be either $\mathrm{i\hat{d}}.\mathrm{A}$ or $\mathrm{i\check{d}}.\mathrm{A}$. Either choice gives a unitor isomorphism for "mixed" composition of an arrow and an op-arrow, and an equality for same-type composition. Below are the choices of right unitor; the left is analogous.

$$
\begin{array}{ccc}
\check{\eta}(\hat{\mathrm{a}}) :\ \hat{\mathrm{a}} = \hat{\mathrm{a}} \circ \mathrm{i\hat{d}}.\mathrm{A}_1 \cong \hat{\mathrm{a}} \circ \mathrm{i\check{d}}.\mathrm{A}_1 & & \hat{\eta}(\check{\mathrm{a}}) :\ \check{\mathrm{a}} = \check{\mathrm{a}} \circ \mathrm{i\check{d}}.\mathrm{A}_1 \cong \check{\mathrm{a}} \circ \mathrm{i\hat{d}}.\mathrm{A}_1 \\
\text{and} & \text{or} & \text{and} \\
\check{\eta}(\check{\mathrm{a}}) : \check{\mathrm{a}} = \check{\mathrm{a}} \circ \mathrm{i\check{d}}.\mathrm{A}_1 & & \hat{\eta}(\hat{\mathrm{a}}) : \hat{\mathrm{a}} = \hat{\mathrm{a}} \circ \mathrm{i\hat{d}}.\mathrm{A}_1
\end{array}
$$

The naturality of each unitor has been imposed by fiat. The coherence with composition, i.e. triangle identity, follows from associativity of horizontal composition in $w(\mathbb{A})$.

$$
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"] \arrow[d, equal] & \mathrm{A}_1^0 \arrow[rr, "\hat{\mathrm{a}}_2^0"] \arrow[d, equal] & & \mathrm{A}_2^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"] \arrow[d, equal] & \mathrm{A}_1^0 \arrow[r, dotted, "\hat{\mathrm{id}}.\mathrm{A}_1^0"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^0 \arrow[r, "\hat{\mathrm{a}}_2^0"] \arrow[d, equal] & \mathrm{A}_2^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"'] & \mathrm{A}_1^0 & \mathrm{A}_1^0 \arrow[l, dotted, "\check{\mathrm{id}}.\mathrm{A}_1^0"] \arrow[r, "\hat{\mathrm{a}}_2^0"'] & \mathrm{A}_2^0
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[rr, "\hat{\mathrm{a}}_1^0"] \arrow[d, equal] & & \mathrm{A}_1^0 \arrow[r, "\hat{\mathrm{a}}_2^0"] \arrow[d, equal] & \mathrm{A}_2^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"] \arrow[d, equal] & \mathrm{A}_1^0 \arrow[r, dotted, "\hat{\mathrm{id}}.\mathrm{A}_1^0"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^0 \arrow[r, "\hat{\mathrm{a}}_2^0"] \arrow[d, equal] & \mathrm{A}_2^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "\hat{\mathrm{a}}_1^0"'] & \mathrm{A}_1^0 & \mathrm{A}_1^0 \arrow[l, dotted, "\check{\mathrm{id}}.\mathrm{A}_1^0"] \arrow[r, "\hat{\mathrm{a}}_2^0"'] & \mathrm{A}_2^0
\end{tikzcd}
$$

Hence $w(\mathbb{A})$ is a double category.

The inclusions $i_0 : \overrightarrow{\mathbb{A}} \to w(\mathbb{A})$ and $i_1 : \overleftarrow{\mathbb{A}} \to w(\mathbb{A})$ preserve horizontal composition strictly, and one preserves the horizontal unit strictly while the other does up to the isomorphism $\hat{\mathrm{id}}.\mathrm{A} \cong \check{\mathrm{id}}.\mathrm{A}$; the coherence of the latter is automatic by strict composition preservation of i_0 and i_1 and strict unitality of $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$. Hence i_0 and i_1 are (pseudo) double functors.

We demonstrate the universal property of these inclusions.

Let $\mathcal{A}$ be a double category on $\mathbb{A}$, and $f : \overrightarrow{\mathbb{A}} \to \mathcal{A}$, $g : \overleftarrow{\mathbb{A}} \to \mathcal{A}$ be (pseudo) double functors from the arrow and oparrow double categories. We define the copairing $\langle f, g \rangle : w(\mathbb{A}) \to \mathcal{A}$.

$$
\begin{tikzcd}
\overrightarrow{\mathbb{A}} \arrow[r, "i_0"] \arrow[d, equal] & w(\mathbb{A}) \arrow[d, dashed, "\langle f,g \rangle"] & \overleftarrow{\mathbb{A}} \arrow[l, "i_1"'] \arrow[d, equal] \\
\overrightarrow{\mathbb{A}} \arrow[r, "f"'] & \mathcal{A} & \overleftarrow{\mathbb{A}} \arrow[l, "g"]
\end{tikzcd}
$$

Because $w(\mathbb{A})$ is freely generated on a presentation, it suffices to define $\langle f, g \rangle$ on generators, verify it is well-defined with respect to the equations, and then give coherent isomorphisms for composition and unit preservation.

For each square in $\overrightarrow{\mathbb{A}}$ and each opsquare in $\overleftarrow{\mathbb{A}}$, the copairing maps by f and g, respectively.

$$
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[r, "f(\hat{\mathrm{a}}^0)", "|"] \arrow[d, "\mathrm{a}_0"'] \arrow[dr, phantom, "\Downarrow f(\mathrm{a}_0{,}\mathrm{a}_1)"] & \mathrm{A}_1^0 \arrow[d, "\mathrm{a}_1"] \\
\mathrm{A}_0^1 \arrow[r, "f(\hat{\mathrm{a}}^1)"', "|"] & \mathrm{A}_1^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[d, "\mathrm{a}_0"'] \arrow[dr, phantom, "\Downarrow g(\mathrm{a}_0{,}\mathrm{a}_1)"] & \mathrm{A}_1^0 \arrow[l, "g(\check{\mathrm{a}}^0)"', "|"] \arrow[d, "\mathrm{a}_1"] \\
\mathrm{A}_0^1 & \mathrm{A}_1^1 \arrow[l, "g(\check{\mathrm{a}}^1)", "|"]
\end{tikzcd}
$$

Because f and g are double functors, there are isomorphisms $\eta_f : U_{\mathrm{A}} \cong f(\hat{\mathrm{id}}.\mathrm{A})$ and $\eta_g : U_{\mathrm{A}} \cong g(\check{\mathrm{id}}.\mathrm{A})$; hence the copairing maps the unit isomorphism $\hat{\mathrm{id}}.\mathrm{A} \cong \check{\mathrm{id}}.\mathrm{A}$ to $\eta_f^{-1} \cdot \eta_g$.

$$\begin{tikzcd}
\mathrm{A} \arrow[r, dotted, "\hat{\mathrm{id}}.\mathrm{A}"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A} \arrow[d, equal] \\
\mathrm{A} & \mathrm{A} \arrow[l, dotted, "\check{\mathrm{id}}.\mathrm{A}"]
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\mathrm{A} \arrow[r, "f(\hat{\mathrm{id}}.\mathrm{A})", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A} \arrow[d, equal] \\
\mathrm{A} \arrow[r, "U_{\mathrm{A}}"{description}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A} \arrow[d, equal] \\
\mathrm{A} \arrow[r, "g(\check{\mathrm{id}}.\mathrm{A})"', "\shortmid"{marking}] & \mathrm{A}
\end{tikzcd}$$

So in general, $\langle f, g\rangle : w(\mathbb{A}) \to \mathcal{A}$ maps composites of these generators to composites of their images. Because $w(\mathbb{A})$ is strictly associative and $\mathcal{A}$ may not be, the copairing involves a choice of composition order, i.e. either left- or right-association.

$$\langle f, g\rangle(\langle \bar{\mathrm{a}}_k\rangle) \quad = \quad \langle f, g\rangle(\bar{\mathrm{a}}_1) \circ (\langle f, g\rangle(\bar{\mathrm{a}}_2) \circ (\langle f, g\rangle(\bar{\mathrm{a}}_3) \circ \cdots))$$

or

$$\langle f, g\rangle(\langle \bar{\mathrm{a}}_k\rangle) \quad = \quad ((\cdots \langle f, g\rangle(\bar{\mathrm{a}}_{k-2})) \circ \langle f, g\rangle(\bar{\mathrm{a}}_{k-1})) \circ \langle f, g\rangle(\bar{\mathrm{a}}_k)$$

We verify the mapping is well-defined, and is a double functor which gives a factorization in $\mathrm{Dbl}_{\mathbb{A}}$.

Because f and g preserve vertical composition in $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$, the copairing $\langle f, g\rangle$ respects those equations. Horizontal composition f and g preserve up to isomorphism, whose naturality ensures that $\langle f, g\rangle$ is well-defined on horizontal composites.

$$\begin{tikzcd}
\mathrm{A}_0^0 \arrow[rr, "f(\hat{\mathrm{a}}_1^0\hat{\mathrm{a}}_2^0)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & \mathrm{A}_2^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "f(\hat{\mathrm{a}}_1^0)", "\shortmid"{marking}] \arrow[d, "\mathrm{a}_0"] & \mathrm{A}_1^0 \arrow[r, "f(\hat{\mathrm{a}}_2^0)", "\shortmid"{marking}] \arrow[d, "\mathrm{a}_1"] & \mathrm{A}_2^0 \arrow[d, "\mathrm{a}_2"] \\
\mathrm{A}_0^1 \arrow[r, "f(\hat{\mathrm{a}}_1^1)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & \mathrm{A}_1^1 \arrow[r, "f(\hat{\mathrm{a}}_2^1)"', "\shortmid"{marking}] & \mathrm{A}_2^1 \arrow[d, equal] \\
\mathrm{A}_0^1 \arrow[rr, "f(\hat{\mathrm{a}}_1^1\hat{\mathrm{a}}_2^1)"', "\shortmid"{marking}] & & \mathrm{A}_2^1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[r, "f(\hat{\mathrm{a}}_1^0\hat{\mathrm{a}}_2^0)", "\shortmid"{marking}] \arrow[d, "\mathrm{a}_0"] & \mathrm{A}_2^0 \arrow[d, "\mathrm{a}_2"] \\
\mathrm{A}_0^1 \arrow[r, "f(\hat{\mathrm{a}}_1^1\hat{\mathrm{a}}_2^1)"'] & \mathrm{A}_2^1
\end{tikzcd}$$

Next, $\langle f, g\rangle$ is well-defined for vertical associativity, because it strictly preserves vertical composition; and for horizontal associativity, because it maps any horizontal composite to either the left-associated or right-associated composite of the images. Lastly, $\langle f, g\rangle$ is well-defined for the equations of unitor naturality, by the naturality of the unit preservation of f and g and of the unitor of $\mathcal{A}$.

$$
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[rr, "f(\hat{\mathrm{a}}_1^0)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & \mathrm{A}_1^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "f(\hat{\mathrm{a}}_1^0)", "\shortmid"{marking}] \arrow[d, equal] & \mathrm{A}_1^0 \arrow[r, "f(\hat{\mathrm{id}}.\mathrm{A}_1^0)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "f(\hat{\mathrm{a}}_1^0)", "\shortmid"{marking}] \arrow[d, equal] & \mathrm{A}_1^0 \arrow[r, "U_{\mathrm{A}_1^0}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^0 \arrow[d, equal] \\
\mathrm{A}_0^0 \arrow[r, "f(\hat{\mathrm{a}}_1^0)"', "\shortmid"{marking}] \arrow[d, dashed, "\mathrm{a}_0^1"] & \mathrm{A}_1^0 \arrow[r, "g(\check{\mathrm{id}}.\mathrm{A}_1^0)"', "\shortmid"{marking}] \arrow[d, dashed, "\mathrm{a}_1^1"] & \mathrm{A}_1^0 \arrow[d, dashed, "\mathrm{a}_1^1"] \\
\mathrm{A}_0^1 \arrow[r, "f(\hat{\mathrm{a}}_1^1)", "\shortmid"{marking}] \arrow[d, equal] & \mathrm{A}_1^1 \arrow[r, "g(\check{\mathrm{id}}.\mathrm{A}_1^1)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^1 \arrow[d, equal] \\
\mathrm{A}_0^1 \arrow[r, "f(\hat{\mathrm{a}}_1^1)", "\shortmid"{marking}] \arrow[d, equal] & \mathrm{A}_1^1 \arrow[r, "U_{\mathrm{A}_1^1}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \mathrm{A}_1^1 \arrow[d, equal] \\
\mathrm{A}_0^1 \arrow[r, "f(\hat{\mathrm{a}}_1^1)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & \mathrm{A}_1^1 \arrow[r, "f(\hat{\mathrm{id}}.\mathrm{A}_1^0)"', "\shortmid"{marking}] & \mathrm{A}_1^1 \arrow[d, equal] \\
\mathrm{A}_0^1 \arrow[rr, "f(\hat{\mathrm{a}}_1^1)"', "\shortmid"{marking}] & & \mathrm{A}_1^1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{A}_0^0 \arrow[r, "f(\hat{\mathrm{a}}_1^0)", "\shortmid"{marking}] \arrow[d, "\mathrm{a}_0^1"] & \mathrm{A}_1^0 \arrow[d, "\mathrm{a}_1^1"] \\
\mathrm{A}_0^1 \arrow[r, "f(\hat{\mathrm{a}}_1^1)"', "\shortmid"{marking}] & \mathrm{A}_1^1
\end{tikzcd}
$$

Hence $\langle f, g\rangle$ is a well-defined mapping of squares; to prove it is a double functor, it remains to give the coherent isomorphisms for composition and unit preservation.

Because $\langle f, g\rangle$ maps horizontal composites either left-associated or right-associated, it preserves composition up to the associator α of $\mathcal{A}$, and the composition isomorphisms μ_f and μ_g: the isomorphism $\mu_{\langle f,g\rangle}$ is defined casewise as a horizontal composite of these, based on the types of the middle pair of components in each composable pair of zig-zags.

$$
\begin{tikzcd}
\mathrm{A}_0 \arrow[r, "f(\hat{\mathrm{a}}_1)", "\shortmid"{marking}] \arrow[d, equal] & \mathrm{A}_1 \arrow[r, "f(\hat{\mathrm{a}}_2)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mu_f"] & \mathrm{A}_2 \arrow[d, equal] \\
\mathrm{A}_0 \arrow[rr, "f(\hat{\mathrm{a}}_1\hat{\mathrm{a}}_2)"', "\shortmid"{marking}] & & \mathrm{A}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}_0 \arrow[r, "g(\check{\mathrm{a}}_1)", "\shortmid"{marking}] \arrow[d, equal] & \mathrm{A}_1 \arrow[r, "g(\check{\mathrm{a}}_2)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\mu_g"] & \mathrm{A}_2 \arrow[d, equal] \\
\mathrm{A}_0 \arrow[rr, "g(\check{\mathrm{a}}_1\check{\mathrm{a}}_2)"', "\shortmid"{marking}] & & \mathrm{A}_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{A}_0 \arrow[r, "f(\hat{\mathrm{a}}_1)", "\shortmid"{marking}] \arrow[d, equal] & \mathrm{A}_1 \arrow[r, "g(\check{\mathrm{a}}_2)\circ f(\hat{\mathrm{a}}_3)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\alpha"] & \mathrm{A}_3 \arrow[d, equal] \\
\mathrm{A}_0 \arrow[r, "f(\hat{\mathrm{a}}_1)\circ g(\check{\mathrm{a}}_2)"', "\shortmid"{marking}] & \mathrm{A}_2 \arrow[r, "f(\hat{\mathrm{a}}_3)"', "\shortmid"{marking}] & \mathrm{A}_3
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}_0 \arrow[r, "g(\check{\mathrm{a}}_1)", "\shortmid"{marking}] \arrow[d, equal] & \mathrm{A}_1 \arrow[r, "f(\hat{\mathrm{a}}_2)\circ g(\check{\mathrm{a}}_3)", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\alpha"] & \mathrm{A}_3 \arrow[d, equal] \\
\mathrm{A}_0 \arrow[r, "g(\check{\mathrm{a}}_1)\circ f(\hat{\mathrm{a}}_2)"', "\shortmid"{marking}] & \mathrm{A}_2 \arrow[r, "g(\check{\mathrm{a}}_3)"'] & \mathrm{A}_3
\end{tikzcd}
$$

The associativity coherence of μ_f and μ_g, together with the pentagon identity of α, provide the associativity coherence of $\mu_{\langle f,g\rangle}$.

For unit preservation, f and g provide the isomorphisms

$$\eta_f : U_{\mathrm{A}} \cong f(\hat{\mathrm{id}}.\mathrm{A}) = \langle f,g\rangle(\hat{\mathrm{id}}.\mathrm{A}) \qquad \text{and} \qquad \eta_g : U_{\mathrm{A}} \cong g(\check{\mathrm{id}}.\mathrm{A}) = \langle f,g\rangle(\check{\mathrm{id}}.\mathrm{A}),$$

so for either choice of unit, the coherence of η_f or η_g entails the coherence for $\eta_{\langle f,g\rangle}$.

So the copairing is a double functor $\langle f,g\rangle : w(\mathbb{A}) \to \mathcal{A}$, which by construction gives a strict factorization of f and g through the inclusions of $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$.

Now, it remains to verify the two-dimensional universal property. Let $h,k : w(\mathbb{A}) \to \mathcal{A}$ be a pair of double functors, and let $\gamma_0 : h(i_0) \Rightarrow k(i_0)$ and $\gamma_1 : h(i_1) \Rightarrow k(i_1)$ be icons, as given below.

$$\begin{tikzcd}
\overrightarrow{\mathbb{A}} \arrow[r, "i_0"] \arrow[d, equal] & w(\mathbb{A}) \arrow[d, bend right=50, "h"'] \arrow[d, bend left=50, "k"] \arrow[d, phantom, "\overset{\langle\gamma_0,\gamma_1\rangle}{\Longrightarrow}"] & \overleftarrow{\mathbb{A}} \arrow[l, "i_1"'] \arrow[d, equal] \\
\overrightarrow{\mathbb{A}} \arrow[r, bend left, "h(i_0)"] \arrow[r, bend right, "k(i_0)"'] \arrow[r, phantom, "\gamma_0\Downarrow"] & \mathcal{A} & \overleftarrow{\mathbb{A}} \arrow[l, bend right, "h(i_1)"'] \arrow[l, bend left, "k(i_1)"] \arrow[l, phantom, "\Downarrow\gamma_1"]
\end{tikzcd}$$

Each icon is a natural family of 2-cells $\gamma_0(\hat{\mathrm{a}}) : h(\hat{\mathrm{a}}) \Rightarrow k(\hat{\mathrm{a}})$ and $\gamma_1 : h(\check{\mathrm{a}}) \Rightarrow k(\check{\mathrm{a}})$. Just as for double functors, we define the copairing for each arrow and op-arrow:

$$\langle\gamma_0,\gamma_1\rangle(\hat{\mathrm{a}}) = \gamma_0(\hat{\mathrm{a}}) \qquad \text{and} \qquad \langle\gamma_0,\gamma_1\rangle(\check{\mathrm{a}}) = \gamma_1(\check{\mathrm{a}}).$$

In general, $\langle\gamma_0,\gamma_1\rangle(\langle\bar{\mathrm{a}}_n\rangle)$ is the horizontal composite of the unary images, conjugated by μ_h and μ_k.

$$\begin{tikzcd}
\mathrm{A}_0 \arrow[rrrr, "h(\langle\bar{\mathrm{a}}_n\rangle)"] \arrow[d, equal] & & \arrow[d, phantom, "\Downarrow\mu_h^{-1}"] & & \mathrm{A}_n \arrow[d, equal] \\
\mathrm{A}_0 \arrow[r, "h(\bar{\mathrm{a}}_1)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\langle\gamma_0,\gamma_1\rangle(\bar{\mathrm{a}}_1)"] & \mathrm{A}_1 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \mathrm{A}_{n-1} \arrow[r, "h(\bar{\mathrm{a}}_n)"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\langle\gamma_0,\gamma_1\rangle(\bar{\mathrm{a}}_n)"] & \mathrm{A}_n \arrow[d, equal] \\
\mathrm{A}_0 \arrow[r, "k(\bar{\mathrm{a}}_1)"'] \arrow[d, equal] & \mathrm{A}_1 \arrow[r] & \cdots \arrow[r] \arrow[d, phantom, "\Downarrow\mu_k"] & \mathrm{A}_{n-1} \arrow[r, "k(\bar{\mathrm{a}}_n)"'] & \mathrm{A}_n \arrow[d, equal] \\
\mathrm{A}_0 \arrow[rrrr, "k(\langle\bar{\mathrm{a}}_n\rangle)"'] & & & & \mathrm{A}_n
\end{tikzcd}$$

The naturality of γ_0 and γ_1, and that of μ_h and μ_k, provide the naturality of the copairing $\langle\gamma_0,\gamma_1\rangle$.

This defines a factorization $\langle\gamma_0,\gamma_1\rangle(i_0) = \gamma_0 : h(i_0) \Rightarrow k(i_0)$ and $\langle\gamma_0,\gamma_1\rangle(i_1) = \gamma_1 : h(i_1) \Rightarrow k(i_1)$.

Last, we verify that this factorization is unique. Let $\delta : h \Rightarrow k$ be a transformation such that $\delta(i_0) = \gamma_0$ and $\delta(i_1) = \gamma_1$. Then $\delta(\hat{\mathrm{a}}) = \gamma_0(\hat{\mathrm{a}})$ and $\delta(\check{\mathrm{a}}) = \gamma_1(\check{\mathrm{a}})$. Yet because $w(\mathbb{A})$ is generated by arrows and op-arrows, this characterizes the transformation; hence we have $\delta = \langle \gamma_0, \gamma_1 \rangle$.

Thus $w(\mathbb{A})$ is a coproduct, i.e. $w(\mathbb{A}) \simeq \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}} \equiv \langle \mathbb{A} \rangle$; and so the weave double category can be constructed from a presentation of squares, opsquares, and unit isomorphisms. □

The weave double category contains *all equational reasoning* of $\mathbb{A}$, in that it contains the four kinds of squares and their composites: the sequential composite of a factorization and a composition square is below.

$$\begin{tikzcd}
\mathrm{A}^0_0 \arrow[r, "\hat{\mathrm{a}}_0"] \arrow[d, "\mathrm{a}^0_1"'] & \mathrm{A}^0_1 \arrow[d, "\mathrm{a}^1_1"] \\
\mathrm{A}^1_0 \arrow[d, "\mathrm{a}^0_2"'] & \mathrm{A}^1_1 \arrow[l, "\check{\mathrm{a}}_1"'] \arrow[d, "\mathrm{a}^1_2"] \\
\mathrm{A}^2_0 \arrow[r, "\hat{\mathrm{a}}_2"'] & \mathrm{A}^2_1
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{A}^0_0 \arrow[r, "\hat{\mathrm{a}}_0"] \arrow[dd, "\mathrm{a}^0_1"'] & \mathrm{A}^0_1 \arrow[d, "\mathrm{a}^1_1"'] & \mathrm{A}^0_1 \arrow[l, dotted] \arrow[r, dotted] \arrow[d, "\mathrm{a}^1_1"] & \mathrm{A}^0_1 \arrow[dd, "\mathrm{a}^1_1"] \\
& \mathrm{A}^1_1 \arrow[d, "\mathrm{a}_1"'] & \mathrm{A}^1_1 \arrow[l, dotted] \arrow[d, equal] & \\
\mathrm{A}^1_0 \arrow[r, dotted] \arrow[dd, "\mathrm{a}^0_2"'] & \mathrm{A}^1_0 \arrow[d, equal] & \mathrm{A}^1_1 \arrow[l, "\check{\mathrm{a}}_1"'] \arrow[r, dotted] \arrow[d, "\mathrm{a}_1"] & \mathrm{A}^1_1 \arrow[dd, "\mathrm{a}^1_2"] \\
& \mathrm{A}^1_0 \arrow[d, "\mathrm{a}^0_2"'] & \mathrm{A}^1_0 \arrow[l, dotted] \arrow[d, "\mathrm{a}^0_2"] & \\
\mathrm{A}^2_0 \arrow[r, dotted] & \mathrm{A}^2_0 & \mathrm{A}^2_0 \arrow[l, dotted] \arrow[r, "\mathrm{a}_2"'] & \mathrm{A}^2_1
\end{tikzcd}$$

The arrows and oparrows of $\langle \mathbb{A} \rangle$ are companions and conjoints.

Proposition 24. $\langle \mathbb{A} \rangle$ is a bifibrant double category, i.e. a logic.

By coproduct, actions by the weave double category $\langle \mathbb{A} \rangle$ are equivalent to pairs of actions by the arrow and oparrow double categories $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$: so left modules are right modules are *bifibrations*.

To show this by universal property, we have to determine how $\mathcal{R}$ forms a double category over $\mathbb{A}$ that represents actions on $\mathcal{R}$. The key is to see that an action $\odot : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_0) \to \mathcal{R}(\mathrm{A}_1)$ is equivalent to a displayed transformation of the following form.

So, we define the double category as the universal comma square.

$$\begin{tikzcd}
& \overrightarrow{\mathbb{A}} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
\mathbb{A} \arrow[rr, phantom, "\Longrightarrow\odot\Longrightarrow"] \arrow[dr, "\mathcal{R}"'] & & \mathbb{A} \arrow[dl, "\mathcal{R}"] \\
& \mathbb{C}\mathrm{at} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overrightarrow{\mathbb{A}} \arrow[ddl, "\mathrm{dom}"'] \arrow[ddr, "\mathrm{cod}"] \arrow[d, dashed] & \\
& \mathcal{R}.\mathcal{R} \arrow[dl] \arrow[dr] & \\
\mathbb{A} \arrow[rr, phantom, "\Longrightarrow"] \arrow[dr, "\mathcal{R}"'] & & \mathbb{A} \arrow[dl, "\mathcal{R}"] \\
& \mathbb{C}\mathrm{at} &
\end{tikzcd}$$

$$\mathcal{R}(\mathrm{A}_0) \xrightarrow{-\odot\hat{\mathrm{a}}} \mathcal{R}(\mathrm{A}_1)$$

To define sequential composition of $\mathcal{R}.\mathcal{R}$, the displayed category $\mathcal{R} : \mathbb{A} \to \mathbb{C}\mathrm{at}$ must be a *pseudofunctor*, i.e. the composition transformation $\mathcal{R}(\mathrm{a}_1) \circ \mathcal{R}(\mathrm{a}_2) \Rightarrow \mathcal{R}(\mathrm{a}_1\mathrm{a}_2)$ must be invertible. This is known as an **exponentiable category** [18], a generalization of fibered and opfibered category.

Definition 25. Let $\mathcal{R}$ be an exponentiable category over $\mathbb{A}$. The **fiber-hom double category** $\mathbb{A} \leftarrow \mathcal{R}.\mathcal{R} \to \mathbb{A}$ is the collage of the comma object of the displayed category $\mathcal{R} : \mathbb{A} \to \mathbb{C}\mathrm{at}$ along itself.

$$\begin{tikzcd}
\mathrm{A}_0^0 \arrow[rrr, "f_0", "\shortmid"{marking}] \arrow[ddd, "\mathrm{a}_0"'] & & & \mathrm{A}_1^0 \arrow[ddd, "\mathrm{a}_1"] \\
& \mathcal{R}(\mathrm{A}_0^0) \arrow[r, "f_0"] \arrow[d, "\mathcal{R}(\mathrm{a}_0)"', "\shortmid"{marking}] \arrow[dr, phantom, "\Longrightarrow\varphi\Longrightarrow"] & \mathcal{R}(\mathrm{A}_1^0) \arrow[d, "\mathcal{R}(\mathrm{a}_1)", "\shortmid"{marking}] & \\
& \mathcal{R}(\mathrm{A}_0^1) \arrow[r, "f_1"'] & \mathcal{R}(\mathrm{A}_1^1) & \\
\mathrm{A}_0^1 \arrow[rrr, "f_1"', "\shortmid"{marking}] & & & \mathrm{A}_1^1
\end{tikzcd}$$

The base category is $\mathbb{A}$; a loose morphism over $(\mathrm{A}_0, \mathrm{A}_1)$ is a functor $f : \mathcal{R}(\mathrm{A}_0) \to \mathcal{R}(\mathrm{A}_1)$, and a square over $(\mathrm{a}_0, \mathrm{a}_1)$ is a transformation $\varphi(f_0, f_1) : \mathcal{R}(\mathrm{a}_0) \Rightarrow \mathcal{R}(\mathrm{a}_1)$. Parallel composition is sequential composition in $\mathbb{C}\mathrm{at}$.

Sequential composition is parallel composition of $\mathbb{C}\mathrm{at}$, conjugated by composition isomorphisms of $\mathcal{R}$. Composing in sequence and parallel, the middle isomorphisms cancel, giving interchange.

$$\begin{tikzcd}
\mathcal{R}(\mathrm{A}_0^0) \arrow[r, equal] \arrow[dd, "\mathcal{R}(\mathrm{a}_0^1\mathrm{a}_0^2)"'] & \mathcal{R}(\mathrm{A}_0^0) \arrow[r, "f_0"] \arrow[d, "\mathcal{R}(\mathrm{a}_0^1)"'] \arrow[dr, phantom, "\overset{\varphi_1}{\Longrightarrow}"] & \mathcal{R}(\mathrm{A}_1^0) \arrow[r, equal] \arrow[d, "\mathcal{R}(\mathrm{a}_1^1)"] & \mathcal{R}(\mathrm{A}_1^0) \arrow[dd, "\mathcal{R}(\mathrm{a}_1^1\mathrm{a}_1^2)"] \\
{} \arrow[r, phantom, "\overset{\cong}{\Longrightarrow}"] & \mathcal{R}(\mathrm{A}_0^1) \arrow[r, "f_1"] \arrow[d, "\mathcal{R}(\mathrm{a}_0^2)"'] \arrow[dr, phantom, "\overset{\varphi_2}{\Longrightarrow}"] & \mathcal{R}(\mathrm{A}_1^1) \arrow[d, "\mathcal{R}(\mathrm{a}_1^2)"] \arrow[r, phantom, "\overset{\cong}{\Longrightarrow}"] & {} \\
\mathcal{R}(\mathrm{A}_0^2) \arrow[r, equal] & \mathcal{R}(\mathrm{A}_0^2) \arrow[r, "f_2"'] & \mathcal{R}(\mathrm{A}_1^2) \arrow[r, equal] & \mathcal{R}(\mathrm{A}_1^2)
\end{tikzcd}$$

Proposition 26. Let $\mathcal{R} \to \mathbb{A}$ be an exponentiable category over $\mathbb{A}$. A right action on $\mathcal{R}$ by a double category $\mathcal{A}$ over $\mathbb{A}$ is equivalent to a double functor $\mathcal{A} \to \mathcal{R}.\mathcal{R}$.

A left action $\mathcal{A} * \mathcal{R} \to \mathcal{R}$ is equivalent to a double functor $\mathcal{A}^{\mathsf{op}} \to \mathcal{R}.\mathcal{R}$.

Proof. Let $\mathcal{A} : \mathrm{Dbl}_{\mathbb{A}}$, and $\odot : \mathcal{R} * \mathcal{A} \to \mathcal{R}$ be a module action. Then mapping

$$\begin{array}{lll} A : \mathcal{A} & \text{to} & - \odot A : \mathcal{R}(\mathrm{A}_0) \to \mathcal{R}(\mathrm{A}_1) \quad \text{and} \\ \alpha : \mathcal{A}(A, A') & \text{to} & - \odot \alpha : \mathcal{R}(\mathrm{a}_0) \Rightarrow \mathcal{R}(\mathrm{a}_1) \end{array}$$

defines a double functor $\mathcal{A} \to \mathcal{R}.\mathcal{R}$: the associator $(R \odot A_1) \odot A_2 \cong R \odot (A_1 \circ A_2)$ defines the composition isomorphism, and the unitor $R \cong R \odot U_{\mathrm{A}}$ defines the unit isomorphism; the coherence equations correspond.

□

Theorem 27. $\langle \mathbb{A} \rangle$-modules are equivalent to bifibrations.

Proof. By coproduct, we have the following equivalence.

$$\mathrm{Dbl}_{\mathbb{A}}(\overleftarrow{\mathbb{A}} + \overrightarrow{\mathbb{A}}, \mathcal{R}.\mathcal{R}) \simeq \mathrm{Dbl}_{\mathbb{A}}(\overrightarrow{\mathbb{A}}, \mathcal{R}.\mathcal{R}) \times \mathrm{Dbl}_{\mathbb{A}}(\overleftarrow{\mathbb{A}}, \mathcal{R}.\mathcal{R})$$

This means that a right action by $\langle \mathbb{A} \rangle$ is equivalent to a pair of right actions by $\overleftarrow{\mathbb{A}}$ and $\overrightarrow{\mathbb{A}}$; these give $\mathcal{R}$ the structures of a fibration and opfibration. □

In the next section, we define *matrix categories* as bimodules of weave double categories. These form a double category over that of categories; so we have to determine how the "weave construction" applies to categories and functors, profunctors and transformations.

First, how does the notion of "arrow category" generalize to profunctors?

Definition 28. Let $f : \mathbb{X} \mid \mathbb{A}$ be a profunctor. The **arrow profunctor** of f is the profunctor of arrow categories $\overrightarrow{f} : \overrightarrow{\mathbb{X}} \mid \overrightarrow{\mathbb{A}}$ consisting of commutative squares; it forms a span profunctor $f \leftarrow \overrightarrow{f} \rightarrow f$.

$$\overrightarrow{f}(\hat{x}, \hat{a}) = \{(f_0 : f(X_0, A_0), f_1 : f(X_1, A_1)) \mid a \cdot f_0 = f_1 \cdot x\}$$

Dually, the **oparrow profunctor** of f is the profunctor of oparrow categories $\overleftarrow{f} : \overleftarrow{\mathbb{X}} \mid \overleftarrow{\mathbb{A}}$.

$$\overleftarrow{f}(\check{x}, \check{a}) = \{f_0 : f(X_0, A_0), f_1 : f(X_1, A_1) \mid x \cdot f_0 = f_1 \cdot a\}$$

$$\begin{tikzcd}
X_0 \arrow[r, "\hat{x}"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\overrightarrow{f}"] & X_1 \arrow[d, "f_1"] \\
A_0 \arrow[r, "\hat{a}"'] & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[d, "f_0"'] \arrow[dr, phantom, "\overleftarrow{f}"] & X_1 \arrow[l, "\check{x}"'] \arrow[d, "f_1"] \\
A_0 & A_1 \arrow[l, "\check{a}"]
\end{tikzcd}$$

Note the only difference between the arrow and oparrow profunctors is which morphism acts on which element of f, i.e. "natural" squares versus "conatural" opsquares.

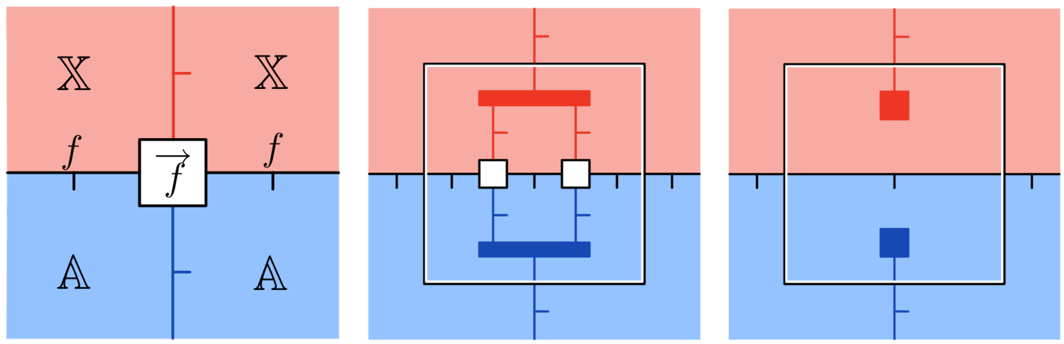
Just as commutative squares of a category compose in parallel, commutative squares of a profunctor compose in parallel. This defines a "vertical profunctor" from one arrow double category to another; see Chapter 3.

Proposition 29. Let $f : \mathbb{X} \mid \mathbb{A}$ be a profunctor. The arrow profunctor $f \leftarrow \overrightarrow{f} \rightarrow f$ is a monad in $\mathrm{Span}(\mathrm{Prof})$. Composition $\overrightarrow{f} * \overrightarrow{f} \Rightarrow \overrightarrow{f}$ is that of commutative squares, and the unit is given by that of $\mathbb{X}$ and $\mathbb{A}$.

$$\begin{tikzcd}
X_0 \arrow[r, "\hat{x}_1"] \arrow[d, "f_0"] \arrow[dr, phantom, "f_1"] & X_1 \arrow[r, "\hat{x}_2"] \arrow[d, "f_1"] \arrow[dr, phantom, "f_2"] & X_2 \arrow[d, "f_2"] \\
A_0 \arrow[r, "\hat{a}_1"] & A_1 \arrow[r, "\hat{a}_2"] & A_2
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\widehat{id}.X"] \arrow[d, "f"] \arrow[dr, phantom, "id.f"] & X \arrow[d, "f"] \\
A \arrow[r, "\widehat{id}.A"] & A
\end{tikzcd}$$

Dually, the oparrow profunctor is a monad in $\mathrm{Span}(\mathrm{Prof})$.

In string diagrams, the arrow profunctor is drawn as follows, and the oparrow profunctor is dual.



arrow profunctor **composition** **unit**

Now in the same way, a profunctor of categories forms a "weave profunctor" of double categories.

Definition 30. Let $f : \mathbb{X} \mid \mathbb{A}$ be a profunctor. Define the **weave vertical profunctor** between weave double categories $\langle f \rangle : \langle \mathbb{X} \rangle \mid \langle \mathbb{A} \rangle$ to be the coproduct of $\overrightarrow{f}$ and $\overleftarrow{f}$ in the 2-category of monads on f.

Hence $\langle f \rangle$ is constructed from a presentation, in the same way as the weave double category $\langle \mathbb{A} \rangle$.

Generators. Squares of $\overrightarrow{f}$ and opsquares of $\overleftarrow{f}$; and for each with identity domain or codomain, a vertical composite with the unit isomorphism.

$$\begin{tikzcd} X \arrow[d, equal] \arrow[r, phantom, ""] & X \arrow[l, dotted, "\check{\mathrm{id}}.X"'] \arrow[d, equal] \\ X \arrow[r, dotted, "\hat{\mathrm{id}}.X"] \arrow[d, "f_0"'] \arrow[ur, phantom, "\cong"] & X \arrow[d, "f_1"] \\ A_0 \arrow[r, "\hat{a}"'] & A_1 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dotted, "\hat{\mathrm{id}}.X"] \arrow[d, equal] & X \arrow[d, equal] \\ X \arrow[d, "f_0"'] \arrow[ur, phantom, "\cong"] & X \arrow[l, dotted, "\check{\mathrm{id}}.X"'] \arrow[d, "f_1"] \\ A_0 & A_1 \arrow[l, "\check{a}"] \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[r, "\hat{x}"] \arrow[d, "f_0"'] & X_1 \arrow[d, "f_1"] \\ A \arrow[r, dotted, "\hat{\mathrm{id}}.A"'] \arrow[d, equal] & A \arrow[d, equal] \\ A \arrow[ur, phantom, "\cong"] & A \arrow[l, dotted, "\check{\mathrm{id}}.A"] \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "\check{x}"'] \arrow[d, "f_1"] \\ A \arrow[d, equal] & A \arrow[l, dotted, "\check{\mathrm{id}}.A"] \arrow[d, equal] \\ A \arrow[r, dotted, "\hat{\mathrm{id}}.A"'] \arrow[ur, phantom, "\cong"] & A \end{tikzcd}$$

- For each vertical-composable pair of an element of $\langle f \rangle$ and a square in $\langle \mathbb{X} \rangle$, a vertical composite.
- For each vertical-composable pair of an element of $\langle f \rangle$ and a square in $\langle \mathbb{A} \rangle$, a vertical composite.
- For each horizontal-composable pair of elements of $\langle f \rangle$, a horizontal composite.

Equations.

- For each vertical-composable pair of $\overrightarrow{\mathbb{X}}$ and $\overrightarrow{f}$, the vertical composite in $\langle f \rangle$ equals that of $\overrightarrow{f}$.
- For each vertical-composable pair of $\overleftarrow{\mathbb{X}}$ and $\overleftarrow{f}$, the vertical composite in $\langle f \rangle$ equals that of $\overleftarrow{f}$.
- For each vertical-composable pair of $\overrightarrow{f}$ and $\overrightarrow{\mathbb{A}}$, the vertical composite in $\langle f \rangle$ equals that of $\overrightarrow{f}$.

- For each vertical-composable pair of $\overleftarrow{f}$ and $\overleftarrow{\mathbb{A}}$, the vertical composite in $\langle f \rangle$ equals that of $\overleftarrow{f}$.

$$
\begin{tikzcd}
X_0^0 \arrow[r, "\hat{x}_1^0"] \arrow[d, "x_0"'] & X_1^0 \arrow[d, "x_1"] \\
X_0^1 \arrow[r, dashed, "\hat{x}_1^1"] \arrow[d, "f_0"'] & X_1^1 \arrow[d, "f_1"] \\
A_0 \arrow[r, "\hat{a}"'] & A_1
\end{tikzcd}
\equiv
\begin{tikzcd}
X_0^0 \arrow[r, "\hat{x}_1^0"] \arrow[dd, "x_0 \cdot f_0"] & X_1^0 \arrow[dd, "x_1 \cdot f_1"] \\
& \\
A_0 \arrow[r, "\hat{a}"'] & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0^0 \arrow[d, "x_0"'] & X_1^0 \arrow[l, "\check{x}_1^0"'] \arrow[d, "x_1"] \\
X_0^1 \arrow[d, "f_0"'] & X_1^1 \arrow[l, dashed, "\check{x}_1^1"'] \arrow[d, "f_1"] \\
A_0 & A_1 \arrow[l, "\check{a}"]
\end{tikzcd}
\equiv
\begin{tikzcd}
X_0^0 \arrow[dd, "x_0 \cdot f_0"] & X_1^0 \arrow[l, "\check{x}_1^0"'] \arrow[dd, "x_1 \cdot f_1"] \\
& \\
A_0 & A_1 \arrow[l, "\check{a}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
X_0 \arrow[r, "\hat{x}"] \arrow[d, "f_0"'] & X_1 \arrow[d, "f_1"] \\
A_0^0 \arrow[r, dashed, "\hat{a}_1^0"] \arrow[d, "a_0"'] & A_1^0 \arrow[d, "a_1"] \\
A_0^1 \arrow[r, "\hat{a}_1^1"'] & A_1^1
\end{tikzcd}
\equiv
\begin{tikzcd}
X_0 \arrow[r, "\hat{x}"] \arrow[dd, "f_0 \cdot a_0"] & X_1 \arrow[dd, "f_1 \cdot a_1"] \\
& \\
A_0^1 \arrow[r, "\hat{a}_1^1"'] & A_1^1
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "\check{x}"'] \arrow[d, "f_1"] \\
A_0^0 \arrow[d, "a_0"'] & A_1^0 \arrow[d, "a_1"] \\
A_0^1 & A_1^1 \arrow[l, "\check{a}_1^1"]
\end{tikzcd}
\equiv
\begin{tikzcd}
X_0 \arrow[dd, "f_0 \cdot a_0"] & X_1 \arrow[l, "\check{x}"'] \arrow[dd, "f_1 \cdot a_1"] \\
& \\
A_0^1 & A_1^1 \arrow[l, "\check{a}_1^1"]
\end{tikzcd}
$$

So vertical composition of $\langle f \rangle$ is unital, by inheriting the vertical units of $\overrightarrow{f}$ and $\overleftarrow{f}$.

- Vertical composition by the inverse of a unit isomorphism is the inverse of vertical composition by the unit isomorphism.

- For each vertical-composable triple, vertical composition is associative.

- For each horizontal-composable pair of $\overrightarrow{f}$, the horizontal composite in $\langle f \rangle$ equals that of $\overrightarrow{f}$.

- For each horizontal-composable pair of $\overleftarrow{f}$, the horizontal composite in $\langle f \rangle$ equals that of $\overleftarrow{f}$.

$$
\begin{array}{c}
\begin{tikzcd}
X_0 \arrow[r, "\hat{x}_1"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\hat{x}_2"] \arrow[d, dashed, "f_1"] & X_2 \arrow[d, "f_2"] \\
A_0 \arrow[r, "\hat{a}_1"'] & A_1 \arrow[r, "\hat{a}_2"'] & A_2
\end{tikzcd} \\
\equiv \\
\begin{tikzcd}
X_0 \arrow[rr, "\hat{x}_1 \hat{x}_2"] \arrow[d, "f_0"'] & & X_2 \arrow[d, "f_2"] \\
A_0 \arrow[rr, "\hat{a}_1 \hat{a}_2"'] & & A_2
\end{tikzcd}
\end{array}
\qquad
\begin{array}{c}
\begin{tikzcd}
X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "\check{x}_1"'] \arrow[d, dashed, "f_1"] & X_2 \arrow[l, "\check{x}_2"'] \arrow[d, "f_2"] \\
A_0 & A_1 \arrow[l, "\check{a}_1"] & A_2 \arrow[l, "\check{a}_2"]
\end{tikzcd} \\
\equiv \\
\begin{tikzcd}
X_0 \arrow[d, "f_0"'] & & X_2 \arrow[ll, "\check{x}_1 \check{x}_2"'] \arrow[d, "f_2"] \\
A_0 & & A_2 \arrow[ll, "\check{a}_1 \check{a}_2"]
\end{tikzcd}
\end{array}
$$

- For each horizontal-composable triple of $\langle f \rangle$, horizontal composition is associative.

- For each horizontal-composable pair of vertical-composable pairs in $\langle f \rangle$, the interchange law.

- The unit isomorphisms are natural with respect to identity squares and opsquares of $\langle f\rangle$.

$$
\begin{tikzcd}
X \arrow[r, dotted, "\widehat{\mathrm{id}}.X"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & X \arrow[d, equal] \\
X \arrow[d, "f"'] & X \arrow[l, dotted, "\check{\mathrm{id}}.X"'] \arrow[d, "f"] \\
A & A \arrow[l, dotted, "\check{\mathrm{id}}.A"]
\end{tikzcd}
\equiv
\begin{tikzcd}
X_0 \arrow[r, dotted, "\widehat{\mathrm{id}}.X"] \arrow[d, "f"'] & X_1 \arrow[d, "f"] \\
A \arrow[r, dotted, "\widehat{\mathrm{id}}.A"'] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & A \arrow[d, equal] \\
A & A \arrow[l, dotted, "\check{\mathrm{id}}.A"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, equal] \arrow[dr, phantom, "\cong"] & X \arrow[l, dotted, "\check{\mathrm{id}}.X"'] \arrow[d, equal] \\
X \arrow[r, dotted, "\widehat{\mathrm{id}}.X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\
A \arrow[r, dotted, "\widehat{\mathrm{id}}.A"'] & A
\end{tikzcd}
\equiv
\begin{tikzcd}
X \arrow[d, "f"'] & X \arrow[l, "\check{\mathrm{id}}.X"'] \arrow[d, "f"] \\
A \arrow[d, equal] \arrow[dr, phantom, "\cong"] & A \arrow[l, dotted, "\check{\mathrm{id}}.A"] \arrow[d, equal] \\
A \arrow[r, dotted, "\widehat{\mathrm{id}}.A"'] & A
\end{tikzcd}
$$

Then $\langle f\rangle$ is a vertical profunctor from $\langle\mathbb{X}\rangle$ to $\langle\mathbb{A}\rangle$, essentially by definition, as follows.

The vertical actions of $\langle\mathbb{X}\rangle$ on $\langle f\rangle$ and $\langle\mathbb{A}\rangle$ on $\langle f\rangle$ is defined in generating $\langle f\rangle$; and they are associative and unital, with the vertical unit of a zig-zag being the horizontal composite of vertical identities.

$$
\begin{tikzcd}
X_0 \arrow[rrrr, leftrightarrow, "\langle\bar{x}_k\rangle"] \arrow[d, "f_0"'] & & & & X_k \arrow[d, "f_i"] \\
A_0 \arrow[r, leftrightarrow, "\bar{a}_1"'] \arrow[d, equal] & A_1 \arrow[r, leftrightarrow] \arrow[d, equal] & \cdots \arrow[r, leftrightarrow] & A_{\ell-1} \arrow[r, leftrightarrow, "\bar{a}_\ell"'] \arrow[d, equal] & A_\ell \arrow[d, equal] \\
A_0 \arrow[r, leftrightarrow, "\bar{a}_1"'] & A_1 \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & A_{\ell-1} \arrow[r, leftrightarrow, "\bar{a}_\ell"'] & A_\ell
\end{tikzcd}
$$

Horizontal composition is defined in generating $\langle f\rangle$, and it satisfies associativity and interchange. The unitors of $\langle\mathbb{X}\rangle$ and $\langle\mathbb{A}\rangle$ are natural with respect to elements of $\langle f\rangle$, by the naturality of the unit isomorphisms with respect to identities of $\langle f\rangle$.

$$
\begin{tikzcd}
X_0 \arrow[r, leftrightarrow, "\langle\bar{x}_{k-1}\rangle"] \arrow[d, equal] & X_{k-1} \arrow[rr, "\hat{x}_k"] \arrow[d, equal] & & X_k \arrow[d, equal] \\
X_0 \arrow[r, leftrightarrow, "\langle\bar{x}_{k-1}\rangle"] \arrow[d, equal] & X_{k-1} \arrow[r, "\hat{x}_k"] \arrow[d, equal] & X_k \arrow[r, dotted, "\widehat{\mathrm{id}}.X_k"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & X_k \arrow[d, equal] \\
X_0 \arrow[r, leftrightarrow, "\langle\bar{x}_{k-1}\rangle"] \arrow[d, "f_0"'] & X_{k-1} \arrow[r, "\hat{x}_k"] \arrow[d, dashed, "f_{i-1}"] & X_k \arrow[d, dashed, "f_i"] & X_k \arrow[l, dotted, "\check{\mathrm{id}}.X_k"'] \arrow[d, "f_i"] \\
A_0 \arrow[r, "\langle\bar{a}_{\ell-1}\rangle"'] \arrow[d, equal] & A_{\ell-1} \arrow[r, "\hat{a}_\ell"'] \arrow[d, equal] & A_\ell \arrow[d, equal] \arrow[dr, phantom, "\cong"] & A_\ell \arrow[l, dotted, "\check{\mathrm{id}}.A_\ell"] \arrow[d, equal] \\
A_0 \arrow[r, "\langle\bar{a}_{\ell-1}\rangle"'] \arrow[d, equal] & A_{\ell-1} \arrow[r, "\hat{a}_\ell"'] \arrow[d, equal] & A_\ell \arrow[r, dotted, "\widehat{\mathrm{id}}.A_\ell"'] & A_\ell \arrow[d, equal] \\
A_0 \arrow[r, "\langle\bar{a}_{\ell-1}\rangle"'] & A_{\ell-1} \arrow[rr, "\hat{a}_\ell"'] & & A_\ell
\end{tikzcd}
=
\begin{tikzcd}
X_0 \arrow[r, leftrightarrow, "\langle\bar{x}_k\rangle"] \arrow[d, "f_0"'] & X_k \arrow[d, "f_i"] \\
A_0 \arrow[r, leftrightarrow, "\langle\bar{a}_\ell\rangle"'] & A_\ell
\end{tikzcd}
$$

This completes the definition of the weave vertical profunctor $\langle f\rangle : \langle\mathbb{X}\rangle \mid \langle\mathbb{A}\rangle$.

Finally, we extend the "weave construction" to functors and transformations. We denote each by double brackets, $[\![\mathbb{X}]\!] : \mathbb{X}_0 \to \mathbb{X}_1$, with application $[\![\mathbb{X}]\!](\mathrm{X}_0) \equiv [\![\mathrm{X}_0]\!]$.

Definition 31. Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ be a functor; this induces an **arrow double functor** $\overrightarrow{[\![\mathbb{A}]\!]} : \overrightarrow{\mathbb{A}_0} \to \overrightarrow{\mathbb{A}_1}$ and an **oparrow double functor** $\overleftarrow{[\![\mathbb{A}]\!]} : \overleftarrow{\mathbb{A}_0} \to \overleftarrow{\mathbb{A}_1}$.

Define the **weave double functor** $\langle[\![\mathbb{A}]\!]\rangle : \langle\mathbb{A}_0\rangle \to \langle\mathbb{A}_1\rangle$ to be their coproduct. Hence $\langle[\![\mathbb{A}]\!]\rangle$ maps squares to squares, opsquares to opsquares, and unit isomorphisms to unit isomorphisms.

Definition 32. Let $\mathbb{X}_0, \mathbb{X}_1, \mathbb{A}_0, \mathbb{A}_1$ be categories, and let $[\![\mathbb{X}]\!] : \mathbb{X}_0 \to \mathbb{X}_1$ and $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ be functors.

Let $f_0 : \mathbb{X}_0 \mid \mathbb{A}_0$, $f_1 : \mathbb{X}_1 \mid \mathbb{A}_1$ be profunctors, and $[\![f]\!] : f_0 \Rightarrow f_1$ a transformation over $([\![\mathbb{X}]\!], [\![\mathbb{A}]\!])$.

$$
\begin{tikzcd}
\mathbb{X}_0 \arrow[r, "f_0", "\shortmid"{marking}] \arrow[d, "[\![\mathbb{X}]\!]"'] \arrow[dr, phantom, "\Downarrow [\![f]\!]"] & \mathbb{A}_0 \arrow[d, "[\![\mathbb{A}]\!]"] \\
\mathbb{X}_1 \arrow[r, "f_1"', "\shortmid"{marking}] & \mathbb{A}_1
\end{tikzcd}
$$

Then $[\![f]\!]$ gives a transformation of squares $\overrightarrow{[\![f]\!]} : \overrightarrow{f_0} \Rightarrow \overrightarrow{f_1}$ and opsquares $\overleftarrow{[\![f]\!]} : \overleftarrow{f_0} \Rightarrow \overleftarrow{f_1}$.

$$
\begin{tikzcd}
{[\![\mathrm{X}_0]\!]} \arrow[r, "{[\![\hat{\mathrm{x}}]\!]}"] \arrow[d, "{[\![\mathrm{f}_0]\!]}", "\shortmid"{marking}] & {[\![\mathrm{X}_1]\!]} \arrow[d, "{[\![\mathrm{f}_1]\!]}", "\shortmid"{marking}] \\
{[\![\mathrm{A}_0]\!]} \arrow[r, "{[\![\hat{\mathrm{a}}]\!]}"'] & {[\![\mathrm{A}_1]\!]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\![\mathrm{X}_0]\!]} \arrow[d, "{[\![\mathrm{f}_0]\!]}", "\shortmid"{marking}] & {[\![\mathrm{X}_1]\!]} \arrow[l, "{[\![\check{\mathrm{x}}]\!]}"'] \arrow[d, "{[\![\mathrm{f}_1]\!]}", "\shortmid"{marking}] \\
{[\![\mathrm{A}_0]\!]} & {[\![\mathrm{A}_1]\!]} \arrow[l, "{[\![\check{\mathrm{a}}]\!]}"]
\end{tikzcd}
$$

Each square commutes by naturality: if $\mathrm{x}\cdot\mathrm{f}_1 = \mathrm{f}_0\cdot\mathrm{a}$, then we have $[\![\mathrm{x}]\!]\cdot[\![\mathrm{f}_1]\!] = [\![\mathrm{x}\cdot\mathrm{f}_1]\!] = [\![\mathrm{f}_0\cdot\mathrm{a}]\!] = [\![\mathrm{f}_0]\!]\cdot[\![\mathrm{a}]\!]$.

The **weave vertical transformation** $\langle[\![f]\!]\rangle : \langle f_0\rangle(\langle\mathbb{X}_0\rangle, \langle\mathbb{A}_0\rangle) \Rightarrow \langle f_1\rangle(\langle\mathbb{X}_1\rangle, \langle\mathbb{A}_1\rangle)$ is the coproduct of these transformations, defined by mapping squares and opsquares of $\mathbb{X}$ and f and $\mathbb{A}$.

This defines the "weave construction" by a mapping of squares from $\mathbb{C}\mathrm{at}$ to $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$: bifibrant double categories and double functors, vertical profunctors and transformations; see Def. 58.

So the question is, does $\langle-\rangle$ form a double functor? i.e. how does it interact with profunctor composition? Here we find that the *associativity quotient* of $f \circ g$ introduces significant complexity to the construction.

The complexity of weaves and composition

Let $f : \mathbb{X} \,|\, \mathbb{Y}$ and $g : \mathbb{Y} \,|\, \mathbb{Z}$ be profunctors. The composite $f \circ g : \mathbb{X} \,|\, \mathbb{Z}$ consists of pairs (f, g) quotiented by associativity: $(\mathrm{f}, \mathrm{y} \cdot \mathrm{g}) = (\mathrm{f} \cdot \mathrm{y}, \mathrm{g})$, i.e. equivalence classes of "pairs up to associativity" $[(\mathrm{f}, \mathrm{g})]$.

Yet two pairs $(\mathrm{f}_0, \mathrm{g}_0)$ and $(\mathrm{f}_1, \mathrm{g}_1)$ may be equivalent via many distinct zig-zags, while in the composite we have only that $[(\mathrm{f}_0, \mathrm{g}_0)] = [(\mathrm{f}_1, \mathrm{g}_1)]$, with no specific zig-zag. This means that all structures defined on $f \circ g$, i.e. *actions* of a matrix profunctor, must be independent of any choice of pair *and* any choice of zig-zag.

Fortunately, the associativity quotient can be clearly characterized in the weave of the composite, $\langle f \circ g \rangle$: the inner actions by zig-zags in $\mathbb{Y}$ are precisely the *identity squares*.

$$
\begin{tikzcd}
\mathrm{X} \arrow[rrrr, dotted, no head] \arrow[d, "\mathrm{f}_0"] & & & & \mathrm{X} \arrow[d, "\mathrm{f}_1"] \\
\mathrm{Y}_0 \arrow[r, leftrightarrow] & \cdots & \arrow[l, phantom] \arrow[r, leftrightarrow, "\mathrm{id}.[(\mathrm{f},\mathrm{g})]"] & \cdots \arrow[r, leftrightarrow] & \mathrm{Y}_1 \\
\mathrm{Z} \arrow[u, phantom] \arrow[rrrr, dotted, no head] & & & & \mathrm{Z}
\arrow[from=2-1, to=3-1, "\mathrm{g}_0"]
\arrow[from=2-5, to=3-5, "\mathrm{g}_1"]
\end{tikzcd}
$$

Hence to define sequential composition of matrix profunctors, we must *quotient* by the action of these zig-zags, to make these identity squares act as the identity; see Def. 43.

So, is $\langle - \rangle$ a double functor? The answer is *no*. Above, there are many distinct representations of each identity square, so there is no transformation $\langle f \circ g \rangle \Rightarrow \langle f \rangle \circ \langle g \rangle$. Yet the other direction is also obstructed, as the following composites of weaves cannot be expressed as squares in $\langle f \circ g \rangle$.

$$
\begin{tikzcd}
\mathrm{X}_0 \arrow[r, "\hat{\mathrm{x}}_1"] \arrow[d, "\mathrm{f}_0"'] & \mathrm{X}_1 \arrow[d, "\mathrm{f}_1"] & \mathrm{X}_2 \arrow[l, "\check{\mathrm{x}}_2"'] \arrow[r, "\hat{\mathrm{x}}_3"] \arrow[d, "\mathrm{f}_2"] & \mathrm{X}_3 \arrow[d, "\mathrm{f}_3"] \\
\mathrm{Y}_0 \arrow[r, "\hat{\mathrm{y}}_1"'] \arrow[d, equal] & \mathrm{Y}_1 \arrow[dr, phantom, "\cong"] & \mathrm{Y}_1 \arrow[l, dotted] \arrow[r, "\hat{\mathrm{y}}_2"'] & \mathrm{Y}_2 \arrow[d, equal] \\
\mathrm{Y}_0 \arrow[rrr, "\hat{\mathrm{y}}_1\hat{\mathrm{y}}_2"'] \arrow[d, "\mathrm{g}_0"'] & & & \mathrm{Y}_2 \arrow[d, "\mathrm{g}_1"] \\
\mathrm{Z}_0 \arrow[rrr, "\hat{\mathrm{z}}"'] & & & \mathrm{Z}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X}_0 \arrow[rrr, "\hat{\mathrm{x}}"] \arrow[d, "\mathrm{f}_0"'] & & & \mathrm{X}_1 \arrow[d, "\mathrm{f}_1"] \\
\mathrm{Y}_0 \arrow[rrr, "\hat{\mathrm{y}}_1\hat{\mathrm{y}}_2"] \arrow[d, equal] & & & \mathrm{Y}_2 \arrow[d, equal] \\
\mathrm{Y}_0 \arrow[r, "\hat{\mathrm{y}}_1"] \arrow[d, "\mathrm{g}_0"'] & \mathrm{Y}_1 \arrow[d, "\mathrm{g}_1"] \arrow[ur, phantom, "\cong"] & \mathrm{Y}_1 \arrow[l, dotted] \arrow[r, "\hat{\mathrm{y}}_2"] \arrow[d, "\mathrm{g}_2"] & \mathrm{Y}_2 \arrow[d, "\mathrm{g}_3"] \\
\mathrm{Z}_0 \arrow[r, "\hat{\mathrm{z}}_1"'] & \mathrm{Z}_1 & \mathrm{Z}_2 \arrow[l, "\check{\mathrm{z}}_2"'] \arrow[r, "\hat{\mathrm{z}}_3"'] & \mathrm{Z}_3
\end{tikzcd}
$$

Proposition 33. Mapping a category $\mathbb{A}$ to the weave double category $\langle \mathbb{A} \rangle$ defines a span functor from $\mathbb{C}\mathrm{at}$ to $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$, which is neither a lax nor colax double functor.

2.2 Matrix categories

We are now ready to define the primary concepts which underlie a logic.

We simplify the presentation of structures and coherences in two ways.

(1) We denote a transformation by its components, e.g. the associator of a matrix category is

$$(\mathrm{a} \odot R) \odot \mathrm{b} \cong \mathrm{a} \odot (R \odot \mathrm{b}).$$

(2) We use the symbol $x \rightrightarrows y$ to denote that the two transformations from x to y, inferrable from context, are equal; e.g. the two ways to reassociate four elements are equal.

$$((\mathrm{a}_1 \odot \mathrm{a}_2) \odot \mathrm{a}_3) \odot R \rightrightarrows \mathrm{a}_1 \odot (\mathrm{a}_2 \odot (\mathrm{a}_3 \odot R))$$

Additionally, we elide the associators and unitors of $\mathrm{Span}\mathbb{C}\mathrm{at}$; they can be inferred.

Definition 34. Let $\mathbb{A}$ and $\mathbb{B}$ be categories, with weave double categories $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$.

A **matrix category** or **two-sided bifibration** $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ which forms a bimodule from $\langle\mathbb{A}\rangle$ to $\langle\mathbb{B}\rangle$.

Hence a matrix category $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is a span category, with a pair of span functors for actions

$$\begin{tikzcd}
\mathbb{A} \arrow[d, equal] & \langle\mathbb{A}\rangle * \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "\odot_{\mathbb{A}}"] & \mathbb{B} \arrow[d, equal] \\
\mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[d, equal] & \mathcal{R} * \langle\mathbb{B}\rangle \arrow[l] \arrow[r] \arrow[d, "\odot_{\mathbb{B}}"] & \mathbb{B} \arrow[d, equal] \\
\mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{B}
\end{tikzcd}$$

and three invertible span transformations for associativity

$$\begin{tikzcd}
\langle\mathbb{A}\rangle * \mathcal{R} * \langle\mathbb{B}\rangle \arrow[r, "\langle\mathbb{A}\rangle * \odot_{\mathbb{B}}"] \arrow[d, "\odot_{\mathbb{A}} * \langle\mathbb{B}\rangle"'] \arrow[dr, phantom, "\Downarrow \alpha_{\mathcal{R}}"] & \langle\mathbb{A}\rangle * \mathcal{R} \arrow[d, "\odot_{\mathbb{A}}"] \\
\mathcal{R} * \langle\mathbb{B}\rangle \arrow[r, "\odot_{\mathbb{B}}"] & \mathcal{R}
\end{tikzcd}$$

$$
\begin{tikzcd}
\langle\mathbb{A}\rangle * \langle\mathbb{A}\rangle * \mathcal{R} \arrow[r, "\circ*\mathcal{R}"] \arrow[d, "\langle\mathbb{A}\rangle*\odot"'] \arrow[dr, phantom, "\Downarrow\alpha_{\mathbb{A}}"] & \langle\mathbb{A}\rangle * \mathcal{R} \arrow[d, "\odot"] \\
\langle\mathbb{A}\rangle * \mathcal{R} \arrow[r, "\odot"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle \arrow[r, "\mathcal{R}*\circ"] \arrow[d, "\odot*\langle\mathbb{B}\rangle"'] \arrow[dr, phantom, "\Downarrow\alpha_{\mathbb{B}}"] & \mathcal{R} * \langle\mathbb{B}\rangle \arrow[d, "\odot"] \\
\mathcal{R} * \langle\mathbb{B}\rangle \arrow[r, "\odot"] & \mathcal{R}
\end{tikzcd}
$$

and two invertible span transformations for unitality

$$
\begin{tikzcd}
\mathcal{R} \arrow[r, equal] \arrow[d, "\mathrm{id}.\mathbb{A}*\mathcal{R}"'] \arrow[dr, phantom, "\upsilon_A"] & \mathcal{R} \arrow[d, equal] \\
\langle\mathbb{A}\rangle * \mathcal{R} \arrow[r, "\odot"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \arrow[r, equal] \arrow[d, "\mathcal{R}*\mathrm{id}.\mathbb{B}"'] \arrow[dr, phantom, "\upsilon_B"] & \mathcal{R} \arrow[d, equal] \\
\mathcal{R} * B \arrow[r, "\odot"] & \mathcal{R}
\end{tikzcd}
$$

so that the following transformations are well-defined, for associativity

$$
\begin{tikzcd}
\langle\mathbb{A}\rangle * \langle\mathbb{A}\rangle * \langle\mathbb{A}\rangle * \mathcal{R} \arrow[r, shift left=2, "(\langle\bar{a}_k\rangle\circ\langle\bar{a}_\ell\rangle\circ\langle\bar{a}_m\rangle)\odot R"] \arrow[r, shift right=2, "\langle\bar{a}_k\rangle\odot(\langle\bar{a}_\ell\rangle\odot(\langle\bar{a}_m\rangle\odot R))"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle\mathbb{A}\rangle * \mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle \arrow[r, shift left=2, "\langle\bar{a}_k\rangle\odot(R\odot(\langle\bar{b}_\ell\rangle\circ\langle\bar{b}_m\rangle))"] \arrow[r, shift right=2, "((\langle\bar{a}_k\rangle\odot R)\odot\langle\bar{b}_\ell\rangle)\odot\langle\bar{b}_m\rangle"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}
\end{tikzcd}
$$

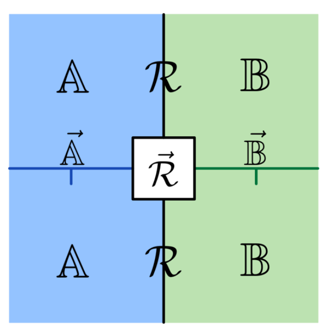
$$
\begin{tikzcd}
\langle\mathbb{A}\rangle * \langle\mathbb{A}\rangle * \mathcal{R} * \langle\mathbb{B}\rangle \arrow[r, shift left=2, "(\langle\bar{a}_k\rangle\circ\langle\bar{a}_\ell\rangle)\odot(R\odot\langle\bar{b}_m\rangle)"] \arrow[r, shift right=2, "(\langle\bar{a}_k\rangle\odot(\langle\bar{a}_\ell\rangle\odot R))\odot\langle\bar{b}_m\rangle"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle \arrow[r, shift left=2, "R\odot(\langle\bar{b}_k\rangle\circ\langle\bar{b}_\ell\rangle\circ\langle\bar{b}_m\rangle)"] \arrow[r, shift right=2, "((R\odot\langle\bar{b}_k\rangle)\odot\langle\bar{b}_\ell\rangle)\odot\langle\bar{b}_m\rangle"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}
\end{tikzcd}
$$

and for unitality.

$$
\begin{tikzcd}
\langle\mathbb{A}\rangle * \mathcal{R} \arrow[r, shift left=2, "(\langle\bar{a}_k\rangle\circ\overline{\mathrm{id}}.\mathrm{A}_k)\odot R"] \arrow[r, shift right=2, "\langle\bar{a}_k\rangle\odot(\overline{\mathrm{id}}.\mathrm{A}_k\odot R)"'] \arrow[r, phantom, "\Downarrow"] & \langle\mathbb{A}\rangle * \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} * \langle\mathbb{B}\rangle \arrow[r, shift left=2, "R\odot(\overline{\mathrm{id}}.\mathrm{B}_0\circ\langle\bar{b}_k\rangle)"] \arrow[r, shift right=2, "(R\odot\overline{\mathrm{id}}.\mathrm{B}_0)\odot\langle\bar{b}_k\rangle"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R} * \langle\mathbb{B}\rangle
\end{tikzcd}
$$

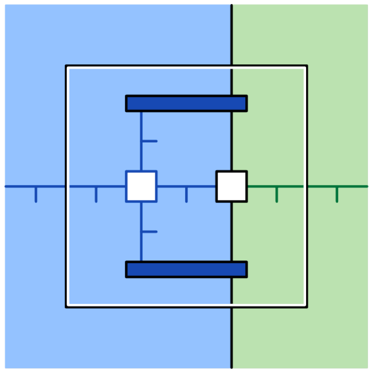
The objects and morphisms of a matrix category are the loose morphisms and squares of a bifibrant double category, i.e. relations and inferences of a logic, via the *collage*; see Prop. 35.

$$
\begin{tikzcd}
\mathrm{A}_0 \arrow[r, "R_0"] \arrow[d, "\mathrm{a}"'] \arrow[dr, phantom, "\Downarrow r"] & \mathrm{B}_0 \arrow[d, "\mathrm{b}"] \\
\mathrm{A}_1 \arrow[r, "R_1"'] & \mathrm{B}_1
\end{tikzcd}
$$

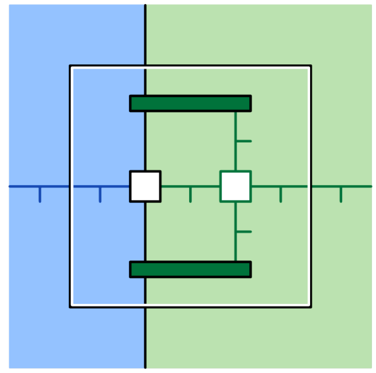


The actions by $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$ define parallel composition of this double category, as we soon expound.

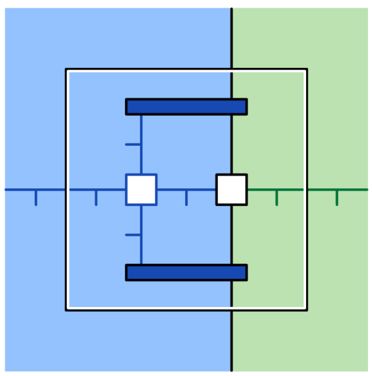
Because a weave double category is a coproduct, an action by $\langle \mathbb{A} \rangle$ defines a pair of actions by $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$, and a bimodule structure defines *four* actions. These are drawn as follows.



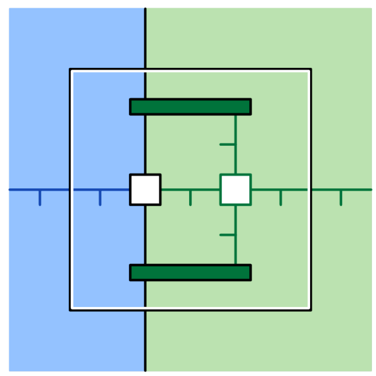
$\overrightarrow{\mathbb{A}}$-substitution



$\overrightarrow{\mathbb{B}}$-image



$\overleftarrow{\mathbb{A}}$-image



$\overleftarrow{\mathbb{B}}$-substitution

Combining these pairwise, there are four distinct bimodule structures, which we name as follows.

$\overrightarrow{\mathbb{A}}, \overrightarrow{\mathbb{B}}$-bimodule	$\overrightarrow{\mathbb{A}}, \overleftarrow{\mathbb{B}}$-bimodule	$\overleftarrow{\mathbb{A}}, \overrightarrow{\mathbb{B}}$-bimodule	$\overleftarrow{\mathbb{A}}, \overleftarrow{\mathbb{B}}$-bimodule
companion	**fibration**	**opfibration**	**conjoint**

Each action defines parallel composition by squares in $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{B}}$ or opsquares in $\overleftarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{B}}$.

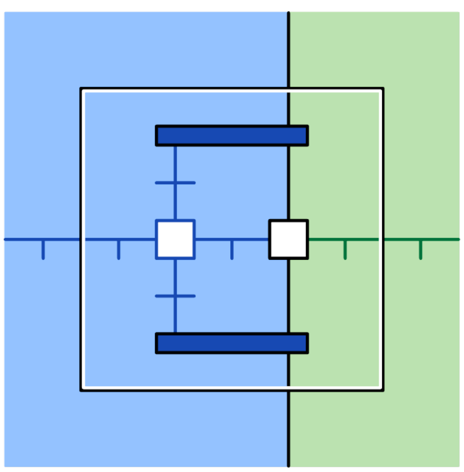
$$\begin{tikzcd}
\mathrm{A}^0_0 \arrow[r, "\hat{\mathrm{a}}^0_1"] \arrow[d, "\mathrm{a}_0"'] & \mathrm{A}^0_1 \arrow[r, "R_0", "|" marking] \arrow[d, "\mathrm{a}_1"'] \arrow[dr, phantom, "\Downarrow r"] & \mathrm{B}^0_0 \arrow[d, "b_0"] \\
\mathrm{A}^1_0 \arrow[r, "\hat{\mathrm{a}}^1_1"'] & \mathrm{A}^1_1 \arrow[r, "R_1"', "|" marking] & \mathrm{B}^1_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}^0_1 \arrow[r, "R_0", "|" marking] \arrow[d, "\mathrm{a}_1"'] \arrow[dr, phantom, "\Downarrow r"] & \mathrm{B}^0_0 \arrow[r, "\hat{\mathrm{b}}^0_1"] \arrow[d, "\mathrm{b}_0"] & \mathrm{B}^0_1 \arrow[d, "\mathrm{b}_1"] \\
\mathrm{A}^1_1 \arrow[r, "R_1"', "|" marking] & \mathrm{B}^1_0 \arrow[r, "\hat{\mathrm{b}}^1_1"'] & \mathrm{B}^1_1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{A}^0_0 \arrow[d, "\mathrm{a}_0"'] & \mathrm{A}^0_1 \arrow[l, "\check{\mathrm{a}}^0_1"'] \arrow[r, "R_0", "|" marking] \arrow[d, "\mathrm{a}_1"'] \arrow[dr, phantom, "\Downarrow r"] & \mathrm{B}^0_0 \arrow[d, "\mathrm{b}_0"] \\
\mathrm{A}^1_0 & \mathrm{A}^1_1 \arrow[l, "\check{\mathrm{a}}^1_1"] \arrow[r, "R_1"', "|" marking] & \mathrm{B}^1_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}^0_1 \arrow[r, "R_0", "|" marking] \arrow[d, "\mathrm{a}_1"'] \arrow[dr, phantom, "\Downarrow r"] & \mathrm{B}^0_0 \arrow[d, "\mathrm{b}_0"] & \mathrm{B}^0_1 \arrow[l, "\check{\mathrm{b}}^0_1"'] \arrow[d, "\mathrm{b}_1"] \\
\mathrm{A}^1_1 \arrow[r, "R_1"', "|" marking] & \mathrm{A}^1_1 & \mathrm{B}^1_1 \arrow[l, "\check{\mathrm{b}}^1_1"]
\end{tikzcd}$$

We draw a zig-zag as an arrow pointing in both directions, and denote the action as follows.

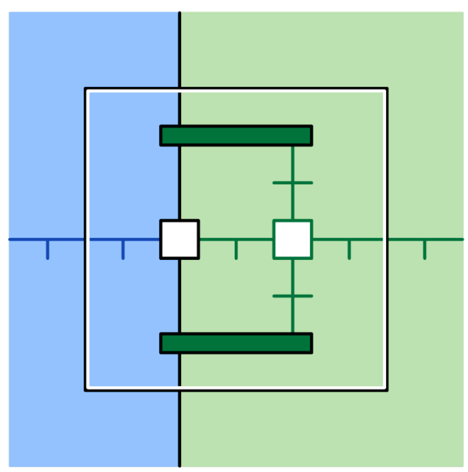
$$\begin{tikzcd}
\mathrm{A}^0_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{a}}_k \rangle"] \arrow[d, "\mathrm{a}_0"'] \arrow[dr, phantom, "w_{\mathrm{A}}"] & \mathrm{A}^0_k \arrow[r, "R_0", "|" marking] \arrow[d, "\mathrm{a}_k"'] \arrow[dr, phantom, "\Downarrow r"] & \mathrm{B}^0_0 \arrow[d, "\mathrm{b}_0"] \\
\mathrm{A}^1_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{a}}_\ell \rangle"'] & \mathrm{A}^1_k \arrow[r, "R_1"', "|" marking] & \mathrm{B}^1_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}^0_k \arrow[r, "R_0", "|" marking] \arrow[d, "\mathrm{a}_k"'] \arrow[dr, phantom, "\Downarrow r"] & \mathrm{B}^0_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{b}}_k \rangle"] \arrow[d, "\mathrm{b}_0"] \arrow[dr, phantom, "w_{\mathrm{B}}"] & \mathrm{B}^0_k \arrow[d, "\mathrm{b}_k"] \\
\mathrm{A}^1_k \arrow[r, "R_1"', "|" marking] & \mathrm{B}^1_0 \arrow[r, leftrightarrow, "\langle \mathrm{b} \rangle_\ell"'] & \mathrm{B}^1_k
\end{tikzcd}$$

$w_{\mathrm{A}} \odot r : \mathcal{R}(\langle \bar{\mathrm{a}}_k \rangle \odot R_0, \langle \bar{\mathrm{a}}_\ell \rangle \odot R_1)$



left action by $\langle \mathbb{A} \rangle$

$r \odot w_{\mathrm{B}} : \mathcal{R}(R_0 \odot \langle \bar{\mathrm{b}}_k \rangle, R_1 \odot \langle \bar{\mathrm{b}}_\ell \rangle)$



right action by $\langle \mathbb{B} \rangle$

Yet apart from functoriality, which involves weaves in $\mathbb{A}$ and $\mathbb{B}$, the action is a structure on objects; and an action by zig-zags is equivalent to a pair of actions by arrows and oparrows. Hence for many definitions, particularly the coherence isomorphisms, we may simplify action notation to $\bar{\mathrm{a}} \odot R$ and $R \odot \bar{\mathrm{b}}$.

We now proceed to draw the coherences of these actions in string diagrams, and show that they define the parallel composition of a bifibrant double category.

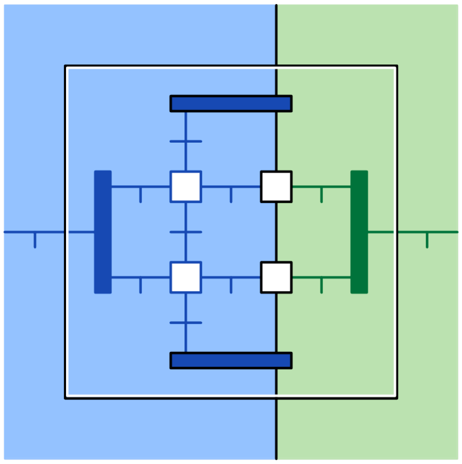
The actions of a matrix category satisfy the following coherence. First, each action is a span functor, i.e. it preserves the sequential composition of the span categories $\langle\mathbb{A}\rangle, \mathcal{R}, \langle\mathbb{B}\rangle$.

Composing in $\langle\mathbb{A}\rangle$ and $\mathcal{R}$, then acting by $\langle\mathbb{A}\rangle$, is equal to acting by $\langle\mathbb{A}\rangle$ then composing in $\mathcal{R}$. Composing in $\mathcal{R}$ and $\langle\mathbb{B}\rangle$ then acting by $\langle\mathbb{B}\rangle$ is equal to acting by $\langle\mathbb{B}\rangle$ then composing in $\mathcal{R}$.

Hence the following two composite squares are well-defined.

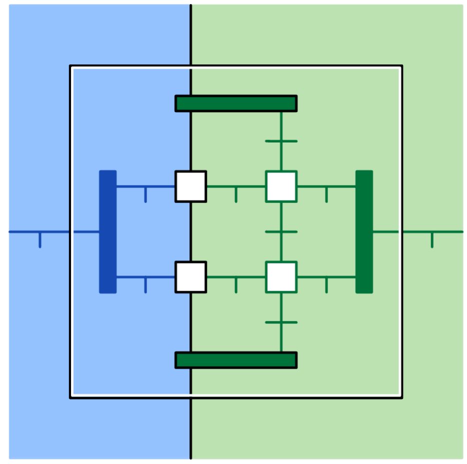
$$
\begin{tikzcd}
\mathrm{A}^0_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{a}}_k \rangle"] \arrow[d, "\mathrm{a}^1_0"'] \arrow[dr, phantom, "w^1_{\mathrm{A}}"] & \mathrm{A}^0_k \arrow[r, "R_0", "\shortmid"{marking}] \arrow[d, "\mathrm{a}^1_i"'] \arrow[dr, phantom, "\Downarrow r_1"] & \mathrm{B}^0_0 \arrow[d, "\mathrm{b}^1_0"] \\
\mathrm{A}^1_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{a}}_\ell \rangle"] \arrow[d, "\mathrm{a}^2_0"'] \arrow[dr, phantom, "w^2_{\mathrm{A}}"] & \mathrm{A}^1_\ell \arrow[r, "R_1"', "\shortmid"{marking}] \arrow[d, "\mathrm{a}^2_j"'] \arrow[dr, phantom, "\Downarrow r_2"] & \mathrm{B}^1_0 \arrow[d, "\mathrm{b}^2_0"] \\
\mathrm{A}^2_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{a}}_m \rangle"'] & \mathrm{A}^2_m \arrow[r, "R_2"', "\shortmid"{marking}] & \mathrm{B}^2_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{A}^0_k \arrow[r, "R_0", "\shortmid"{marking}] \arrow[d, "\mathrm{a}^1_i"'] \arrow[dr, phantom, "\Downarrow r_1"] & \mathrm{B}^0_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{b}}_k \rangle"] \arrow[d, "\mathrm{b}^1_0"'] \arrow[dr, phantom, "w^1_{\mathrm{B}}"] & \mathrm{B}^0_k \arrow[d, "\mathrm{b}^1_i"] \\
\mathrm{A}^1_k \arrow[r, "R_1"', "\shortmid"{marking}] \arrow[d, "\mathrm{a}^2_j"'] \arrow[dr, phantom, "\Downarrow r_2"] & \mathrm{B}^1_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{b}}_\ell \rangle"] \arrow[d, "\mathrm{b}^2_0"'] \arrow[dr, phantom, "w^2_{\mathrm{B}}"] & \mathrm{B}^1_\ell \arrow[d, "\mathrm{b}^2_j"] \\
\mathrm{A}^2_k \arrow[r, "R_2"', "\shortmid"{marking}] & \mathrm{B}^2_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{b}}_m \rangle"'] & \mathrm{B}^2_m
\end{tikzcd}
$$

By the coherence principle, these equations can be expressed by drawing simultaneous sequential and parallel composition. Note that this is the "interchange law" for double categories.



left interchange

$$(w_{\mathrm{A}}^1 \cdot w_{\mathrm{A}}^2) \odot (r_1 \cdot r_2) = (w_{\mathrm{A}}^1 \odot r_1) \cdot (w_{\mathrm{A}}^2 \odot r_2)$$



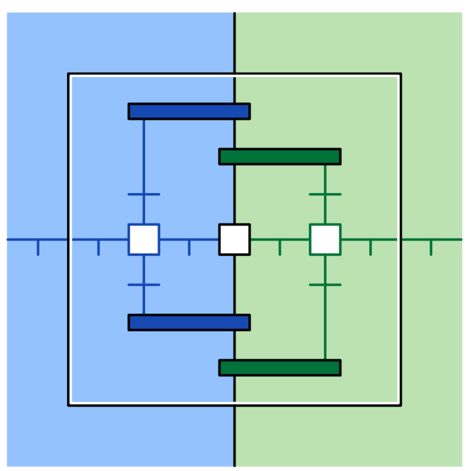
right interchange

$$(r_1 \cdot r_2) \odot (w_{\mathrm{B}}^1 \cdot w_{\mathrm{B}}^2) = (r_1 \odot w_{\mathrm{B}}^1) \cdot (r_2 \odot w_{\mathrm{B}}^2)$$

Next to unpack is the three-dimensional structure. The actions are associative and unital up to coherent isomorphism: there are three "associators" for $\mathbb{A}\mathbb{A}\mathcal{R}$, $\mathbb{A}\mathcal{R}\mathbb{B}$, and $\mathcal{R}\mathbb{B}\mathbb{B}$, and two "unitors" for $\mathrm{id}_{\mathbb{A}}\mathcal{R}$ and $\mathcal{R}\mathrm{id}_{\mathbb{B}}$.

Three-dimensional string diagrams effectively depict the coherence of these isomorphisms. First, each is *natural* with respect to the morphisms of $\langle\mathbb{A}\rangle$, $\mathcal{R}$, and $\langle\mathbb{B}\rangle$.

The center associator is an invertible span transformation $(\langle\mathbb{A}\rangle \odot \mathcal{R}) \odot \langle\mathbb{B}\rangle \cong \langle\mathbb{A}\rangle \odot (\mathcal{R} \odot \langle\mathbb{B}\rangle)$. This can be drawn as a cube, with source on top and target on bottom, connected by the homs of $\langle\mathbb{A}\rangle$, $\mathcal{R}$, and $\langle\mathbb{B}\rangle$.



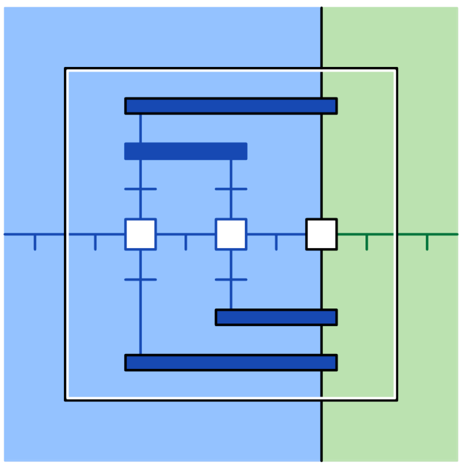
center associator

$$\alpha_{\mathcal{R}} : \bar{\mathrm{a}} \odot (R \odot \bar{\mathrm{b}}) \cong (\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}$$

By the coherence principle, this cube expresses the naturality of the associator with respect to morphisms of $\langle\mathbb{A}\rangle, \mathcal{R}, \langle\mathbb{B}\rangle$: for every pair of weaves $w_{\mathrm{A}} : \langle \bar{\mathrm{a}}^0_k\rangle \to \langle \bar{\mathrm{a}}^1_m\rangle$ and $w_{\mathrm{B}} : \langle \bar{\mathrm{b}}^0_\ell\rangle \to \langle \bar{\mathrm{b}}^1_n\rangle$ the following commutes.

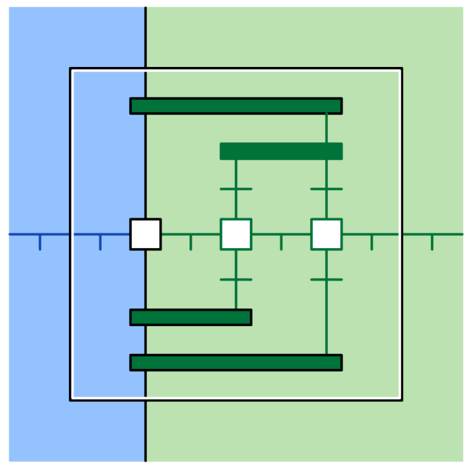
$$\begin{tikzcd}
(\langle \bar{\mathrm{a}}^0_k\rangle \odot R) \odot \langle \bar{\mathrm{b}}^0_\ell\rangle \arrow[r, "\alpha_{\mathcal{R}}"] \arrow[d, "(w_{\mathrm{A}} \odot R) \odot w_{\mathrm{B}}"'] & \langle \bar{\mathrm{a}}^0_k\rangle \odot (R \odot \langle \bar{\mathrm{b}}^0_\ell\rangle) \arrow[d, "w_{\mathrm{A}} \odot (R \odot w_B)"] \\
(\langle \bar{\mathrm{a}}^1_m\rangle \odot R) \odot \langle \bar{\mathrm{b}}^1_n\rangle \arrow[r, "\alpha_{\mathcal{R}}"'] & \langle \bar{\mathrm{a}}^1_m\rangle \odot (R \odot \langle \bar{\mathrm{b}}^1_n\rangle)
\end{tikzcd}$$

Continuing with the isomorphisms, there are associators for each composite action



left associator

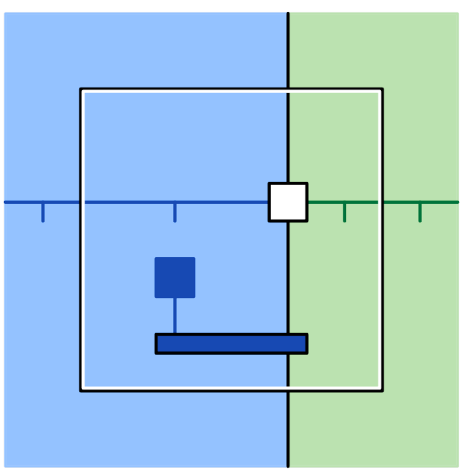
$\alpha_{\mathbb{A}} :\ (\bar{\mathrm{a}}_1 \circ \bar{\mathrm{a}}_2) \odot R \cong \bar{\mathrm{a}}_1 \odot (\bar{\mathrm{a}}_2 \odot R)$



right associator

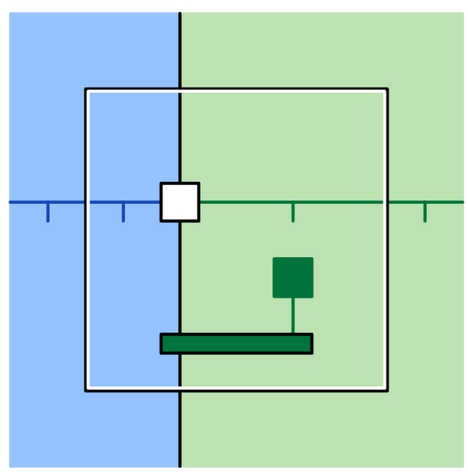
$\alpha_{\mathbb{B}} :\ R \odot (\bar{\mathrm{b}}_1 \circ \bar{\mathrm{b}}_2) \cong (R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2$

and the left and right unitors, which are invertible span transformations.



left unitor

$\upsilon_{\mathbb{A}} : R \cong \overline{\mathrm{id}}.\mathrm{A} \odot R$



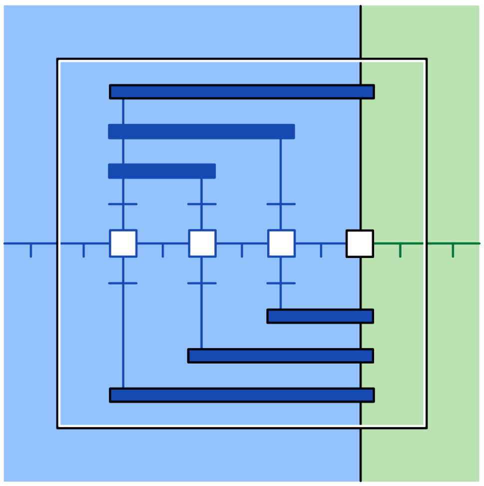
right unitor

$\upsilon_{\mathbb{B}} :\ R \cong R \odot \overline{\mathrm{id}}.\mathrm{B}$

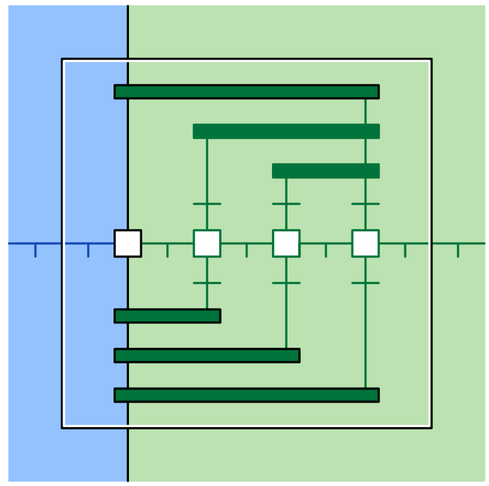
Finally, we have the equations that these isomorphisms satisfy.

For each quadruple in $\langle\mathbb{A}\rangle*\langle\mathbb{A}\rangle*\langle\mathbb{A}\rangle*\mathcal{R}$, $\langle\mathbb{A}\rangle*\langle\mathbb{A}\rangle*\mathcal{R}*\langle\mathbb{B}\rangle$, $\langle\mathbb{A}\rangle*\mathcal{R}*\langle\mathbb{B}\rangle*\langle\mathbb{B}\rangle$, and $\mathcal{R}*\langle\mathbb{B}\rangle*\langle\mathbb{B}\rangle*\langle\mathbb{B}\rangle$, the two ways to reassociate are equal.

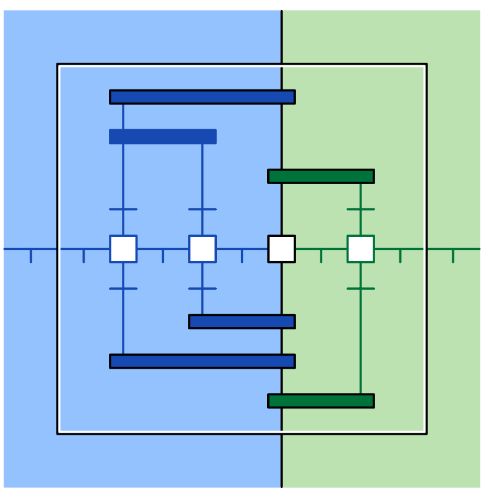
associator coherence



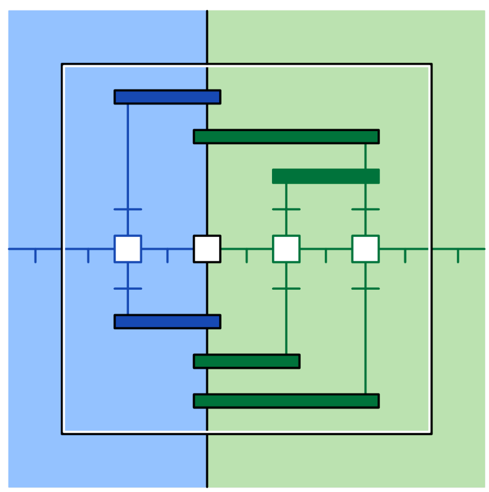
$$(\langle\bar{\mathrm{a}}_k\rangle\circ\langle\bar{\mathrm{a}}_\ell\rangle\circ\langle\bar{\mathrm{a}}_m\rangle)\odot R$$
$$\rightrightarrows\langle\bar{\mathrm{a}}_k\rangle\odot(\langle\bar{\mathrm{a}}_\ell\rangle\odot(\langle\bar{\mathrm{a}}_m\rangle\odot R))$$



$$R\odot(\langle\bar{\mathrm{b}}_k\rangle\circ\langle\bar{\mathrm{b}}_\ell\rangle\circ\langle\bar{\mathrm{b}}_m\rangle)$$
$$\rightrightarrows((R\odot\langle\bar{\mathrm{b}}_k\rangle)\odot\langle\bar{\mathrm{b}}_\ell\rangle)\odot\langle\bar{\mathrm{b}}_m\rangle$$



$$(\langle\bar{\mathrm{a}}_k\rangle\circ\langle\bar{\mathrm{a}}_\ell\rangle)\odot(R\odot\langle\bar{\mathrm{b}}_m\rangle)$$
$$\rightrightarrows(\langle\bar{\mathrm{a}}_k\rangle\odot(\langle\bar{\mathrm{a}}_\ell\rangle\odot R))\odot\langle\bar{\mathrm{b}}_m\rangle$$

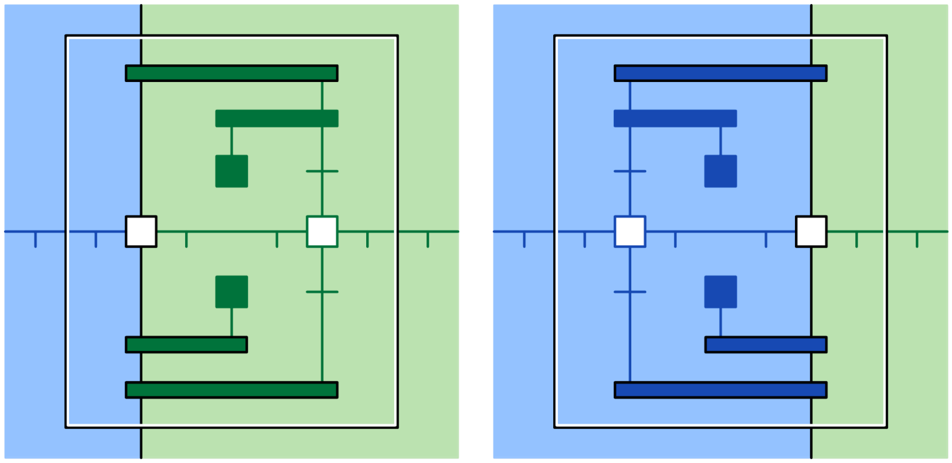


$$\langle\bar{\mathrm{a}}_k\rangle\odot(R\odot(\langle\bar{\mathrm{b}}_\ell\rangle\circ\langle\bar{\mathrm{b}}_m\rangle))$$
$$\rightrightarrows((\langle\bar{\mathrm{a}}_k\rangle\odot R)\odot\langle\bar{\mathrm{b}}_\ell\rangle)\odot\langle\bar{\mathrm{b}}_m\rangle$$

These equations define the "pentagon equations" of a double category.

Last, the left unitor coheres with the left associator, and the right unitor coheres with the right associator.

unitor coherence



$(\langle \bar{\mathrm{a}}_k \rangle \circ \mathrm{id.A}) \odot R \rightrightarrows \langle \bar{\mathrm{a}}_k \rangle \odot (\mathrm{id.A} \odot R)$ $\quad$ $R \odot (\mathrm{id.B} \circ \langle \bar{\mathrm{b}}_k \rangle) \rightrightarrows (R \odot \mathrm{id.B}) \odot \langle \bar{\mathrm{b}}_k \rangle$

These equations define the "triangle equations" of a double category.

We summarize the definition, by dimension: 1 is *data*, 2 and 3 are *structure*, and 4 is *property*.

1.	**matrix category**	a span category	$\mathcal{R} : \mathbb{A} \parallel \mathbb{B}$
2.	precompose action	a span functor	$\langle\mathbb{A}\rangle \odot \mathcal{R} : \langle\mathbb{A}\rangle * \mathcal{R} \to \mathcal{R}$
	postcompose action	a span functor	$\mathcal{R} \odot \langle\mathbb{B}\rangle : \mathcal{R} * \langle\mathbb{B}\rangle \to \mathcal{R}$
3.	associators	inv. span trans.	$\alpha_{\mathbb{A}} : (\bar{a}_1 \odot \bar{a}_2) \odot R \cong \bar{a}_1 \odot (\bar{a}_2 \odot \mathcal{R})$
			$\alpha_{\mathcal{R}} : (\bar{a} \odot R) \odot \bar{a} \cong \bar{a} \odot (R \odot \bar{b})$
			$\alpha_{\mathbb{B}} : (R \odot \bar{b}_1) \odot \bar{b}_2 \cong R \odot (\bar{b}_1 \odot \bar{b}_2)$
	unitors	inv. span trans.	$\upsilon_{\mathbb{A}} : R \cong \overline{\mathrm{id}}.\mathrm{A} \odot R$
			$\upsilon_{\mathbb{B}} : R \cong R \odot \overline{\mathrm{id}}.\mathrm{B}$
4.	assoc. coherence	equations	$(\bar{a}_1 \circ \bar{a} \circ \bar{a}_3) \odot R \rightrightarrows \bar{a}_1 \odot (\bar{a}_2 \odot (\bar{a}_3 \odot R))$
			$\bar{a}_1 \odot (R \odot (\bar{b}_2 \circ \bar{b}_3)) \rightrightarrows ((\bar{a}_1 \odot R) \odot \bar{b}_2) \odot \bar{b}_3$
			$(\bar{a}_1 \circ \bar{a}_2) \odot (R \odot \bar{b}_3) \rightrightarrows (\bar{a}_1 \odot (\bar{a}_2 \odot R)) \odot \bar{b}_3$
			$R \odot (\bar{b}_1 \circ \bar{b}_2 \circ \bar{b}_3) \rightrightarrows ((R \odot \bar{b}_1) \odot \bar{b}_2) \odot \bar{b}_3$
	unit coherence	equations	$(\bar{a} \circ \overline{\mathrm{id}}.\mathrm{A}) \odot R \rightrightarrows \bar{a} \odot (\overline{\mathrm{id}}.\mathrm{A} \odot R)$
			$R \odot (\overline{\mathrm{id}}.\mathrm{B} \circ \bar{b}) \rightrightarrows (R \odot \overline{\mathrm{id}}.\mathrm{B}) \odot \bar{b}$

To complete the section, we show how matrix category forms a logic.

Proposition 35. Let $\mathcal{R} : \mathbb{A} \parallel \mathbb{B}$ be a matrix category, i.e. two-sided bifibration. The **collage** of $\mathcal{R}$, defined as follows, is a bifibrant double category. The base category is $\mathbb{A} + \mathbb{B}$, and the total category is $\langle\mathbb{A}\rangle + \mathcal{R} + \langle\mathbb{B}\rangle$.

$$\mathbb{A} + \mathbb{B} \longleftarrow \langle\mathbb{A}\rangle + \mathcal{R} + \langle\mathbb{B}\rangle \longrightarrow \mathbb{A} + \mathbb{B}$$

Parallel composition is given by the actions of $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$ on $\mathcal{R}$, and parallel composition in $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$. The associators and unitors are defined by the coherence isomorphisms of $\mathcal{R}$, and those of $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$; their equations hold by fiat. The collage is a bifibrant double category, because morphisms of $\mathbb{A}$ and $\mathbb{B}$ induce arrows and oparrows, which are companions and conjoints.

2.2.1 Matrix functor [Descent]

A matrix category is a 2-bimodule, so its actions are associative and unital up to coherent isomorphism. In the same way, a *matrix functor* preserves the actions up to a coherent isomorphism.

Definition 36. Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors, denoted $[\![\mathbb{A}]\!](\mathrm{A}_0) \equiv [\![\mathrm{A}_0]\!] : \mathbb{A}_1$.

Let $\mathcal{R}_0 : \mathbb{A}_0 \,\|\, \mathbb{B}_0$ and $\mathcal{R}_1 : \mathbb{A}_1 \,\|\, \mathbb{B}_1$ be matrix categories. A **matrix functor** $[\![\mathcal{R}(\mathbb{A},\mathbb{B})]\!]$ from $\mathcal{R}_0$ to $\mathcal{R}_1$ is a morphism of 2-bimodules in $\mathrm{Span}\mathbb{C}\mathrm{at}$. This is a span functor

$$\begin{tikzcd}
\mathbb{A}_0 \arrow[d, "{[\![\mathbb{A}]\!]}"] & \mathcal{R}_0 \arrow[l] \arrow[r] \arrow[d, "{[\![\mathcal{R}]\!]}"] & \mathbb{B}_0 \arrow[d, "{[\![\mathbb{B}]\!]}"] \\
\mathbb{A}_1 & \mathcal{R}_1 \arrow[l] \arrow[r] & \mathbb{B}_1
\end{tikzcd}$$

with invertible span transformations called the **left** and **right join**

$$\begin{tikzcd}
\langle \mathbb{A}_0 \rangle * \mathcal{R}_0 \arrow[r, "{\langle [\![\mathbb{A}]\!] \rangle * [\![\mathcal{R}]\!]}"] \arrow[d, "\odot^0_{\mathbb{A}}"'] \arrow[dr, phantom, "{\Downarrow [\![\odot_{\mathbb{A}}]\!]}"] & \langle \mathbb{A}_1 \rangle * \mathcal{R}_1 \arrow[d, "\odot^1_{\mathbb{A}}"] \\
\mathcal{R}_0 \arrow[r, "{[\![\mathcal{R}]\!]}"'] & \mathcal{R}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R}_0 * \langle \mathbb{B}_0 \rangle \arrow[r, "{[\![\mathcal{R}]\!] * \langle [\![\mathbb{B}]\!] \rangle}"] \arrow[d, "\odot^0_{\mathbb{B}}"'] \arrow[dr, phantom, "{\Downarrow [\![\odot_{\mathbb{B}}]\!]}"] & \mathcal{R}_1 * \langle \mathbb{B}_1 \rangle \arrow[d, "\odot^1_{\mathbb{B}}"] \\
\mathcal{R}_0 \arrow[r, "{[\![\mathcal{R}]\!]}"'] & \mathcal{R}_1
\end{tikzcd}$$

$$[\![\odot_{\mathbb{A}}]\!] : [\![\bar{\mathrm{a}}_0]\!] \odot [\![R_0]\!] \cong [\![\bar{\mathrm{a}}_0 \odot R_0]\!] \qquad [\![\odot_{\mathbb{B}}]\!] : [\![R_0]\!] \odot [\![\bar{\mathrm{b}}_0]\!] \cong [\![R_0 \odot \bar{\mathrm{b}}_0]\!]$$

which together are natural with respect to the center associator:

$$\begin{tikzcd}
{[\![\bar{\mathrm{a}}]\!] \odot ([\![R]\!] \odot [\![\bar{\mathrm{b}}]\!])} \arrow[r, "\alpha_{\mathcal{R}}"] \arrow[d, "{[\![\mathbb{A}]\!] \odot_{\mathbb{A}} [\![\odot_{\mathbb{B}}]\!]}"'] & {([\![\bar{\mathrm{a}}]\!] \odot [\![R]\!]) \odot [\![\bar{\mathrm{b}}]\!]} \arrow[d, "{[\![\odot_{\mathbb{A}}]\!] \odot_{\mathbb{B}} [\![\mathbb{B}]\!]}"] \\
{[\![\bar{\mathrm{a}}]\!] \odot [\![R \odot \bar{\mathrm{b}}]\!]} \arrow[d, "{[\![\odot_{\mathbb{A}}]\!]}"'] & {[\![\bar{\mathrm{a}} \odot R]\!] \odot [\![\bar{\mathrm{b}}]\!]} \arrow[d, "{[\![\odot_{\mathbb{B}}]\!]}"] \\
{[\![\bar{\mathrm{a}} \odot (R \odot \bar{\mathrm{b}})]\!]} \arrow[r, "\alpha_{\mathcal{R}}"'] & {[\![(\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}]\!]}
\end{tikzcd}$$

and each is natural with respect to its own associator:

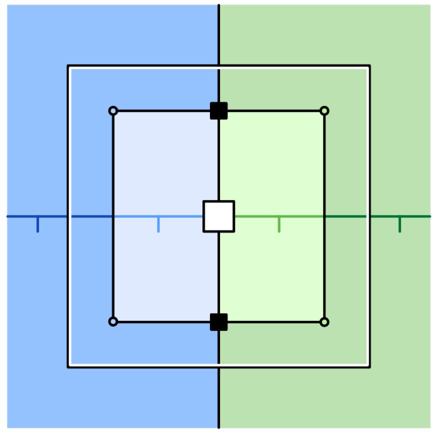
$$
\begin{tikzcd}
(\llbracket \bar{\mathrm{a}}_1 \rrbracket \circ \llbracket \bar{\mathrm{a}}_2 \rrbracket) \odot \llbracket R \rrbracket \arrow[r, "\alpha_{\mathbb{A}}"] \arrow[d, "\llbracket \odot_{\mathbb{A}} \rrbracket \odot_{\mathbb{A}} \llbracket \mathcal{R} \rrbracket"] & \llbracket \bar{\mathrm{a}}_1 \rrbracket \odot (\llbracket \bar{\mathrm{a}}_2 \rrbracket \odot \llbracket R \rrbracket) \arrow[d, "\llbracket \mathbb{A} \rrbracket \odot_{\mathbb{A}} \llbracket \odot_{\mathbb{A}} \rrbracket"] \\
\llbracket \bar{\mathrm{a}}_1 \circ \bar{\mathrm{a}}_2 \rrbracket \odot \llbracket R \rrbracket \arrow[d, "\llbracket \odot_{\mathbb{A}} \rrbracket"] & \llbracket \bar{\mathrm{a}}_1 \rrbracket \odot \llbracket \bar{\mathrm{a}}_2 \odot R \rrbracket \arrow[d, "\llbracket \odot_{\mathbb{A}} \rrbracket"] \\
\llbracket (\bar{\mathrm{a}}_1 \circ \bar{\mathrm{a}}_2) \odot R \rrbracket \arrow[r, "\alpha_{\mathbb{A}}"'] & \llbracket \bar{\mathrm{a}}_1 \odot (\bar{\mathrm{a}}_2 \odot R) \rrbracket
\end{tikzcd}
\qquad
\begin{tikzcd}
\llbracket R \rrbracket \odot (\llbracket \bar{\mathrm{b}}_1 \rrbracket \circ \llbracket \bar{\mathrm{b}}_2 \rrbracket) \arrow[r, "\alpha_{\mathbb{B}}"] \arrow[d, "\llbracket \mathcal{R} \rrbracket \odot_{\mathbb{B}} \llbracket \odot_{\mathbb{B}} \rrbracket"] & (\llbracket R \rrbracket \odot \llbracket \bar{\mathrm{b}}_1 \rrbracket) \odot \llbracket \bar{\mathrm{b}}_2 \rrbracket \arrow[d, "\llbracket \odot_{\mathbb{B}} \rrbracket \odot_{\mathbb{B}} \llbracket \mathbb{B} \rrbracket"] \\
\llbracket R \rrbracket \odot \llbracket \bar{\mathrm{b}}_1 \circ \bar{\mathrm{b}}_2 \rrbracket \arrow[d, "\llbracket \odot_{\mathbb{B}} \rrbracket"] & \llbracket R \odot \bar{\mathrm{b}}_1 \rrbracket \odot \llbracket \bar{\mathrm{b}}_2 \rrbracket \arrow[d, "\llbracket \odot_{\mathbb{B}} \rrbracket"] \\
\llbracket R \odot (\bar{\mathrm{b}}_1 \circ \bar{\mathrm{b}}_2) \rrbracket \arrow[r, "\alpha_{\mathbb{B}}"'] & \llbracket (R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2 \rrbracket
\end{tikzcd}
$$

and each is natural with respect to its own unitor.

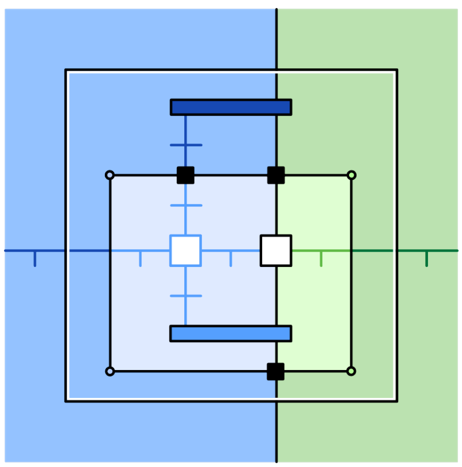
$$
\begin{tikzcd}
\llbracket \overline{\mathrm{id}}.\mathrm{A} \rrbracket \odot \llbracket R \rrbracket \arrow[r, "\upsilon_{\mathbb{A}}"] \arrow[d, "\llbracket \odot_{\mathbb{A}} \rrbracket"] & \llbracket R \rrbracket \arrow[d, equal] \\
\llbracket \overline{\mathrm{id}}.\mathrm{A} \odot R \rrbracket \arrow[r, "\upsilon_{\mathbb{A}}"'] & \llbracket R \rrbracket
\end{tikzcd}
\qquad
\begin{tikzcd}
\llbracket R \rrbracket \odot \llbracket \overline{\mathrm{id}}.\mathrm{B} \rrbracket \arrow[r, "\upsilon_{\mathbb{B}}"] \arrow[d, "\llbracket \odot_{\mathbb{B}} \rrbracket"] & \llbracket R \rrbracket \arrow[d, equal] \\
\llbracket R \odot \overline{\mathrm{id}}.\mathrm{B} \rrbracket \arrow[r, "\upsilon_{\mathbb{B}}"'] & \llbracket R \rrbracket
\end{tikzcd}
$$

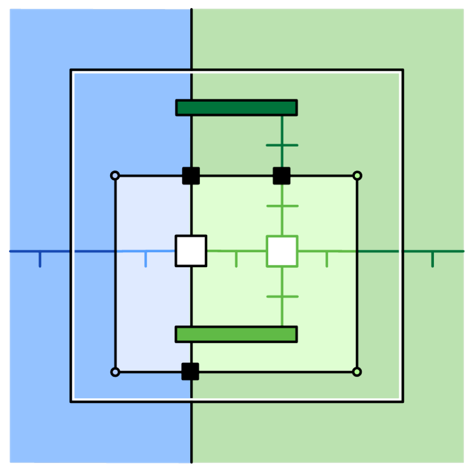
A matrix functor is visualized as follows.

Dimension 2 is the mapping, a span functor with its induced span transformation.



Dimension 3 is the joins, which slide each action through the mapping.





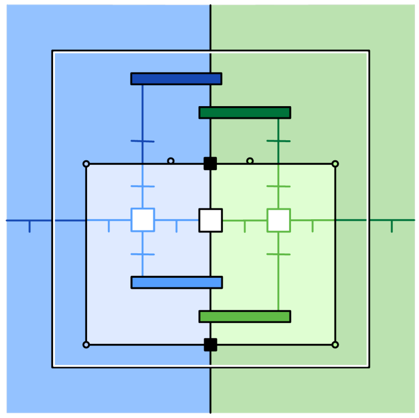
left join

$[\![\langle \bar{\mathrm{a}}_k \rangle]\!] \odot_1 [\![R]\!] \;\cong\; [\![\langle \bar{\mathrm{a}}_k \rangle \odot_0 R]\!]$

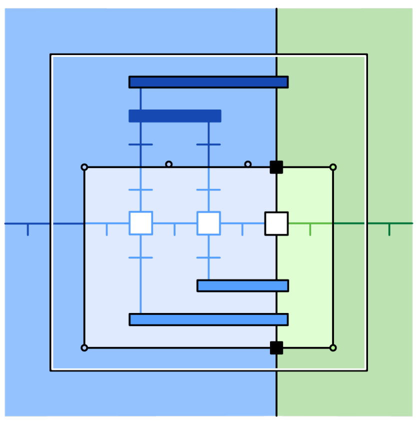
right join

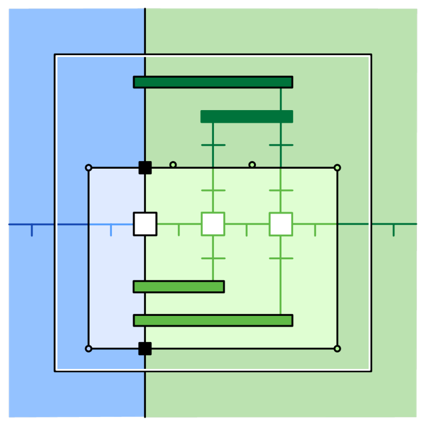
$[\![R]\!] \odot_1 [\![\langle \bar{\mathrm{b}}_\ell \rangle]\!] \cong [\![R \odot_0 \langle \bar{\mathrm{b}}_\ell \rangle]\!]$

Dimension 4 is the coherence equations, for associators and for unitors.

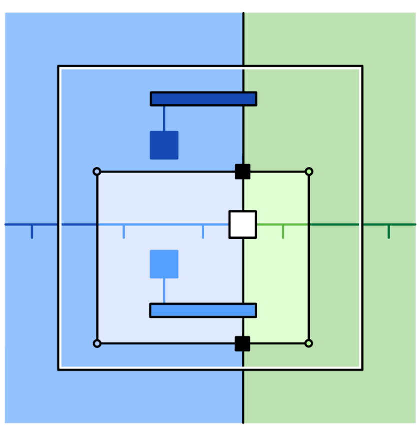


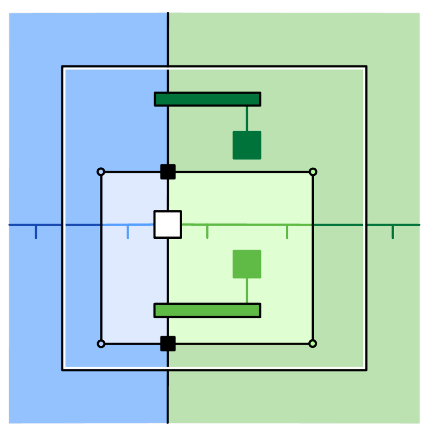
associator coherence





unitor coherence





We summarize the concept of matrix functor.

2.	**matrix functor**	span functor	$[\![\mathcal{R}]\!]([\![\mathbb{A}]\!],[\![\mathbb{B}]\!]) : \mathcal{R}_0(\mathbb{A}_0,\mathbb{B}_0) \to \mathcal{R}_1(\mathbb{A}_1,\mathbb{B}_1)$
3.	left join	inv. span trans.	$[\![\odot_{\mathbb{A}}]\!] : [\![\bar{\mathrm{a}}_0]\!] \odot [\![R_0]\!] \cong [\![\bar{\mathrm{a}}_0 \odot R_0]\!]$
	right join	inv. span trans.	$[\![\odot_{\mathbb{B}}]\!] : [\![R_0]\!] \odot [\![\bar{\mathrm{b}}_0]\!] \cong [\![R_0 \odot \bar{\mathrm{b}}_0]\!]$
4.	left assoc. coherence	equation	$([\![\bar{\mathrm{a}}_1]\!] \circ [\![\bar{\mathrm{a}}_2]\!]) \odot [\![R]\!] \rightrightarrows [\![\bar{\mathrm{a}}_1 \odot (\bar{\mathrm{a}}_2 \odot R)]\!]$
	center assoc. coherence	equation	$[\![\bar{\mathrm{a}}]\!] \odot ([\![R]\!] \odot [\![\bar{\mathrm{b}}]\!]) \rightrightarrows [\![(\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}]\!]$
	right assoc. coherence	equation	$[\![R]\!] \odot ([\![\bar{\mathrm{b}}_1]\!] \circ [\![\bar{\mathrm{b}}_2]\!]) \rightrightarrows [\![(R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2]\!]$
	left unit coherence	equation	$[\![\bar{\mathrm{id}}.\mathrm{A}]\!] \odot [\![R]\!] \rightrightarrows [\![R]\!]$
	right unit coherence	equation	$[\![R]\!] \odot [\![\bar{\mathrm{id}}.\mathrm{B}]\!] \rightrightarrows [\![R]\!]$

To conclude the section, we derive a *formula* for the category of matrix functors between a pair of matrix categories. This is the foundation of the "co/descent calculus" of bifibrant double categories.

The descent formula

In the same way that the set of transformations between profunctors is formed by an *end*, the category of matrix functors between matrix categories is formed by a *descent object* [19].

A transformation of profunctors satisfies a *naturality equation*, and hence the end which forms the set of transformations is an equalizer. By contrast, a matrix functor is only "natural" up to isomorphism: the category of span functors equipped with a pair of joins is formed by the following *iso-inserter*.

$$\begin{tikzcd}
 & \mathbb{S}(\mathcal{R}_0,\mathcal{R}_1) \arrow[r, "{(\,[\![\bar{\mathrm{a}}]\!]\odot[\![R]\!],\ [\![R]\!]\odot[\![\bar{\mathrm{b}}]\!]\,)}"] \arrow[dd, phantom, "{\Downarrow(\,[\![\odot_{\mathbb{A}}]\!],\ [\![\odot_{\mathbb{B}}]\!]\,)}"] & \mathbb{S}(\langle\mathbb{A}_0\rangle * \mathcal{R}_0,\mathcal{R}_1) \times \mathbb{S}(\mathcal{R}_0 * \langle\mathbb{B}_0\rangle,\mathcal{R}_1) \arrow[dd, equal] \\
\mathrm{Nat}(\mathcal{R}_0,\mathcal{R}_1) \arrow[ur, squiggly, "iso.ins"] \arrow[dr, squiggly, "iso.ins"'] & & \\
 & \mathbb{S}(\mathcal{R}_0,\mathcal{R}_1) \arrow[r, "{(\,[\![\bar{\mathrm{a}}\odot R]\!],\ [\![R\odot\bar{\mathrm{b}}]\!]\,)}"'] & \mathbb{S}(\langle\mathbb{A}_0\rangle * \mathcal{R}_0,\mathcal{R}_1) \times \mathbb{S}(\mathcal{R}_0 * \langle\mathbb{B}_0\rangle,\mathcal{R}_1)
\end{tikzcd}$$

Each coherence equation of these joins is then imposed by an *equifier*.

First, joining composites is well-defined:

$$
\begin{tikzcd}[column sep=large]
& \mathrm{Nat}(\mathcal{R}_0, \mathcal{R}_1) \arrow[dd, equal] \arrow[r, shift left=3, "(\llbracket \bar{a}_1 \rrbracket \circ \llbracket \bar{a}_2 \rrbracket) \odot \llbracket \mathcal{R} \rrbracket"] \arrow[r, shift right=3, "\llbracket \bar{a}_1 \odot (\bar{a}_2 \odot \mathcal{R}) \rrbracket"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathbb{S}((\langle \mathbb{A}_0 \rangle * \langle \mathbb{A}_0 \rangle) * \mathcal{R}_0, \mathcal{R}_1) \\
& & \times \\
\mathrm{Nat}(\mathcal{R}_0, \mathcal{R}_1)_\alpha \arrow[r, squiggly, "equif"] & \mathrm{Nat}(\mathcal{R}_0, \mathcal{R}_1) \arrow[dd, equal] \arrow[r, shift left=3, "\llbracket \bar{a} \rrbracket \odot (\llbracket \mathcal{R} \rrbracket \odot \llbracket \bar{b} \rrbracket)"] \arrow[r, shift right=3, "\llbracket (\bar{a} \odot \mathcal{R}) \odot \bar{b} \rrbracket"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathbb{S}(\langle \mathbb{A}_0 \rangle * (\mathcal{R}_0 * \langle \mathbb{B}_0 \rangle), \mathcal{R}_1) \\
& & \times \\
& \mathrm{Nat}(\mathcal{R}_0, \mathcal{R}_1) \arrow[r, shift left=3, "\llbracket \mathcal{R} \rrbracket \odot (\llbracket \bar{b}_1 \rrbracket \circ \llbracket \bar{b}_2 \rrbracket)"] \arrow[r, shift right=3, "\llbracket (\mathcal{R} \odot \bar{b}_1) \odot \bar{b}_2 \rrbracket"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathbb{S}(\mathcal{R}_0 * (\langle \mathbb{B}_0 \rangle * \langle \mathbb{B}_0 \rangle), \mathcal{R}_1)
\end{tikzcd}
$$

and second, joining units is well-defined.

$$
\begin{tikzcd}[column sep=large]
& \mathrm{Nat}(\mathcal{R}_0, \mathcal{R}_1)_\alpha \arrow[d, equal] \arrow[r, shift left=3, "\llbracket \bar{\mathrm{id}}.\mathrm{A} \rrbracket \odot \llbracket \mathcal{R} \rrbracket"] \arrow[r, shift right=3, "\llbracket \mathcal{R} \rrbracket"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathbb{S}(\mathcal{R}_0, \mathcal{R}_1) \\
\mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{R}_0, \mathcal{R}_1) \arrow[r, squiggly, "equif"] & \mathrm{Nat}(\mathcal{R}_0, \mathcal{R}_1)_\alpha \arrow[d, equal] & \times \\
& \mathrm{Nat}(\mathcal{R}_0, \mathcal{R}_1)_\alpha \arrow[r, shift left=3, "\llbracket \mathcal{R} \rrbracket \odot \llbracket \bar{\mathrm{id}}.\mathrm{B} \rrbracket"] \arrow[r, shift right=3, "\llbracket \mathcal{R} \rrbracket"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathbb{S}(\mathcal{R}_0, \mathcal{R}_1)
\end{tikzcd}
$$

All together, this constructs the *descent object* in $\mathbb{C}\mathrm{at}$ of the above functors and transformations.

$$
\begin{tikzcd}
\mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{R}_0, \mathcal{R}_1) \arrow[r, squiggly, "desc"] & \mathbb{S}(\mathcal{R}_0, \mathcal{R}_1) \arrow[r, shift left=4] \arrow[r, shift right=4] & \begin{array}{c} \mathbb{S}(\langle \mathbb{A}_0 \rangle * \mathcal{R}_0, \mathcal{R}_1) \\ \times \\ \mathbb{S}(\mathcal{R}_0 * \langle \mathbb{B}_0 \rangle, \mathcal{R}_1) \end{array} \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \begin{array}{c} \mathbb{S}((\langle \mathbb{A}_0 \rangle * \langle \mathbb{A}_0 \rangle) * \mathcal{R}_0, \mathcal{R}_1) \\ \times \\ \mathbb{S}(\langle \mathbb{A}_0 \rangle * (\mathcal{R}_0 * \langle \mathbb{B}_0 \rangle), \mathcal{R}_1) \\ \times \\ \mathbb{S}(\mathcal{R}_0 * (\langle \mathbb{B}_0 \rangle * \langle \mathbb{B}_0 \rangle), \mathcal{R}_1) \end{array}
\end{tikzcd}
$$

We denote the descent object, an equifier of an iso-inserter, by an "arrow product" notation.

$$
\mathrm{Mat}\mathbb{C}\mathrm{at}[\mathcal{R}_0 \to \mathcal{R}_1] \quad \equiv \quad \vec{\Pi}\mathrm{A} : \mathbb{A}_0, \mathrm{B} : \mathbb{B}_0 \;\; \mathbb{C}\mathrm{at}[\mathcal{R}_0(\mathrm{A}, \mathrm{B}) \to \mathcal{R}_1(\llbracket \mathrm{A} \rrbracket, \llbracket \mathrm{B} \rrbracket)]
$$

As we will see, the "descent" construction is dual to that of *composition* of matrix categories (2.5).

2.3 Matrix profunctors

Just as a matrix category is a bimodule of weave double categories, a matrix profunctor is a bimodule of weave vertical profunctors, which is coherent with the bimodule structures of the source and target matrix categories.

Definition 37. Let $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$ be categories, and $\mathcal{Q} : \mathbb{X} \,\|\, \mathbb{Y}$ $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ be matrix categories.

Let $f : \mathbb{X} \,|\, \mathbb{A}$ and $g : \mathbb{Y} \,|\, \mathbb{B}$ be profunctors, giving weave profunctors $f \leftarrow \langle f \rangle \rightarrow f$ and $g \leftarrow \langle g \rangle \rightarrow g$.

A **matrix profunctor** $i(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A}, \mathbb{B})$ is a span profunctor which is a bimodule from $\langle f \rangle$ to $\langle g \rangle$, which coheres with the associators and unitors of $\mathcal{Q}$ and $\mathcal{R}$.

Hence a matrix profunctor is a span profunctor

$$\begin{tikzcd}
\mathbb{X} \arrow[d, "f"'] & \mathcal{Q} \arrow[l] \arrow[r] \arrow[d, "i"] \arrow[l, phantom, shift right=4, "\Longleftarrow"] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \mathbb{Y} \arrow[d, "g"] \\
\mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{B}
\end{tikzcd}$$

with two span transformations, precompose action by $\langle f \rangle$ and postcompose action by $\langle g \rangle$

$$\begin{tikzcd}
\langle\mathbb{X}\rangle * \mathcal{Q} \arrow[r, "\odot_{\mathbb{X}}"] \arrow[d, "\langle f\rangle * i"'] \arrow[dr, phantom, "\overset{\circ_f}{\Longrightarrow}"] & \mathcal{Q} \arrow[d, "i"] \\
\langle\mathbb{A}\rangle * \mathcal{R} \arrow[r, "\odot_{\mathbb{A}}"'] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Q} * \langle\mathbb{Y}\rangle \arrow[r, "\odot_{\mathbb{Y}}"] \arrow[d, "i * \langle g\rangle"'] \arrow[dr, phantom, "\overset{\circ_g}{\Longrightarrow}"] & \mathcal{Q} \arrow[d, "i"] \\
\mathcal{R} * \langle\mathbb{B}\rangle \arrow[r, "\odot_{\mathbb{B}}"'] & \mathcal{R}
\end{tikzcd}$$

which cohere with the associators and unitors of $\mathcal{Q}$ and $\mathcal{R}$, as follows.

associator coherence

$$\begin{tikzcd}
\langle\mathbb{X}\rangle * \mathcal{Q} \arrow[rrr] \arrow[ddd, "\langle f\rangle * i"'] & & & \mathcal{Q} \arrow[ddd, "i"] \\
& \langle\mathbb{X}\rangle * \mathcal{Q} * \langle\mathbb{Y}\rangle \arrow[ul] \arrow[r] \arrow[d, "\langle f\rangle * i * \langle g\rangle"] \arrow[ur, phantom, "\cong"] \arrow[l, phantom, "\Leftarrow \langle f\rangle * \circ_g"] \arrow[dr, phantom, "\circ_f * \langle g\rangle \Longrightarrow"] & \mathcal{Q} * \langle\mathbb{Y}\rangle \arrow[ur] \arrow[d, "i * \langle g\rangle"] \arrow[r, phantom, "\circ_g \Longrightarrow"] & \\
& \langle\mathbb{A}\rangle * \mathcal{R} * \langle\mathbb{Y}\rangle \arrow[dl] \arrow[r] \arrow[u, phantom, shift left=6, "\overset{\circ_f}{\Longrightarrow}"] \arrow[dr, phantom, "\cong"] & \mathcal{R} * \langle\mathbb{B}\rangle \arrow[dr] & \\
\langle\mathbb{A}\rangle * \mathcal{R} \arrow[rrr] & & & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{\mathrm{x}} \odot (\mathcal{Q} \odot \bar{\mathrm{y}}) \arrow[r, "\alpha_{\mathcal{Q}}"] \arrow[d, "{[\mathrm{f}_0,\mathrm{f}_1]\circ(i\circ[\mathrm{g}_0,\mathrm{g}_1])}"'] & (\bar{\mathrm{x}} \odot \mathcal{Q}) \odot \bar{\mathrm{y}} \arrow[d, "{([\mathrm{f}_0,\mathrm{f}_1]\circ i)\circ[\mathrm{g}_0,\mathrm{g}_1]}"] \\
\bar{\mathrm{a}} \odot (\mathcal{R} \odot \bar{\mathrm{b}}) \arrow[r, "\alpha_{\mathcal{R}}"'] & (\bar{\mathrm{a}} \odot \mathcal{R}) \odot \bar{\mathrm{b}}
\end{tikzcd}$$

$$
\begin{tikzcd}
\langle\mathbb{X}\rangle * \mathcal{Q} \arrow[rrr] \arrow[ddd, "|" marking, "\langle f\rangle * i"'] & & & \mathcal{Q} \arrow[ddd, "|" marking, "i"] \\
& \langle\mathbb{X}\rangle * \langle\mathbb{X}\rangle * \mathcal{Q} \arrow[ul] \arrow[r] \arrow[d, "|" marking, "\langle f\rangle * \langle f\rangle * i"] \arrow[u, phantom, "\cong", shift right=8ex] \arrow[l, phantom, "\Leftarrow \langle f\rangle * \circ_f", shift right=5ex] & \langle\mathbb{X}\rangle * \mathcal{Q} \arrow[ur] \arrow[d, "|" marking, "\langle f\rangle * i"] \arrow[r, phantom, "\circ_f \Rightarrow", shift right=5ex] & \\
& \langle\mathbb{A}\rangle * \langle\mathbb{A}\rangle * \mathcal{R} \arrow[dl] \arrow[r] \arrow[ur, phantom, "\circ * i \Rightarrow"] \arrow[r, phantom, "\circ_f \Rightarrow", shift left=3ex] \arrow[d, phantom, "\cong", shift right=8ex] & \langle\mathbb{A}\rangle * \mathcal{R} \arrow[dr] & \\
\langle\mathbb{A}\rangle * \mathcal{R} \arrow[rrr] & & & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
(\bar{\mathrm{x}}_1 \circ \bar{\mathrm{x}}_2) \odot Q \arrow[r, "\alpha_{\mathbb{X}}"] \arrow[d, "|" marking, "{[\mathrm{f}_0,\mathrm{f}_1,\mathrm{f}_2] \circ i}"] & \bar{\mathrm{x}}_1 \odot (\bar{\mathrm{x}}_2 \odot Q) \arrow[d, "|" marking, "{[\mathrm{f}_0,\mathrm{f}_1] \circ ([\mathrm{f}_1,\mathrm{f}_2] \circ i)}"] \\
(\bar{\mathrm{a}}_1 \circ \bar{\mathrm{a}}_2) \odot R \arrow[r, "\alpha_{\mathbb{A}}"'] & \bar{\mathrm{a}}_1 \odot (\bar{\mathrm{a}}_2 \odot R)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{Q} * \langle\mathbb{Y}\rangle \arrow[rrr] \arrow[ddd, "|" marking, "i * \langle g\rangle"'] & & & \mathcal{Q} \arrow[ddd, "|" marking, "i"] \\
& \mathcal{Q} * \langle\mathbb{Y}\rangle * \langle\mathbb{Y}\rangle \arrow[ul] \arrow[r] \arrow[d, "|" marking, "i * \langle g\rangle * \langle g\rangle"] \arrow[u, phantom, "\cong", shift right=8ex] \arrow[l, phantom, "\Leftarrow \circ_g * \langle g\rangle", shift right=5ex] & \mathcal{Q} * \langle\mathbb{Y}\rangle \arrow[ur] \arrow[d, "|" marking, "i * \langle g\rangle"] \arrow[r, phantom, "\circ_g \Rightarrow", shift right=5ex] & \\
& \mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle \arrow[dl] \arrow[r] \arrow[ur, phantom, "i * \circ \Rightarrow"] \arrow[r, phantom, "\circ_g \Rightarrow", shift left=3ex] \arrow[d, phantom, "\cong", shift right=8ex] & \mathcal{R} * \langle\mathbb{B}\rangle \arrow[dr] & \\
\mathcal{R} * \langle\mathbb{B}\rangle \arrow[rrr] & & & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \odot (\bar{\mathrm{y}}_1 \circ \bar{\mathrm{y}}_2) \arrow[r, "\alpha_{\mathbb{Y}}"] \arrow[d, "|" marking, "{i \circ [\mathrm{g}_0,\mathrm{g}_1,\mathrm{g}_2]}"] & (Q \odot \bar{\mathrm{y}}_1) \odot \bar{\mathrm{y}}_2 \arrow[d, "|" marking, "{(i \circ [\mathrm{g}_0,\mathrm{g}_1]) \circ [\mathrm{g}_1,\mathrm{g}_2]}"] \\
R \odot (\bar{\mathrm{b}}_1 \circ \bar{\mathrm{b}}_2) \arrow[r, "\alpha_{\mathbb{B}}"'] & (R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2
\end{tikzcd}
$$

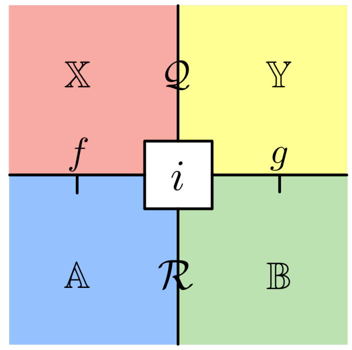
unitor coherence

$$
\begin{tikzcd}
\mathcal{Q} \arrow[rrr, equal] \arrow[ddd, "|" marking, "i"'] \arrow[dr, equal] & & & \mathcal{Q} \arrow[ddd, "|" marking, "i"] \\
& \mathcal{Q} \arrow[r] \arrow[d, "|" marking, "i"] \arrow[u, phantom, "\cong", shift right=8ex] & \langle\mathbb{X}\rangle * \mathcal{Q} \arrow[ur] \arrow[d, "|" marking, "\langle f\rangle * i"] \arrow[r, phantom, "\circ_f \Rightarrow", shift right=5ex] & \\
& \mathcal{R} \arrow[r] \arrow[dl, equal] \arrow[ur, phantom, "\mathrm{id} * i \Rightarrow"] \arrow[d, phantom, "\cong", shift right=8ex] & \langle\mathbb{A}\rangle * \mathcal{R} \arrow[dr] & \\
\mathcal{R} \arrow[rrr, equal] & & & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathrm{id}}.\mathrm{X} \odot Q \arrow[r, "\upsilon_{\mathbb{X}}"] \arrow[d, "|" marking, "\mathrm{id}.\mathrm{f} \circ i"] & Q \arrow[d, "i"] \\
\overline{\mathrm{id}}.\mathrm{A} \odot R \arrow[r, "\upsilon_{\mathbb{A}}"'] & R
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{Q} \arrow[rrr, equal] \arrow[ddd, "|" marking, "i"'] \arrow[dr, equal] & & & \mathcal{Q} \arrow[ddd, "|" marking, "i"] \\
& \mathcal{Q} \arrow[r] \arrow[d, "|" marking, "i"] \arrow[u, phantom, "\cong", shift right=8ex] & \mathcal{Q} * \langle\mathbb{Y}\rangle \arrow[ur] \arrow[d, "|" marking, "i * \langle g\rangle"] \arrow[r, phantom, "\circ_g \Rightarrow", shift right=5ex] & \\
& \mathcal{R} \arrow[r] \arrow[dl, equal] \arrow[ur, phantom, "i * \mathrm{id} \Rightarrow"] \arrow[d, phantom, "\cong", shift right=8ex] & \mathcal{R} * \langle\mathbb{B}\rangle \arrow[dr] & \\
\mathcal{R} \arrow[rrr, equal] & & & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \odot \overline{\mathrm{id}}.\mathrm{Y} \arrow[r, "\upsilon_{\mathbb{Y}}"] \arrow[d, "|" marking, "i \circ \mathrm{id}.\mathrm{g}"] & Q \arrow[d, "i"] \\
R \odot \overline{\mathrm{id}}.\mathrm{B} \arrow[r, "\upsilon_{\mathbb{B}}"'] & R
\end{tikzcd}
$$

To unpack the definition, matrix profunctor elements are seen as squares of a double category.

$$\begin{tikzcd}
\mathrm{X} \arrow[r, "Q" description] \arrow[d, "\mathrm{f}" description] \arrow[dr, phantom, "\overset{\parallel}{\underset{\Downarrow}{i}}"] & \mathrm{Y} \arrow[d, "\mathrm{g}" description] \\
\mathrm{A} \arrow[r, "R" description] & \mathrm{B}
\end{tikzcd}$$

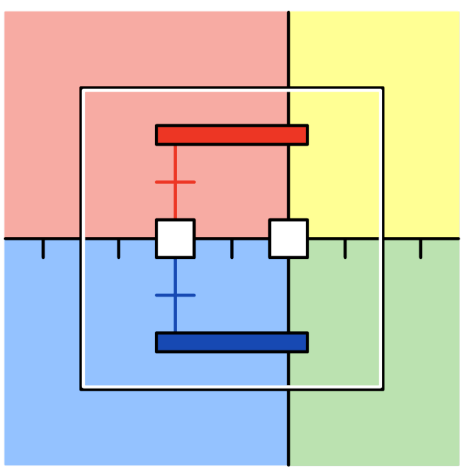


matrix profunctor

$i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$

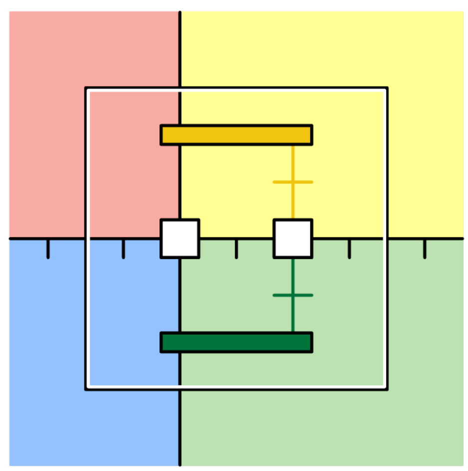
The actions of arrow profunctors $\langle f\rangle$ and $\langle g\rangle$ on i define parallel composition of squares:

$$\begin{tikzcd}
\mathrm{X}_0 \arrow[r, leftrightarrow, "\bar{\mathrm{x}}"] \arrow[d, "\mathrm{f}_0" description] & \mathrm{X}_1 \arrow[r, "Q" description] \arrow[d, "\mathrm{f}_1" description] \arrow[dr, phantom, "\overset{\parallel}{\underset{\Downarrow}{i}}"] & \mathrm{Y}_0 \arrow[d, "\mathrm{g}_0" description] \\
\mathrm{A}_0 \arrow[r, leftrightarrow, "\bar{\mathrm{a}}"'] & \mathrm{A}_1 \arrow[r, "R" description] & \mathrm{B}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X}_1 \arrow[r, "Q" description] \arrow[d, "\mathrm{f}_1" description] \arrow[dr, phantom, "\overset{\parallel}{\underset{\Downarrow}{i}}"] & \mathrm{Y}_0 \arrow[r, leftrightarrow, "\bar{\mathrm{y}}"] \arrow[d, "\mathrm{g}_0" description] & \mathrm{Y}_1 \arrow[d, "\mathrm{g}_1" description] \\
\mathrm{A}_1 \arrow[r, "R" description] & \mathrm{B}_0 \arrow[r, leftrightarrow, "\bar{\mathrm{b}}"'] & \mathrm{B}_1
\end{tikzcd}$$



precompose action

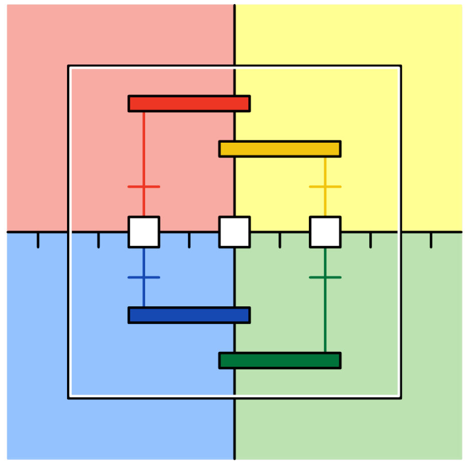
$\circ_f : \langle f\rangle * i \to i$



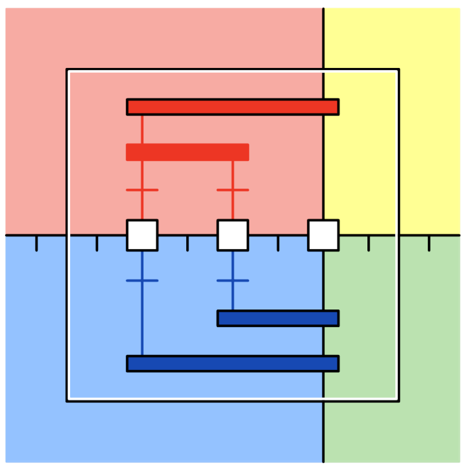
postcompose action

$\circ_g : i * \langle g\rangle \to i$

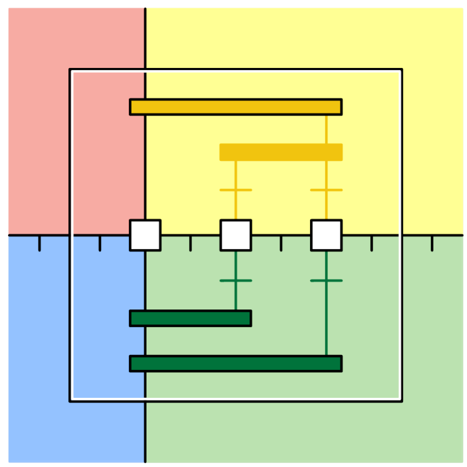
and the associators and unitors of $\mathcal{Q}$ and $\mathcal{R}$ are natural with respect to these actions. By the coherence principle, each equation can be drawn as a single string diagram.



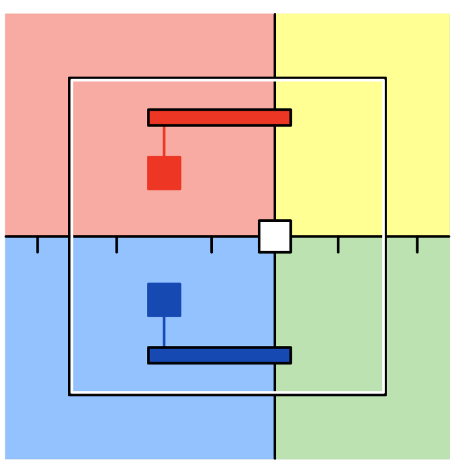
center associator coherence



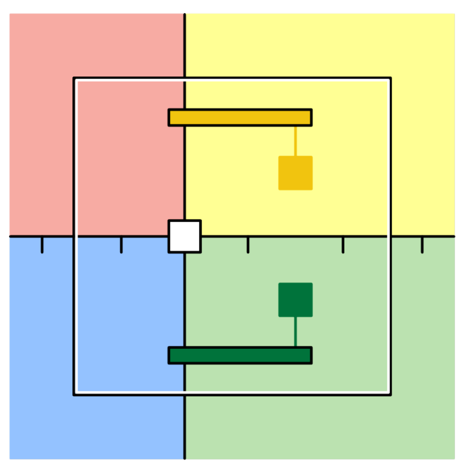
left assoc. coherence



right assoc. coherence



left unit coherence



right unit coherence

We summarize the concept of matrix profunctor, ordered by dimension.

2.	**matrix profunctor**	a span profunctor	$i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$
3.	precompose action	a span transformation	$\langle f\rangle \odot i : \langle f\rangle * i \Rightarrow i$
	postcompose action	a span transformation	$i \odot \langle g\rangle : i * \langle g\rangle \Rightarrow i$
4.	assoc. coherence	equations	$(\bar{\mathrm{x}}_1 \odot \bar{\mathrm{x}}_2) \odot Q \rightrightarrows \bar{\mathrm{a}}_1 \odot (\bar{\mathrm{a}}_2 \odot R)$
			$\bar{\mathrm{x}} \odot (Q \odot \bar{\mathrm{y}}) \rightrightarrows (\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}$
			$Q \odot (\bar{\mathrm{y}}_1 \odot \bar{\mathrm{y}}_2) \rightrightarrows (R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2$
	unit coherence	equations	$\overline{\mathrm{id}}.\mathrm{X} \odot Q \rightrightarrows \overline{\mathrm{id}}.\mathrm{A} \odot R$
			$Q \odot \overline{\mathrm{id}}.\mathrm{Y} \rightrightarrows R \odot \overline{\mathrm{id}}.\mathrm{B}$

Note. A matrix profunctor $i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$ does not include nor entail any action of the elements of f or g on $\mathcal{Q}$ or $\mathcal{R}$. Visually, this means that in general the "bars" of f and g connecting $\mathbb{X}$ to $\mathbb{A}$ and $\mathbb{Y}$ to $\mathbb{B}$ do not bend; formally it means that the *collage* is not a bifibrant double category. It is a special property when such actions do exist.

Last, we verify a key fact about matrix profunctors which is needed for the coherence of the three-dimensional category of matrix categories [Theorem 54]. A span $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ is **exponentiable** or "powerful" if pre- and post-composition by $\mathcal{R}$ have right adjoints [18].

Theorem 38. Matrix profunctors are exponentiable.

Proof. We follow the reasoning of Street in [18]. Let $i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$ be a matrix profunctor, as defined above. This determines a displayed profunctor $i : f \times g \to \mathrm{Prof}$ with actions

$$\mathcal{Q}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg}) \quad \text{and} \quad i(\mathrm{f},\mathrm{g}) \circ \mathcal{R}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb}).$$

These actions are invertible, because $\mathcal{Q}$ and $\mathcal{R}$ are bifibrations: each $i : i(\mathrm{xf},\mathrm{yg})$ and each $i : i(\mathrm{fa},\mathrm{gb})$ factor as the following elements of $\mathcal{Q}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g})$ and $i(\mathrm{f},\mathrm{g}) \circ \mathcal{R}(\mathrm{a},\mathrm{b})$, respectively.

$$\begin{tikzcd}
X_0 \arrow[rrr, "Q", "\shortmid" marking] \arrow[d, equal] \arrow[drrr, phantom, "\cong"] & & & Y_0 \arrow[d, equal] \\
X_0 \arrow[d, "x"'] & X_0 \arrow[l, dotted] \arrow[r, "Q", "\shortmid" marking] \arrow[d, equal] & Y_0 \arrow[r, dotted] \arrow[d, equal] & Y_0 \arrow[d, "y"] \\
X_1 \arrow[d, equal] & X_0 \arrow[l, "\check{x}"'] \arrow[r, "Q", "\shortmid" marking] \arrow[d, "x"'] \arrow[ddr, phantom, "\Downarrow i"] & Y_0 \arrow[r, "\hat{y}"] \arrow[d, "y"] & Y_1 \arrow[d, equal] \\
X_1 \arrow[d, "f"'] & X_1 \arrow[l, dotted] \arrow[d, "f"'] & Y_1 \arrow[r, dotted] \arrow[d, "g"] & Y_1 \arrow[d, "g"] \\
A \arrow[d, equal] \arrow[drrr, phantom, "\cong"] & A \arrow[l, dotted] \arrow[r, "R"', "\shortmid" marking] & B \arrow[r, dotted] & B \arrow[d, equal] \\
A \arrow[rrr, "R"', "\shortmid" marking] & & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rrrr, "Q", "\shortmid" marking] \arrow[d, equal] \arrow[drrrr, phantom, "\cong"] & & & & Y \arrow[d, equal] \\
X \arrow[r, dotted] \arrow[d, "f"'] & X \arrow[rr, "Q", "\shortmid" marking] \arrow[d, "f"'] \arrow[ddrr, phantom, "\Downarrow i"] & & Y \arrow[d, "g"] & Y \arrow[l, dotted] \arrow[d, "g"] \\
A_0 \arrow[r, dotted] \arrow[d, equal] & A_0 \arrow[d, "a"'] & & B_0 \arrow[d, "b"] & B_0 \arrow[l, dotted] \arrow[d, equal] \\
A_0 \arrow[r, "\hat{a}"'] \arrow[d, "a"'] & A_1 \arrow[rr, "R"', "\shortmid" marking] \arrow[d, equal] & & B_1 \arrow[d, equal] & B_0 \arrow[l, "\check{b}"] \arrow[d, "b"] \\
A_1 \arrow[r, dotted] \arrow[d, equal] \arrow[drrrr, phantom, "\cong"] & A_1 \arrow[rr, "R"', "\shortmid" marking] & & B_1 & B_1 \arrow[l, dotted] \arrow[d, equal] \\
A_1 \arrow[rrrr, "R"', "\shortmid" marking] & & & & B_1
\end{tikzcd}$$

These inverses serve to define right adjoints to composition by $f \leftarrow i \rightarrow g$: given a span profunctor $j(f,h) : \mathcal{S}(\mathbb{X},\mathbb{Z}) \mid \mathcal{T}(\mathbb{A},\mathbb{C})$, the right extension $[i \rightarrow j](g,h) : [\mathcal{Q} \rightarrow \mathcal{S}](\mathbb{Y},\mathbb{Z}) \mid [\mathcal{R} \rightarrow \mathcal{T}](\mathbb{B},\mathbb{C})$ consists of transformations $i(-,g) \Rightarrow j(-,h)$ and actions as follows.

$$\begin{tikzcd}
\mathcal{Q}(-,Y_0) \arrow[rr, "i(-{,}yg)", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & \mathcal{R}(-,B) \arrow[d, equal] \\
\mathcal{Q}(-,Y_0) \arrow[r, "\mathcal{Q}(-{,}y)", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathcal{Q}(-,Y_1) \arrow[r, "i(-{,}g)", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathcal{R}(-,B) \arrow[d] \\
\mathcal{S}(-,Z_0) \arrow[r, "\mathcal{S}(-{,}z)"', "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & \mathcal{S}(-,Z_1) \arrow[r, "j(-{,}h)"', "\shortmid" marking] & \mathcal{T}(-,C) \arrow[d, equal] \\
\mathcal{S}(-,Z_0) \arrow[rr, "j(-{,}zh)"', "\shortmid" marking] & & \mathcal{T}(-,C)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Q}(-,Y) \arrow[rr, "i(-{,}gb)", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & \mathcal{R}(-,B_1) \arrow[d, equal] \\
\mathcal{Q}(-,Y) \arrow[r, "i(-{,}g)", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathcal{R}(-,B_0) \arrow[r, "\mathcal{R}(-{,}b)", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathcal{R}(-,B_1) \arrow[d] \\
\mathcal{S}(-,Z) \arrow[r, "j(-{,}h)"', "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & \mathcal{T}(-,C_0) \arrow[r, "\mathcal{T}(-{,}c)"', "\shortmid" marking] & \mathcal{T}(-,C_1) \arrow[d, equal] \\
\mathcal{S}(-,Z) \arrow[rr, "j(-{,}hc)"', "\shortmid" marking] & & \mathcal{T}(-,C_1)
\end{tikzcd}$$

Hence by reasoning exactly analogous to that of Street [18], matrix profunctors are exponentiable.

□

2.3.1 Matrix transformation

Just as a matrix profunctor is a bimodule of weave profunctors, which coheres with the associators of its source and target matrix categories, a matrix transformation is a homomorphism of these bimodules, which coheres with the joins of the source and target matrix functors.

Definition 39. Let $[\![\mathbb{X}]\!] : \mathbb{X}_0 \rightarrow \mathbb{X}_1$, $[\![\mathbb{Y}]\!] : \mathbb{Y}_0 \rightarrow \mathbb{Y}_1$, $[\![\mathbb{A}]\!] : \mathbb{A}_0 \rightarrow \mathbb{A}_1$, $[\![\mathbb{B}]\!] : \mathbb{B}_0 \rightarrow \mathbb{B}_1$ be functors, $f_0 : \mathbb{X}_0 \mid \mathbb{A}_0$, $f_1 : \mathbb{X}_1 \mid \mathbb{A}_1$, $g_0 : \mathbb{Y}_0 \mid \mathbb{B}_0$, $g_1 : \mathbb{Y}_1 \mid \mathbb{B}_1$ profunctors, and $[\![f]\!](\mathbb{X},\mathbb{A}) : f_0 \Rightarrow f_1$, $[\![g]\!](\mathbb{Y},\mathbb{B}) : g_0 \Rightarrow$

g_1 transformations.

Let $\mathcal{Q}_0 : \mathbb{X}_0 \,\|\, \mathbb{Y}_0$, $\mathcal{Q}_1 : \mathbb{X}_1 \,\|\, \mathbb{Y}_1$, $\mathcal{R}_0 : \mathbb{A}_0 \,\|\, \mathbb{B}_0$, $\mathcal{R}_1 : \mathbb{A}_1 \,\|\, \mathbb{B}_1$ be matrix categories, and $\llbracket \mathcal{Q} \rrbracket : \mathcal{Q}_0 \to \mathcal{Q}_1$, $\llbracket \mathcal{R} \rrbracket : \mathcal{R}_0 \to \mathcal{R}_1$ be matrix functors. Let $i_0(f_0, g_0) : \mathcal{Q}_0 \mid \mathcal{R}_0$ and $i_1(f_1, g_1) : \mathcal{Q}_1 \mid \mathcal{R}_1$ be matrix profunctors.

A **matrix transformation** $\llbracket i \rrbracket(f, g) : i_0 \to i_1$ is a span transformation

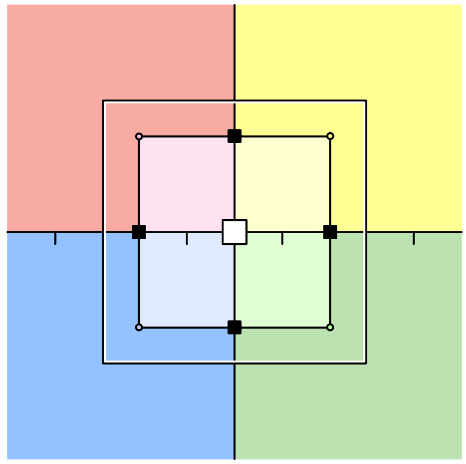
$$\begin{tikzcd}
f_0 \arrow[d, "\llbracket f \rrbracket"] & i_0 \arrow[l] \arrow[r] \arrow[d, "\llbracket i \rrbracket"] & g_0 \arrow[d, "\llbracket g \rrbracket"] \\
f_1 & i_1 \arrow[l] \arrow[r] & g_1
\end{tikzcd}$$

which coheres with the left and right joins of $\llbracket \mathcal{Q} \rrbracket$ and $\llbracket \mathcal{R} \rrbracket$.

$$\begin{tikzcd}
\mathcal{Q}_0 \arrow[rrr] \arrow[ddd, "i_0"'] & & & \mathcal{Q}_1 \arrow[ddd, "i_1"] \\
& \langle \mathbb{X}_0 \rangle * \mathcal{Q}_0 \arrow[ul] \arrow[r, "\langle \llbracket \mathbb{X} \rrbracket \rangle * \llbracket \mathcal{Q} \rrbracket"] \arrow[d, "\langle f_0 \rangle * i_0"'] \arrow[r, phantom, shift left=4ex, "\cong"] & \langle \mathbb{X}_1 \rangle * \mathcal{Q}_1 \arrow[ur] \arrow[d, "\langle f_1 \rangle * i_1"] & \\
& \langle \mathbb{A}_0 \rangle * \mathcal{R}_0 \arrow[dl] \arrow[r, "\langle \llbracket \mathbb{A} \rrbracket \rangle * \llbracket \mathcal{R} \rrbracket"'] \arrow[r, phantom, shift right=4ex, "\cong"] \arrow[u, phantom, shift left=6ex, "\xRightarrow{\langle \llbracket f \rrbracket \rangle * \llbracket i \rrbracket}"] & \langle \mathbb{A}_1 \rangle * \mathcal{R}_1 \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrr] \arrow[rrruu, phantom, "\xRightarrow{\llbracket i \rrbracket}"] & & & \mathcal{R}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\llbracket \mathrm{x} \rrbracket \odot \llbracket Q \rrbracket \arrow[r, "\llbracket \odot_{\mathbb{X}} \rrbracket"] \arrow[d, "\llbracket \mathrm{f} \rrbracket \odot \llbracket i \rrbracket"'] & \llbracket \mathrm{x} \odot Q \rrbracket \arrow[d, "\llbracket \mathrm{f} \odot i \rrbracket"] \\
\llbracket \mathrm{a} \rrbracket \odot \llbracket R \rrbracket \arrow[r, "\llbracket \odot_{\mathbb{A}} \rrbracket"'] & \llbracket \mathrm{a} \odot R \rrbracket
\end{tikzcd}$$

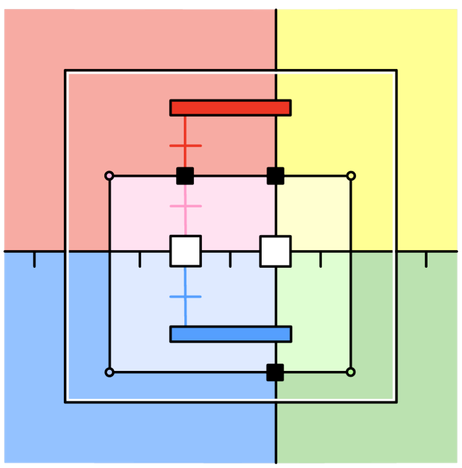
$$\begin{tikzcd}
\mathcal{Q}_0 \arrow[rrr, "\llbracket \mathcal{Q} \rrbracket"] \arrow[ddd, "i_0"'] & & & \mathcal{Q}_1 \arrow[ddd, "i_1"] \\
& \mathcal{Q}_0 * \langle \mathbb{Y}_0 \rangle \arrow[ul] \arrow[r, "\llbracket \mathcal{Q} \rrbracket * \langle \llbracket \mathbb{Y} \rrbracket \rangle"] \arrow[d, "i_0 * \langle g_0 \rangle"'] \arrow[r, phantom, shift left=4ex, "\cong"] & \mathcal{Q}_1 * \langle \mathbb{Y}_1 \rangle \arrow[ur] \arrow[d, "i_1 * \langle g_1 \rangle"] & \\
& \mathcal{R}_0 * \langle \mathbb{B}_0 \rangle \arrow[dl] \arrow[r, "\llbracket \mathcal{R} \rrbracket * \langle \llbracket \mathbb{B} \rrbracket \rangle"'] \arrow[r, phantom, shift right=4ex, "\cong"] \arrow[u, phantom, shift left=6ex, "\xRightarrow{\llbracket i \rrbracket * \langle \llbracket g \rrbracket \rangle}"] & \mathcal{R}_1 * \langle \mathbb{B}_1 \rangle \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrr, "\llbracket \mathcal{R} \rrbracket"'] \arrow[rrruu, phantom, "\xRightarrow{\llbracket i \rrbracket}"] & & & \mathcal{R}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\llbracket Q \rrbracket \odot \llbracket \mathrm{y} \rrbracket \arrow[r, "\llbracket \odot_{\mathbb{Y}} \rrbracket"] \arrow[d, "\llbracket \mathrm{g} \rrbracket \odot \llbracket i \rrbracket"'] & \llbracket Q \odot \mathrm{y} \rrbracket \arrow[d, "\llbracket i \odot \mathrm{g} \rrbracket"] \\
\llbracket R \rrbracket \odot \llbracket \mathrm{b} \rrbracket \arrow[r, "\llbracket \odot_{\mathbb{B}} \rrbracket"'] & \llbracket R \odot \mathrm{b} \rrbracket
\end{tikzcd}$$

In string diagrams, a matrix transformation is drawn as:

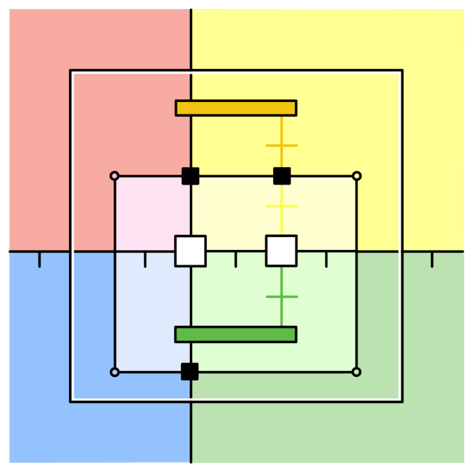


matrix transformation

and the coherence with the joins of $\overline{Q}$ and $\overline{R}$ is drawn as follows.



left join coherence



right join coherence

We summarize the concept of matrix transformation.

3.	**matrix transformation**	a span transformation	$[\![i]\!]([\![f]\!],[\![g]\!]) : i_0(f_0,g_0) \Rightarrow i_1(f_1,g_1)$
4.	left join coherence	equation	$[\![\mathrm{x}]\!] \odot [\![Q]\!] \rightrightarrows [\![\mathrm{a} \odot R]\!]$
	right join cohreence	equation	$[\![Q]\!] \odot [\![\mathrm{y}]\!] \rightrightarrows [\![R \odot \mathrm{b}]\!]$

2.4 $\mathrm{Mat}\mathbb{C}\mathrm{at}$ over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$

Matrix categories and matrix functors, matrix profunctors and matrix transformations form $\mathrm{Mat}\mathbb{C}\mathrm{at}$, a bifibrant double category which is fibered over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$.

Definition 40. Define MatCat to be the category of matrix categories and matrix functors. Composition of matrix functors is defined by that of span functors, and that of joins; one can verify this satisfies the necessary coherence, and that matrix functor composition is associative and unital.

Definition 41. Define $\mathrm{MatProf}$ to be the category of matrix profunctors and matrix transformations. Composition is defined by that of span transformations, and the coherence of the composite follows from that of its factors. $\mathrm{MatProf}$ is equipped with projections to MatCat, giving a span of categories.

$$\mathrm{MatCat} \longleftarrow \mathrm{MatProf} \longrightarrow \mathrm{MatCat}$$

Theorem 42. $\mathrm{MatProf}$ is fibered over $\mathrm{MatCat} \times \mathrm{MatCat}$.

Proof. Let $\mathcal{Q}_0(\mathbb{X}_0, \mathbb{Y}_0)$, $\mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$, $\mathcal{Q}_1(\mathbb{X}_1, \mathbb{Y}_1)$ and $\mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ be matrix categories.

Let $\llbracket\mathcal{Q}\rrbracket(\llbracket\mathbb{X}\rrbracket, \llbracket\mathbb{Y}\rrbracket) : \mathcal{Q}_0 \to \mathcal{Q}_1$ and $\llbracket\mathcal{R}\rrbracket(\llbracket\mathbb{A}\rrbracket, \llbracket\mathbb{B}\rrbracket) : \mathcal{R}_0 \to \mathcal{R}_1$ be matrix functors.

The **matrix functor substitution** matrix profunctor $i_1(f_1, g_1)(\llbracket\mathcal{Q}\rrbracket, \llbracket\mathcal{R}\rrbracket) : \mathcal{Q}_0(\mathbb{X}_0, \mathbb{Y}_0) \mid \mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$ is defined by substituting functors into profunctors: $f_1(\llbracket\mathbb{X}\rrbracket, \llbracket\mathbb{A}\rrbracket), i_1(\llbracket\mathcal{Q}\rrbracket, \llbracket\mathcal{Q}\rrbracket), g_1(\llbracket\mathbb{Y}\rrbracket, \llbracket\mathbb{B}\rrbracket)$.

$$\begin{tikzcd}
\mathbb{X}_1 \arrow[dddd, "f_1" description] & & \mathcal{Q}_1 \arrow[ll] \arrow[rr] & & \mathbb{Y}_1 \arrow[dddd, "g_1" description] \\
& \mathbb{X}_0 \arrow[ul] \arrow[dd, "f_1(\overline{\mathbb{X}},\overline{\mathbb{A}})" description] & \mathcal{Q}_0 \arrow[l] \arrow[r] \arrow[u] \arrow[dd, "{i_1}^{f_1}_{g_1}(\overline{\mathcal{Q}},\overline{\mathcal{R}})" description] & \mathbb{Y}_0 \arrow[ur] \arrow[dd, "g_1(\overline{\mathbb{Y}},\overline{\mathbb{B}})" description] & \\
{} \arrow[r, phantom, "\Leftarrow cart ="] & {} \arrow[r, phantom, "\Longleftarrow"] & {} \arrow[r, phantom, "\Longrightarrow"] & {} \arrow[r, phantom, "= cart \Rightarrow"] & {} \\
& \mathbb{A}_0 \arrow[dl] & \mathcal{R}_0 \arrow[l] \arrow[r] \arrow[d] & \mathbb{B}_0 \arrow[dr] & \\
\mathbb{A}_1 & & \mathcal{R}_1 \arrow[ll] \arrow[rr] & & \mathbb{B}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Q}_0 \arrow[r, "{i_1}^{f_1}_{g_1}(\overline{\mathcal{Q}},\overline{\mathcal{R}})"] \arrow[d, "\overline{\mathcal{Q}}" description] \arrow[dr, phantom, "\Downarrow cart"] & \mathcal{R}_0 \arrow[d, "\overline{\mathcal{R}}"] \\
\mathcal{Q}_1 \arrow[r, "i_1"'] & \mathcal{R}_1
\end{tikzcd}$$

Hence it consists of elements

$${i_1}^{f_1}_{g_1}(\llbracket\mathcal{Q}\rrbracket, \llbracket\mathcal{R}\rrbracket)(\mathrm{f}_1, \mathrm{g}_1)(Q_0, R_0) = i_1(\mathrm{f}_1, \mathrm{g}_1)(\llbracket Q_0\rrbracket, \llbracket R_0\rrbracket)$$

which can be understood as squares of the following form.

$$\begin{tikzcd}
{[\![\mathrm{X}_0]\!]} \arrow[r, "{[\![Q_0]\!]}", "\shortmid"{marking}] \arrow[d, "\mathrm{f}_1"'] \arrow[dr, phantom, "\overset{i_1}{\Downarrow}"] & {[\![\mathrm{Y}_0]\!]} \arrow[d, "\mathrm{g}_1"] \\
{[\![\mathrm{A}_0]\!]} \arrow[r, "{[\![R_0]\!]}"', "\shortmid"{marking}] & {[\![\mathrm{B}_0]\!]}
\end{tikzcd}$$

The substitution ${i_1}^{f_1}_{g_1}(\overline{\mathcal{Q}}, \overline{\mathcal{R}})$ is a matrix profunctor, because it is a restriction of the matrix profunctor i_1; its actions by the arrow profunctors of f_1 and g_1 are inherited, as well as their coherence. It is equipped with a cartesian morphism to i_1, by universal property of pullback.

Hence $\mathrm{MatProf}$ is fibered over $\mathrm{MatCat} \times \mathrm{MatCat}$. $\square$

Now to complete the double category, we need only to define horizontal composition: *sequential* matrix profunctor composition, in the direction of profunctors, as opposed to span composition.

To compose matrix profunctors m over f and n over g, we have to define an action by $\langle f \circ g\rangle$. We can use the actions of m and n, because squares of $\langle f \circ g\rangle$ are composites in $\langle f\rangle \circ \langle g\rangle$, as follows.

A square of $\langle f \circ g\rangle$ from $\hat{\mathrm{x}} : \langle\mathbb{X}\rangle(\mathrm{X}_0, \mathrm{X}_1)$ to $\hat{\mathrm{z}} : \langle\mathbb{Z}\rangle(\mathrm{Z}_0, \mathrm{Z}_1)$ is a pair of elements of $f \circ g$ so that $(\mathrm{f}_0, \mathrm{g}_0 \cdot \mathrm{z}) = (\mathrm{x} \cdot \mathrm{f}_1, \mathrm{g}_1)$. By the definition of equality in $f \circ g$, this means there is a zig-zag of arrows $\hat{\mathrm{y}} : \overrightarrow{\mathbb{Y}}(\mathrm{Y}_0, \mathrm{Y}_1)$ or oparrows $\check{\mathrm{y}} : \overleftarrow{\mathbb{Y}}(\mathrm{Y}_0, \mathrm{Y}_1)$ so that each square commutes.

$$\begin{tikzcd}
\mathrm{X}_0 \arrow[r, "\hat{\mathrm{x}}"] \arrow[d, "\mathrm{f}_0"'] & \mathrm{X}_1 \arrow[d, "\mathrm{f}_1"] \\
\mathrm{Y}_0 \arrow[r, "\hat{\mathrm{y}}"] \arrow[d, "\mathrm{g}_0"'] & \mathrm{Y}_1 \arrow[d, "\mathrm{g}_1"] \\
\mathrm{Z}_0 \arrow[r, "\hat{\mathrm{z}}"'] & \mathrm{Z}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X}_0 \arrow[r, "\hat{\mathrm{x}}"] \arrow[d, "\mathrm{f}_0"'] & \mathrm{X}_1 \arrow[d, "\mathrm{f}_1"] \\
\mathrm{Y}_0 \arrow[d, "\mathrm{g}_0"'] & \mathrm{Y}_1 \arrow[l, "\check{\mathrm{y}}"'] \arrow[d, "\mathrm{g}_1"] \\
\mathrm{Z}_0 \arrow[r, "\hat{\mathrm{z}}"'] & \mathrm{Z}_1
\end{tikzcd}$$

Such a square equals the following sequential composite of a weave in f and a weave in g.

$$
\begin{tikzcd}
\mathrm{X}_0 \arrow[r, "\hat{\mathrm{x}}"] \arrow[d, "\mathrm{f}_0"] & \mathrm{X}_1 \arrow[d, "\mathrm{f}_1"] \\
\mathrm{Y}_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{y}}_k \rangle"] \arrow[d, "\mathrm{g}_0"] & \mathrm{Y}_k \arrow[d, "\mathrm{g}_1"] \\
\mathrm{Z}_0 \arrow[r, "\hat{\mathrm{z}}"'] & \mathrm{Z}_1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{X}_0 \arrow[rrr, "\hat{\mathrm{x}}"] \arrow[d, equal] & {} \arrow[dr, phantom, "\cong"] & & \mathrm{X}_1 \arrow[d, equal] \\
\mathrm{X}_0 \arrow[r, dotted] \arrow[dd, "\mathrm{f}_0"] & \mathrm{X}_0 \arrow[r, dotted, leftrightarrow, "\langle \overline{\mathrm{id}}.\mathrm{X} \rangle"] \arrow[dd, "\mathrm{f}_0"] & \mathrm{X}_0 \arrow[r, "\hat{\mathrm{x}}"] \arrow[d, dashed, "\mathrm{x}"] & \mathrm{X}_1 \arrow[d, equal] \\
& & \mathrm{X}_1 \arrow[r, dotted] \arrow[d, dashed, "\mathrm{f}_1"] & \mathrm{X}_1 \arrow[d, dashed, "\mathrm{f}_1"] \\
\mathrm{Y}_0 \arrow[r, dotted] \arrow[d, dashed, "\mathrm{g}_0"] & \mathrm{Y}_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{y}}_k \rangle"] \arrow[d, dashed, "\mathrm{g}_0"] & \mathrm{Y}_k \arrow[r, dotted] \arrow[dd, "\mathrm{g}_1"] & \mathrm{Y}_k \arrow[dd, "\mathrm{g}_1"] \\
\mathrm{Z}_0 \arrow[r, dotted] \arrow[d, equal] & \mathrm{Z}_0 \arrow[d, dashed, "\mathrm{z}"] & & \\
\mathrm{Z}_0 \arrow[r, "\hat{\mathrm{z}}"'] \arrow[d, equal] & \mathrm{Z}_1 \arrow[r, dotted, leftrightarrow, "\langle \overline{\mathrm{id}}.\mathrm{Z} \rangle"'] \arrow[dr, phantom, "\cong"] & \mathrm{Z}_1 \arrow[r, dotted] & \mathrm{Z}_1 \arrow[d, equal] \\
\mathrm{Z}_0 \arrow[rrr, "\hat{\mathrm{z}}"'] & & & \mathrm{Z}_1
\end{tikzcd}
$$

So a square $\langle \bar{\mathrm{y}}_k \rangle : \langle f \circ g \rangle((\mathrm{f}_0, \mathrm{g}_0), (\mathrm{f}_1, \mathrm{g}_1))(\hat{\mathrm{x}}, \hat{\mathrm{z}})$ factors as the sequential composite of the following.

$$
\begin{aligned}
\upsilon(\hat{\mathrm{x}}) \cdot (\mathrm{f}_0, \mathrm{f}_0, \ldots, \mathrm{x} \cdot \mathrm{f}_1, \mathrm{f}_1) \quad &: \quad \langle f \rangle(\mathrm{f}_0, \mathrm{f}_1)(\hat{\mathrm{x}}, \langle \bar{\mathrm{y}}_k \rangle) \\
(\mathrm{g}_0, \mathrm{g}_0 \cdot \mathrm{z}, \ldots, \mathrm{g}_1, \mathrm{g}_1) \cdot \upsilon(\hat{\mathrm{z}}) \quad &: \quad \langle g \rangle(\mathrm{g}_0, \mathrm{g}_1)(\langle \bar{\mathrm{y}}_k \rangle, \hat{\mathrm{z}})
\end{aligned}
$$

An opsquare $\langle \bar{\mathrm{y}}_k \rangle : \langle f \circ g \rangle((\mathrm{f}_0, \mathrm{g}_0), (\mathrm{f}_1, \mathrm{g}_1))(\check{\mathrm{x}}, \check{\mathrm{z}})$ factors as the sequential composite of the following.

$$
\begin{aligned}
\upsilon(\check{\mathrm{x}}) \cdot (\mathrm{f}_0, \mathrm{x} \cdot \mathrm{f}_0, \ldots, \mathrm{f}_1, \mathrm{f}_1) \quad &: \quad \langle f \rangle(\mathrm{f}_0, \mathrm{f}_1)(\check{\mathrm{x}}, \langle \bar{\mathrm{y}}_k \rangle) \\
(\mathrm{g}_0, \mathrm{g}_0, \ldots, \mathrm{g}_1 \cdot \mathrm{z}, \mathrm{g}_1) \cdot \upsilon(\check{\mathrm{z}}) \quad &: \quad \langle g \rangle(\mathrm{g}_0, \mathrm{g}_1)(\langle \bar{\mathrm{y}}_k \rangle, \check{\mathrm{z}}).
\end{aligned}
$$

In general, a weave in $f \circ g$ is a composite of these squares and opsquares with weaves in $\mathbb{X}$ and $\mathbb{Z}$. For any weave $w : \langle f \circ g \rangle$, denote by $w(f) : \langle f \rangle$ the weave in f obtained by factoring each square and opsquare as above; similarly denote the factor of g by $w(g) : \langle g \rangle$.

These provide concise notation for defining the actions of $\langle f \circ g \rangle$.

This ensures the totality of the actions; so in fact, the crux of sequential composition is to ensure that the actions are *well-defined* over the *identities*. Recall from 2.1.2 we noted that the associativity quotient $(\mathrm{f}, \mathrm{y} \cdot \mathrm{g}) \equiv (\mathrm{f} \cdot \mathrm{y}, \mathrm{g})$ defines the identity squares of $\langle f \circ g \rangle$.

Elements of $f \circ g$ and $k \circ \ell$ are determined only up to associativity, and distinct zig-zags give distinct actions; so to compose matrix profunctors $m(f, k) : \mathcal{R}(\mathbb{X}, \mathbb{A}) \mid \mathcal{S}(\mathbb{Y}, \mathbb{B})$ and $n(g, \ell) : \mathcal{S}(\mathbb{Y}, \mathbb{B}) \mid \mathcal{T}(\mathbb{Z}, \mathbb{C})$, we need to quotient $m \circ n$ by the actions of these identity squares in $\langle f \circ g \rangle$ and $\langle k \circ \ell \rangle$.

Definition 43. Let $m(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,\ell) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$ be matrix profunctors.

The **sequential composite** matrix profunctor $(m \diamond n)(f \circ g, k \circ \ell) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$ is defined to be the following coequalizer.

$$\begin{tikzcd}
\mathcal{R} \arrow[dr, equal] \arrow[drrr, equal] \arrow[dddd, "m \circ n" description, "+" marking] & & & & \\
& \mathcal{R} \arrow[r, "\mathrm{id} * \mathcal{R} * \mathrm{id}"] \arrow[dd, "\mathrm{id}.mn.\mathrm{id}" description, "+" marking] \arrow[u, phantom, "\cong"] & \langle\mathbb{X}\rangle * \mathcal{R} * \langle\mathbb{A}\rangle \arrow[r, "\odot"] \arrow[d, "\langle f\rangle * m * \langle k\rangle", "+" marking] \arrow[dr, phantom, "= \odot \Rightarrow"] & \mathcal{R} \arrow[r, equal] \arrow[d, "m" description, "+" marking] & \mathcal{R} \arrow[dd, "+" marking] \\
\arrow[r, phantom, "\Leftarrow"] & {} \arrow[r, phantom, "= cart \Rightarrow"] & \langle\mathbb{Y}\rangle * \mathcal{S} * \langle\mathbb{B}\rangle \arrow[r, "\odot"] \arrow[d, "\langle g\rangle * n * \langle \ell\rangle", "+" marking] \arrow[dr, phantom, "= \odot \Rightarrow"] & \mathcal{S} \arrow[d, "n" description, "+" marking] \arrow[r, dashed, "coeq"] & m \diamond n \\
& \mathcal{T} \arrow[r, "\mathrm{id} * \mathcal{T} * \mathrm{id}"'] \arrow[d, phantom, "\cong"] & \langle\mathbb{Z}\rangle * \mathcal{T} * \langle\mathbb{C}\rangle \arrow[r, "\odot"] & \mathcal{T} \arrow[r, equal] & \mathcal{T} \\
\mathcal{T} \arrow[ur, equal] \arrow[urrr, equal] & & & &
\end{tikzcd}$$

Hence elements are equivalence classes $[S.(m,n)] : m \circ n$, such that for each pair of zig-zags, and each pair of pairs of weaves, the following are equated.

$$[S.(m,n)] \;\equiv\; [\upsilon_{\mathcal{R}} \cdot (\langle \bar{\mathrm{y}}_i \rangle \odot S \odot \langle \bar{\mathrm{b}}_j \rangle).(w_f \odot m \odot w_k,\; w_g \odot n \odot w_\ell) \cdot \upsilon_{\mathcal{T}}^{-1}]$$

$$\begin{tikzcd}
\mathrm{X} \arrow[rrr, "R", "+" marking] \arrow[d, equal] & & & \mathrm{A} \arrow[d, equal] \\
\mathrm{X} \arrow[r, dotted, no head] \arrow[d, "\mathrm{f}_0"] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{w_f}}"] & \mathrm{X} \arrow[r, "R", "+" marking] \arrow[d, "\mathrm{f}_1"] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{m}}"] \arrow[u, phantom, "\cong"] & \mathrm{A} \arrow[r, dotted, no head] \arrow[d, "\mathrm{k}_0"] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{w_k}}"] & \mathrm{A} \arrow[d, "\mathrm{k}_1"] \\
\mathrm{Y}_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{y}}_i \rangle"] \arrow[d, "\mathrm{g}_0"] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{w_g}}"] & \mathrm{Y}_1 \arrow[r, "S", "+" marking] \arrow[d, "\mathrm{g}_1"] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{n}}"] & \mathrm{B}_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{b}}_j \rangle"] \arrow[d, "\mathrm{l}_0"] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{w_\ell}}"] & \mathrm{B}_1 \arrow[d, "\mathrm{l}_1"] \\
\mathrm{Z} \arrow[r, dotted, no head] \arrow[d, equal] & \mathrm{Z} \arrow[r, "T", "+" marking] \arrow[d, phantom, "\cong"] & \mathrm{C} \arrow[r, dotted, no head] & \mathrm{C} \arrow[d, equal] \\
\mathrm{Z} \arrow[rrr, "T"', "+" marking] & & & \mathrm{C}
\end{tikzcd}$$

This is a span profunctor $f \circ g \leftarrow m \diamond n \rightarrow k \circ \ell$ mapping each $[S.(m,n)]$ to $[\mathrm{Y}_1.(\mathrm{f}_1,\mathrm{g}_1)]$ and $[\mathrm{B}_0.(\mathrm{k}_0,\mathrm{l}_0)]$; this is well-defined because any other representative lies over equivalent pairs $\mathrm{Y}_0.(\mathrm{f}_0,\mathrm{g}_0)$ and $\mathrm{B}_1.(\mathrm{k}_1,\mathrm{l}_1)$.

Moreover, $m \diamond n$ is a matrix profunctor from $f \circ g$ to $k \circ \ell$: as described above, every square and opsquare in $f \circ g$ is a composite of a weave in f and a weave in g. Because a weave in $f \circ g$ is in general a horizontal and vertical composite of squares and opsquares of $f \circ g$ composed with weaves in $\mathbb{X}$ and $\mathbb{Z}$, we define the action inductively over the structure of a composite weave. Then for the base generators, the quotient ensures that the action is well-defined.

- The action of a horizontal composite is the horizontal composite of the actions of each factor.

$$
\begin{tikzcd}
\mathrm{X}_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow f_1"] & \mathrm{X}_1 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow f_2"] & \mathrm{X}_2 \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow m"] & \mathrm{A} \arrow[d] \\
\mathrm{Y}_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow g_1"] & \mathrm{Y}_1 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow g_2"] & \mathrm{Y}_2 \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow n"] & \mathrm{B} \arrow[d] \\
\mathrm{Z}_0 \arrow[r, leftrightarrow] & \mathrm{Z}_1 \arrow[r, leftrightarrow] & \mathrm{Z}_2 \arrow[r, "|" marking] & \mathrm{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{X}_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow f_1"] & \mathrm{X}_1 \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow f_2 \odot m"] & \mathrm{A} \arrow[d] \\
\mathrm{Y}_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow g_1"] & \mathrm{Y}_1 \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow g_2 \odot n"] & \mathrm{B} \arrow[d] \\
\mathrm{Z}_0 \arrow[r, leftrightarrow] & \mathrm{Z}_1 \arrow[r, "|" marking] & \mathrm{C}
\end{tikzcd}
$$

- The action of a vertical composite of weaves in $\mathbb{X}$ and $\mathbb{Z}$ with a weave in $f \circ g$ is the vertical composite of the actions of the following factorization by op/cartesian squares.

$$
\begin{tikzcd}
\mathrm{X}^0_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow x"] & \mathrm{X}^0_1 \arrow[r, "|" marking] \arrow[d] \arrow[ddr, phantom, "\Downarrow m"] & \mathrm{A} \arrow[dd] \\
\mathrm{X}^1_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow f"] & \mathrm{X}^1_1 \arrow[d] & \\
\mathrm{Y}_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow g"] & \mathrm{Y}_1 \arrow[r, "|" marking] \arrow[d] \arrow[ddr, phantom, "\Downarrow n"] & \mathrm{B} \arrow[dd] \\
\mathrm{Z}^0_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow z"] & \mathrm{Z}^0_1 \arrow[d] & \\
\mathrm{Z}^1_0 \arrow[r, leftrightarrow] & \mathrm{Z}^1_1 \arrow[r, "|" marking] & \mathrm{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{X}^0_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow x"] & \mathrm{X}^0_1 \arrow[r, dotted, no head] \arrow[d] \arrow[dr, phantom, "\eta"] & \mathrm{X}^0_1 \arrow[r, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "R"] & \mathrm{A} \arrow[d, equal] \\
\mathrm{X}^1_0 \arrow[r, leftrightarrow] \arrow[d, equal] & \mathrm{X}^1_1 \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \mathrm{X}^0_1 \arrow[l] \arrow[r, "|" marking] \arrow[d] \arrow[ddr, phantom, "\Downarrow m"] & \mathrm{A} \arrow[dd] \\
\mathrm{X}^1_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow f"] & \mathrm{X}^1_1 \arrow[r, dotted, no head] \arrow[d] & \mathrm{X}^1_1 \arrow[d] & \\
\mathrm{Y}_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow g"] & \mathrm{Y}_1 \arrow[r, dotted, no head] \arrow[d] & \mathrm{Y}_1 \arrow[r, "|" marking] \arrow[d] \arrow[ddr, phantom, "\Downarrow n"] & \mathrm{B} \arrow[dd] \\
\mathrm{Z}^0_0 \arrow[r, leftrightarrow] \arrow[d, equal] & \mathrm{Z}^0_1 \arrow[r, dotted, no head] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & \mathrm{Z}^0_1 \arrow[d] & \\
\mathrm{Z}^0_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\Downarrow z"] & \mathrm{Z}^0_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\varepsilon"] & \mathrm{Z}^1_1 \arrow[r, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "T"] & \mathrm{C} \arrow[d, equal] \\
\mathrm{Z}^1_0 \arrow[r, leftrightarrow] & \mathrm{Z}^1_1 \arrow[r, dotted, no head] & \mathrm{Z}^1_1 \arrow[r, "|" marking] & \mathrm{C}
\end{tikzcd}
$$

- The action by a square or opsquare is the action of its factorization into a weave in f and a weave in g, on m and n respectively. The case of a square is given as follows, and an opsquare dually.

$$
\begin{tikzcd}
\mathrm{X}_0 \arrow[r, "\hat{\mathrm{x}}"] \arrow[dd, "\mathrm{f}_0"] & \mathrm{X}_1 \arrow[r, "R", "\shortmid"{marking}] \arrow[dd, "\mathrm{f}_1"] & \mathrm{A} \arrow[dd, "\mathrm{k}"] \\
& \arrow[u, phantom, "\Downarrow m"] & \\
\mathrm{Y}_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{y}}_i \rangle"] \arrow[dd, "\mathrm{g}_0"] & \mathrm{Y}_i \arrow[r, "S", "\shortmid"{marking}] \arrow[dd, "\mathrm{g}_1"] & \mathrm{B} \arrow[dd, "\mathrm{l}"] \\
& \arrow[d, phantom, "\Downarrow n"] & \\
\mathrm{Z}_0 \arrow[r, "\hat{\mathrm{z}}"'] & \mathrm{Z}_1 \arrow[r, "T"', "\shortmid"{marking}] & \mathrm{C}
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
\mathrm{X}_0 \arrow[rrr, "\hat{\mathrm{x}}"] \arrow[d, equal] & \arrow[d, phantom, "\cong"] & & \mathrm{X}_1 \arrow[rr, "R", "\shortmid"{marking}] \arrow[d, equal] & \arrow[ddd, Rightarrow, "m"] & \mathrm{A} \arrow[ddd, "\mathrm{k}"] \\
\mathrm{X}_0 \arrow[r, dotted] \arrow[dd, "\mathrm{f}_0"] & \mathrm{X}_0 \arrow[r, dotted] \arrow[dd, "\mathrm{f}_0"] & \mathrm{X}_0 \arrow[r, "\hat{\mathrm{x}}"] \arrow[d, "\mathrm{x}"] \arrow[dr, phantom, "\varepsilon"] & \mathrm{X}_1 \arrow[d, equal] & & \\
& & \mathrm{X}_1 \arrow[r, dotted] \arrow[d, "\mathrm{f}_1"] & \mathrm{X}_1 \arrow[d, "\mathrm{f}_1"] & & \\
\mathrm{Y}_0 \arrow[r, dotted] \arrow[d, "\mathrm{g}_0"] & \mathrm{Y}_0 \arrow[r, leftrightarrow, "\langle \bar{\mathrm{y}}_i \rangle"] \arrow[d, "\mathrm{g}_0"] & \mathrm{Y}_i \arrow[r, dotted] \arrow[dd, "\mathrm{g}_1"] & \mathrm{Y}_i \arrow[rr, "S", "\shortmid"{marking}] \arrow[dd, "\mathrm{g}_1"] & \arrow[ddd, Rightarrow, "n"] & \mathrm{B} \arrow[ddd, "\mathrm{l}"] \\
\mathrm{Z}_0 \arrow[r, dotted] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & \mathrm{Z}_0 \arrow[d, "\mathrm{z}"] & & & & \\
\mathrm{Z}_0 \arrow[r, "\hat{\mathrm{z}}"] \arrow[d, equal] & \mathrm{Z}_1 \arrow[r, dotted] \arrow[d, phantom, "\cong"] & \mathrm{Z}_1 \arrow[r, dotted] & \mathrm{Z}_1 \arrow[d, equal] & & \\
\mathrm{Z}_0 \arrow[rrr, "\hat{\mathrm{z}}"'] & & & \mathrm{Z}_1 \arrow[rr, "T"', "\shortmid"{marking}] & & \mathrm{C}
\end{tikzcd}
$$

This action is well-defined by the quotient. Because squares and opsquares are the base generators of weaves, this completes the induction. Hence the actions by $\langle f \circ g \rangle$ and $\langle k \circ \ell \rangle$ are well-defined.

Last, because the actions are defined componentwise, the coherence of $m \diamond n$ with the associators and unitors of $\mathcal{R}$ and $\mathcal{T}$ follows from that of m with $\mathcal{R}$ and $\mathcal{S}$ and that of n with $\mathcal{S}$ and $\mathcal{T}$.

$$
\begin{tikzcd}
\bar{\mathrm{x}} \odot (R \odot \bar{\mathrm{a}}) \arrow[r, "\alpha_{\mathcal{R}}"] \arrow[d, "\bar{\mathrm{y}}_{\mathrm{f}} \odot (m \odot \bar{\mathrm{b}}_{\mathrm{k}})"] & (\bar{\mathrm{x}} \odot R) \odot \bar{\mathrm{a}} \arrow[d, "(\bar{\mathrm{y}}_{\mathrm{f}} \odot m) \odot \bar{\mathrm{b}}_{\mathrm{k}}"] \\
\bar{\mathrm{y}} \odot (S \odot \bar{\mathrm{b}}) \arrow[r, "\alpha_{\mathcal{S}}"] \arrow[d, "\bar{\mathrm{y}}_{\mathrm{g}} \odot (n \odot \bar{\mathrm{b}}_{\mathrm{l}})"] & (\bar{\mathrm{y}} \odot S) \odot \bar{\mathrm{b}} \arrow[d, "(\bar{\mathrm{y}}_{\mathrm{g}} \odot n) \odot \bar{\mathrm{b}}\mathrm{l}"] \\
\bar{\mathrm{z}} \odot (T \odot \bar{\mathrm{c}}) \arrow[r, "\alpha_{\mathcal{T}}"] & (\bar{\mathrm{z}} \odot T) \odot \bar{\mathrm{c}}
\end{tikzcd}
$$

Hence the sequential composite $(m \diamond n)(f \circ g, k \circ \ell) : \mathcal{R}(\mathbb{X}, \mathbb{A}) \,|\, \mathcal{T}(\mathbb{Z}, \mathbb{C})$ is a matrix profunctor.

Theorem 44. Matrix categories and matrix functors, matrix profunctors and matrix transformations form a bifibrant double category, i.e. logic, which we call $\mathrm{Mat}\mathbb{C}\mathrm{at}$.

Proof. Because matrix profunctor composition is defined by coequalizer, it is canonically functorial. Let $[\![m]\!]([\![f]\!],[\![k]\!]) : m_0(f_0,k_0) \Rightarrow m_1(f_1,k_1)$ and $[\![n]\!]([\![g]\!],[\![\ell]\!]) : n_0(g_0,\ell_0) \Rightarrow n_1(g_1,\ell_1)$ be a sequential-composable pair of matrix transformations. The composite is defined as follows.

$$\begin{aligned} ([\![m]\!] \diamond [\![n]\!]) : \quad (m_0 \diamond n_0)(f_0 \circ g_0, k_0 \circ \ell_0) &\Rightarrow (m_1 \diamond n_1)(f_1 \circ g_1, k_1 \circ \ell_1) \\ [S_0.(m_0,n_0)] &\mapsto [[\![S_0]\!].([\![m_0]\!],[\![n_0]\!])] \end{aligned}$$

To be a matrix transformation, this composite must cohere with the left and right joins of the matrix functors $\llbracket\mathcal{R}\rrbracket(\llbracket\mathbb{X}\rrbracket, \llbracket\mathbb{A}\rrbracket)$ and $\mathcal{T}(\llbracket\mathbb{Z}\rrbracket, \llbracket\mathbb{C}\rrbracket)$; yet just as for matrix profunctors, this follows from the coherence of $\llbracket m\rrbracket$ with respect to $\llbracket\mathcal{R}\rrbracket$ and $\llbracket\mathcal{S}\rrbracket$ and that of $\llbracket n\rrbracket$ with respect to $\llbracket\mathcal{S}\rrbracket$ and $\llbracket\mathcal{T}\rrbracket$.

$$
\begin{tikzcd}
\llbracket\bar{\mathrm{x}}\rrbracket \odot \llbracket R\rrbracket \arrow[r, "\llbracket\odot_{\mathbb{X}}\rrbracket"] \arrow[d, "\llbracket\mathrm{f}\rrbracket\odot\llbracket m\rrbracket"] & \llbracket\bar{\mathrm{x}} \odot R\rrbracket \arrow[d, "\llbracket\mathrm{f}\odot m\rrbracket"] \\
\llbracket\bar{\mathrm{y}}\rrbracket \odot \llbracket S\rrbracket \arrow[r, "\llbracket\odot_{\mathbb{Y}}\rrbracket"] \arrow[d, "\llbracket\mathrm{g}\rrbracket\odot\llbracket n\rrbracket"] & \llbracket\bar{\mathrm{y}} \odot S\rrbracket \arrow[d, "\llbracket\mathrm{g}\odot n\rrbracket"] \\
\llbracket\bar{\mathrm{z}}\rrbracket \odot \llbracket T\rrbracket \arrow[r, "\llbracket\odot_{\mathbb{Z}}\rrbracket"] & \llbracket\bar{\mathrm{z}} \odot T\rrbracket
\end{tikzcd}
\qquad
\begin{tikzcd}
\llbracket R\rrbracket \odot \llbracket\bar{\mathrm{a}}\rrbracket \arrow[r, "\llbracket\odot_{\mathbb{A}}\rrbracket"] \arrow[d, "\llbracket m\rrbracket\odot\llbracket\mathrm{k}\rrbracket"] & \llbracket R \odot \bar{\mathrm{a}}\rrbracket \arrow[d, "\llbracket m\odot\mathrm{k}\rrbracket"] \\
\llbracket S\rrbracket \odot \llbracket\bar{\mathrm{b}}\rrbracket \arrow[r, "\llbracket\odot_{\mathbb{B}}\rrbracket"] \arrow[d, "\llbracket n\rrbracket\odot\llbracket\mathrm{l}\rrbracket"] & \llbracket S \odot \bar{\mathrm{b}}\rrbracket \arrow[d, "\llbracket n\odot\mathrm{l}\rrbracket"] \\
\llbracket T\rrbracket \odot \llbracket\bar{\mathrm{c}}\rrbracket \arrow[r, "\llbracket\odot_{\mathbb{C}}\rrbracket"] & \llbracket T \odot \bar{\mathrm{c}}\rrbracket
\end{tikzcd}
$$

This preserves composition of matrix transformations, by canonical functoriality of coequalizer.

The associator and unitors of $\mathrm{Mat}\mathbb{C}\mathrm{at}$ are inherited from $\mathrm{Span}\mathbb{C}\mathrm{at}$: the span transformations

$$
\begin{array}{ccc}
(m \diamond n) \diamond p & \cong & m \diamond (n \diamond p) \\
[((m, n), p)] & \mapsto & [(m, (n, p))]
\end{array}
\qquad
\begin{array}{ccc}
m & \cong & \mathcal{R} \diamond m \\
m & \mapsto & [(\mathrm{id}.R, m)] \\
m & \cong & m \diamond \mathcal{S} \\
m & \mapsto & [(m, \mathrm{id}.S)]
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{R} \diamond m & \cong & m \\
[(r, m)] & \mapsto & r \cdot m \\
m \diamond \mathcal{S} & \cong & m \\
[(m, s)] & \mapsto & m \cdot s
\end{array}
$$

are matrix transformations, and they are well-defined on equivalence classes in the sequential composite because the quotient only reindexes along the base pair of morphisms.

Hence $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a double category. □

We now define *substitution* of functors in matrix categories, and transformations in matrix profunctors; hence MatCat is fibered over $\mathrm{Cat} \times \mathrm{Cat}$, and $\mathrm{MatProf}$ is fibered over $\mathrm{Prof} \times \mathrm{Prof}$.

$$
\begin{tikzcd}
\mathrm{MatCat} \arrow[d] & \mathrm{MatProf} \arrow[l] \arrow[r] \arrow[d] & \mathrm{MatCat} \arrow[d] \\
\mathrm{Cat} \times \mathrm{Cat} & \mathrm{Prof} \times \mathrm{Prof} \arrow[l] \arrow[r] & \mathrm{Cat} \times \mathrm{Cat}
\end{tikzcd}
$$

Definition 45. A **double fibration** is a category in the 2-category of fibrations. See [4].

Proposition 46. Let Cat be the category of categories and functors, and let MatCat be the category of matrix categories and matrix functors. The projection $\mathrm{MatCat} \to \mathrm{Cat} \times \mathrm{Cat}$ is a fibration.

Proof. Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1, [\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors, and let $\mathcal{R}_1 : \mathbb{A}_1 \,\|\, \mathbb{B}_1$ be a matrix category. We define the **substitution** matrix category $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) : \mathbb{A}_0 \,\|\, \mathbb{B}_0$ as follows.

1. The span category $\mathbb{A}_0 \leftarrow \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) \to \mathbb{B}_0$ is the pullback of $\mathcal{R}_1$ along the functors $[\![\mathbb{A}]\!], [\![\mathbb{B}]\!]$. So the category over $\mathrm{A}_0 : \mathbb{A}_0, \mathrm{B}_0 : \mathbb{B}_0$ is $\mathcal{R}_1([\![\mathrm{A}_0]\!], [\![\mathrm{B}_0]\!])$, and similarly for morphisms.

$$\begin{tikzcd}
\mathbb{A}_0 \arrow[d, "{[\![\mathbb{A}]\!]}"] & \mathcal{R}_1([\![\mathbb{A}_0]\!], [\![\mathbb{B}_0]\!]) \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{B}_0 \arrow[d, "{[\![\mathbb{B}]\!]}"] \\
\mathbb{A}_1 & \mathcal{R}_1 \arrow[l] \arrow[r] & \mathbb{B}_1
\end{tikzcd}$$

Hence $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])(\mathrm{a}_0, \mathrm{b}_0)(R_1^0, R_1^1)$ consists of squares $r_1 : \mathcal{R}_1$ over $([\![\mathrm{a}_0]\!], [\![\mathrm{b}_0]\!])$.

$$\begin{tikzcd}
{[\![\mathrm{A}_0^0]\!]} \arrow[r, "R_1^0", "\shortmid"{marking}] \arrow[d, "{[\![\mathrm{a}_0]\!]}"'] \arrow[dr, phantom, "\Downarrow r_1"] & {[\![\mathrm{B}_0^0]\!]} \arrow[d, "{[\![\mathrm{b}_0]\!]}"] \\
{[\![\mathrm{A}_0^1]\!]} \arrow[r, "R_1^1"', "\shortmid"{marking}] & {[\![\mathrm{B}_0^1]\!]}
\end{tikzcd}$$

2. The actions of $\mathbb{A}_0$ and $\mathbb{B}_0$ on $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$, span functors

$$\begin{array}{rcccl}
\langle\mathbb{A}_0\rangle \odot - & : & \langle\mathbb{A}_0\rangle * \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) & \to & \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) \\
- \odot \langle\mathbb{B}_0\rangle & : & \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) * \langle\mathbb{B}_0\rangle & \to & \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])
\end{array}$$

are those induced by pullback: map the arrow or oparrow by the functor, and then act on $\mathcal{R}_1$.

$$\begin{array}{cccc}
\bar{\mathrm{a}}_0 : \langle\mathbb{A}_0\rangle(\mathrm{A}_0^0, \mathrm{A}_0^1) & R_1 : \mathcal{R}_1([\![\mathrm{A}_0^1]\!], [\![\mathrm{B}_0^0]\!]) & \mapsto & [\![\bar{\mathrm{a}}_0]\!] \odot R_1 \ : \ R_1([\![\mathrm{A}_0^0]\!], [\![\mathrm{B}_0^0]\!]) \\
R_1 : \mathcal{R}_1([\![\mathrm{A}_0^1]\!], [\![\mathrm{B}_0^0]\!]) & \bar{\mathrm{b}}_0 : \langle\mathbb{B}_0\rangle(\mathrm{B}_0^0, \mathrm{B}_0^1) & \mapsto & R_1 \odot [\![\bar{\mathrm{b}}_0]\!] \ : \ \mathcal{R}_1([\![\mathrm{A}_0^1]\!], [\![\mathrm{B}_0^1]\!])
\end{array}$$

$$\begin{tikzcd}
{[\![\mathrm{A}_0^0]\!]} \arrow[r, leftrightarrow, "{[\![\bar{\mathrm{a}}_0]\!]}"] & {[\![\mathrm{A}_0^1]\!]} \arrow[r, "R_1", "\shortmid"{marking}] & {[\![\mathrm{B}_0^0]\!]} \arrow[r, leftrightarrow, "{[\![\bar{\mathrm{b}}_0]\!]}"] & {[\![\mathrm{B}_0^1]\!]}
\end{tikzcd}$$

3,4. The associators and unitors are inherited from $\mathcal{R}_1$, satisfying the necessary coherence.

The substitution matrix category $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$ is equipped with a projection matrix functor to $\mathcal{R}_1$, and this is a cartesian morphism over functors $[\![\mathbb{A}]\!], [\![\mathbb{B}]\!]$, by universal property of pullback. □

In the same way, we define substitution of transformations in a matrix profunctor by pullback.

Theorem 47. $\mathrm{MatProf} \to \mathrm{Prof} \times \mathrm{Prof}$ is a fibration.

Proof. Let $[\![\mathbb{X}]\!] : \mathbb{X}_0 \to \mathbb{X}_1$, $[\![\mathbb{Y}]\!] : \mathbb{Y}_0 \to \mathbb{Y}_1$, $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$, $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors, and let $\mathcal{Q}_1 : \mathbb{X}_1 \parallel \mathbb{Y}_1$ and $\mathcal{R}_1 : \mathbb{A}_1 \parallel \mathbb{B}_1$ be matrix categories, determining substitutions $\mathcal{Q}_1([\![\mathbb{X}]\!], [\![\mathbb{Y}]\!]) : \mathbb{X}_0 \parallel \mathbb{Y}_0$ and $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) : \mathbb{A}_0 \parallel \mathbb{B}_0$.

Let $f_0 : \mathbb{X}_0 \mid \mathbb{A}_0$, $f_1 : \mathbb{X}_1 \mid \mathbb{A}_1$, $g_0 : \mathbb{Y}_0 \mid \mathbb{B}_0$, $g_1 : \mathbb{Y}_1 \mid \mathbb{B}_1$ be profunctors, and $[\![f]\!] : f_0 \Rightarrow f_1$ and $[\![g]\!] : g_0 \Rightarrow g_1$ be transformations. For a matrix profunctor $i_1(f_1, g_1) : \mathcal{Q}_1 \mid \mathcal{R}_1$, define the **substitution** matrix profunctor $i_1([\![f]\!], [\![g]\!]) : \mathcal{Q}_1([\![\mathbb{X}]\!], [\![\mathbb{Y}]\!]) \mid \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$ from f_0 to g_0 as follows.

2. The span profunctor $f_0 \leftarrow i_1([\![f]\!], [\![g]\!]) \to g_0$ is the pullback of i_1 along the transformations $[\![f]\!], [\![g]\!]$. This means that the profunctor over $\mathrm{f}_0 : f_0(\mathrm{X}_0, \mathrm{A}_0), \mathrm{g}_0 : g_0(\mathrm{Y}_0, \mathrm{B}_0)$ is $i_1([\![\mathrm{f}_0]\!], [\![\mathrm{g}_0]\!]) : \mathcal{Q}_1([\![\mathrm{X}_0]\!], [\![\mathrm{Y}_0]\!]) \mid \mathcal{R}_1([\![\mathrm{A}_0]\!], [\![$

$$\begin{tikzcd}
f_0 \arrow[d, "{[\![f]\!]}"] & i_1([\![f_0]\!], [\![g_0]\!]) \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & g_0 \arrow[d, "{[\![g]\!]}"] \\
f_1 & i_1 \arrow[l] \arrow[r] & g_1
\end{tikzcd}$$

Hence it consists of squares of the following form.

$$\begin{tikzcd}
{[\![\mathrm{X}_0]\!]} \arrow[r, "Q_1"] \arrow[d, "{[\![\mathrm{f}_0]\!]}"'] \arrow[dr, phantom, "\Downarrow i_1"] & {[\![\mathrm{Y}_0]\!]} \arrow[d, "{[\![\mathrm{g}_0]\!]}"] \\
{[\![\mathrm{A}_0]\!]} \arrow[r, "R_1"'] & {[\![\mathrm{B}_0]\!]}
\end{tikzcd}$$

3. The actions by the weave profunctors $\langle f_0 \rangle$ and $\langle g_0 \rangle$ are those induced by pullback.

$$\begin{tikzcd}
{[\![\mathrm{X}_0^0]\!]} \arrow[r, leftrightarrow, "{[\![\bar{\mathrm{x}}_0]\!]}"] \arrow[d, "{[\![\mathrm{f}_0^0]\!]}"'] & {[\![\mathrm{X}_0^1]\!]} \arrow[r, "Q_1"] \arrow[d, "{[\![\mathrm{f}_0^1]\!]}"'] \arrow[dr, phantom, "\Downarrow i_1"] & {[\![\mathrm{Y}_0^0]\!]} \arrow[r, leftrightarrow, "{[\![\bar{\mathrm{y}}_0]\!]}"] \arrow[d, "{[\![\mathrm{g}_0^0]\!]}"] & {[\![\mathrm{Y}_0^1]\!]} \arrow[d, "{[\![\mathrm{g}_0^1]\!]}"] \\
{[\![\mathrm{A}_0^0]\!]} \arrow[r, leftrightarrow, "{[\![\bar{\mathrm{a}}_0]\!]}"'] & {[\![\mathrm{A}_0^1]\!]} \arrow[r, "R_1"'] & {[\![\mathrm{B}_0^0]\!]} \arrow[r, leftrightarrow, "{[\![\bar{\mathrm{b}}_0]\!]}"'] & {[\![\mathrm{B}_0^1]\!]}
\end{tikzcd}$$

4. Because the associators and unitors of $\mathcal{Q}_1([\![\mathbb{X}]\!], [\![\mathbb{Y}]\!])$ and $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$ are inherited from $\mathcal{Q}_1$ and $\mathcal{R}_1$, their coherence with $i_1([\![f]\!], [\![g]\!])$ is inherited from that of $\mathcal{Q}_1$ and $\mathcal{R}_1$ with i_1.

$\square$

Theorem 48. $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ is a double fibration.

Proof. We show that matrix profunctor composition preserves substitution.

Let $m_i(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n_i(g,\ell) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$, for $i : \{0,1\}$, be matrix profunctors. Let $[\![m]\!] : m_0 \Rightarrow m_1$ and $[\![n]\!] : n_0 \Rightarrow n_1$ be matrix transformations, and form the substitution.

$$\begin{tikzcd}
\mathbb{X}_1 \arrow[dd, "f_1"'] & & \mathcal{R}_1 \arrow[ll] \arrow[rr] & & \mathbb{A}_1 \arrow[dd, "k_1"] \\
& \mathbb{X}_0 \arrow[ul] \arrow[d, "f_1([\![\mathbb{X}]\!],[\![\mathbb{Y}]\!])"'] \arrow[l, phantom, "\Leftarrow cart"] & \mathcal{R}_1([\![\mathbb{X}]\!],[\![\mathbb{A}]\!]) \arrow[l] \arrow[r] \arrow[u] \arrow[d, "m_1([\![f]\!],[\![k]\!])"] \arrow[l, phantom, "\Longleftarrow", shift right=4] \arrow[r, phantom, "\Longrightarrow", shift right=4] & \mathbb{A}_0 \arrow[ur] \arrow[d, "k_1([\![\mathbb{A}]\!],[\![\mathbb{B}]\!])"] \arrow[r, phantom, "cart \Rightarrow"] & \\
\mathbb{Y}_1 \arrow[dd, "g_1"'] & \mathbb{Y}_0 \arrow[l] \arrow[d, "g_1([\![\mathbb{Y}]\!],[\![\mathbb{Z}]\!])"'] \arrow[l, phantom, "\Leftarrow cart", shift right=6] & \mathcal{S}_1([\![\mathbb{Y}]\!],[\![\mathbb{B}]\!]) \arrow[l] \arrow[r] \arrow[d, "n_1([\![g]\!],[\![\ell]\!])"] \arrow[l, phantom, "\Longleftarrow", shift right=6] \arrow[r, phantom, "\Longrightarrow", shift right=6] & \mathbb{B}_0 \arrow[r] \arrow[d, "\ell_1([\![\mathbb{B}]\!],[\![\mathbb{C}]\!])"] \arrow[r, phantom, "cart \Rightarrow", shift right=6] & \mathbb{B}_1 \arrow[dd, "\ell_1"] \\
& \mathbb{Z}_0 \arrow[dl] & \mathcal{T}_1([\![\mathbb{Z}]\!],[\![\mathbb{C}]\!]) \arrow[l] \arrow[r] \arrow[d] & \mathbb{C}_0 \arrow[dr] & \\
\mathbb{Z}_1 & & \mathcal{T}_1 \arrow[ll] \arrow[rr] & & \mathbb{C}_1
\end{tikzcd}$$

The composite $m_1([\![f]\!],[\![k]\!]) \diamond n_1([\![g]\!],[\![\ell]\!])$ consists of equivalence classes $[S_1.(m_1,n_1)]$ over $[([\![\mathrm{f}_0]\!],[\![\mathrm{g}_0]\!])]$ and $[([\![\mathrm{k}_0]\!],[\![\mathrm{l}_0]\!])]$. By comparison, the substitution $(m_1 \diamond n_1)([\![f]\!] \circ [\![g]\!], [\![k]\!] \circ [\![\ell]\!])$ consists of equivalence classes $[S_1.(m_1,n_1)]$ over pairs $[(\mathrm{f}_1,\mathrm{g}_1)]$ and $[(\mathrm{k}_1,\mathrm{l}_1)]$ which are equal to pairs $[([\![\mathrm{f}_0]\!],[\![\mathrm{g}_0]\!])]$ and $[([\![\mathrm{k}_0]\!],[\![\mathrm{l}_0]\!])]$ by associativity.

$$\begin{tikzcd}
{[\![\mathrm{X}_0]\!]} \arrow[r, equal] \arrow[d, "{[\![\mathrm{f}_0]\!]}"] & {[\![\mathrm{X}_0]\!]} \arrow[r, "R_1", "\shortmid"{marking}] \arrow[d, "\mathrm{f}_1"] \arrow[dr, phantom, "\overset{m_1}{\Downarrow}"] & {[\![\mathrm{A}_0]\!]} \arrow[r, equal] \arrow[d, "\mathrm{k}_1"] & {[\![\mathrm{A}_0]\!]} \arrow[d, "{[\![\mathrm{k}_0]\!]}"] \\
{[\![\mathrm{Y}_0]\!]} \arrow[r, leftrightarrow] \arrow[d, "{[\![\mathrm{g}_0]\!]}"] & \mathrm{Y}_1 \arrow[r, "S_1", "\shortmid"{marking}] \arrow[d, "\mathrm{g}_1"] \arrow[dr, phantom, "\overset{n_1}{\Downarrow}"] & \mathrm{B}_1 \arrow[r, leftrightarrow] \arrow[d, "\mathrm{l}_1"] & {[\![\mathrm{B}_0]\!]} \arrow[d, "{[\![\mathrm{l}_0]\!]}"] \\
{[\![\mathrm{Z}_0]\!]} \arrow[r, equal] & {[\![\mathrm{Z}_0]\!]} \arrow[r, "T_1", "\shortmid"{marking}] & {[\![\mathrm{C}_0]\!]} \arrow[r, equal] & {[\![\mathrm{C}_0]\!]}
\end{tikzcd}$$

Hence the two are isomorphic.

$$m_1([\![f]\!],[\![g]\!]) \diamond n_1([\![k]\!],[\![\ell]\!]) \cong (m_1 \diamond n_1)([\![f]\!] \circ [\![g]\!], [\![k]\!] \circ [\![\ell]\!])$$

Thus, sequential composition of matrix profunctors preserves substitution of transformations. This means that $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a weak category in the 2-category of fibered categories, i.e. a *fibered double category*.

$$\begin{tikzcd}
\mathrm{MatCat} \arrow[d] & \mathrm{MatProf} \arrow[l] \arrow[r] \arrow[d] & \mathrm{MatCat} \arrow[d] \\
\mathrm{Cat} \times \mathrm{Cat} & \mathrm{Prof} \times \mathrm{Prof} \arrow[l] \arrow[r] & \mathrm{Cat} \times \mathrm{Cat}
\end{tikzcd}$$

□

As $\mathbb{C}\mathrm{at}$ and $\mathrm{Mat}\mathbb{C}\mathrm{at}$ are bifibrant double categories, we call this structure a **fibered logic**.

2.5 Parallel composition [Codescent]

We now define composition of matrix categories: $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \rightarrow \mathbb{C}\mathrm{at}$ is a *metalogic* [Def. 54].

Matrix categories compose in essentially the same way as profunctors; but rather than a coequalizer, the composite is a *codescent object* [20, Sec. 4]: this adjoins to $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B} \leftarrow \mathcal{S} \rightarrow \mathbb{C}$ a coherent associator of the inner actions of $\langle\mathbb{B}\rangle$.

Definition 49. Let $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \,\|\, \mathbb{C}$ be matrix categories. The **composite** matrix category $\mathcal{R} \otimes \mathcal{S} : \mathbb{A} \,\|\, \mathbb{C}$ is defined as follows. To the composite span category $\mathbb{A} \leftarrow \mathcal{R} * \mathcal{S} \rightarrow \mathbb{C}$, an associator isomorphism is adjoined, by forming the *iso-coinserter* of the inner actions by $\langle\mathbb{B}\rangle$.

$$\begin{tikzcd}
(\mathcal{R} * \langle\mathbb{B}\rangle) * \mathcal{S} \arrow[r, "\odot * \mathcal{S}"] \arrow[dd, "\cong"'] \arrow[ddr, phantom, "\Downarrow \alpha_{\mathcal{R}\mathcal{S}}"] & \mathcal{R} * \mathcal{S} \arrow[dr, squiggly, "\iota"] & \\
 & & (\mathcal{R} * \mathcal{S})_\alpha \\
\mathcal{R} * (\langle\mathbb{B}\rangle * \mathcal{S}) \arrow[r, "\mathcal{R} * \odot"'] & \mathcal{R} * \mathcal{S} \arrow[ur, squiggly, "\iota"'] &
\end{tikzcd}$$

This associator is natural by its universal construction, so for every weave $w_{\mathrm{B}} : \langle\mathbb{B}\rangle(\langle\bar{\mathrm{b}}_k\rangle, \langle\bar{\mathrm{b}}_\ell\rangle)$ and $r : \mathcal{R}(R_0, R_1)$, $s : \mathcal{S}(S_0, S_1)$ the following commutes.

$$\begin{tikzcd}
(R_0, \langle\bar{\mathrm{b}}_k\rangle \odot S_0) \arrow[r, "\alpha_{\mathcal{R}\mathcal{S}}"] \arrow[d, "(r, w_{\mathrm{B}} \odot s)"'] & (R_0 \odot \langle\bar{\mathrm{b}}_k\rangle, S_0) \arrow[d, "(r \odot w_{\mathrm{B}}, s)"] \\
(R_1, \langle\bar{\mathrm{b}}_\ell\rangle \odot S_1) \arrow[r, "\alpha_{\mathcal{R}\mathcal{S}}"] & (R_1 \odot \langle\bar{\mathrm{b}}_\ell\rangle, S_1)
\end{tikzcd}$$

On the associator, two equations are imposed by *coequifier*, for reassociating a composite and a unit.

$$\begin{tikzcd}
\mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle * \mathcal{S} \arrow[r, shift left=2, "B_0.(R, \bar{\mathrm{b}}_1 \odot (\bar{\mathrm{b}}_2 \odot S))"] \arrow[r, shift right=2, "B_2.((R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2, S)"'] \arrow[r, phantom, "\Downarrow\;\Downarrow"] & (\mathcal{R} * \mathcal{S})_\alpha \arrow[r, squiggly, "co.equif"] & (\mathcal{R} * \mathcal{S})_\beta
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{R} * \mathcal{S} \arrow[r, shift left=2, "\mathrm{B}.(R, \overline{\mathrm{id}}.\mathrm{B} \odot S)"] \arrow[r, shift right=2, "\mathrm{B}.(R \odot \overline{\mathrm{id}}.\mathrm{B}, S)"'] \arrow[r, phantom, "\Downarrow\;\Downarrow"] & (\mathcal{R} * \mathcal{S})_\beta \arrow[r, squiggly, "co.equif"] & \mathcal{R} \otimes \mathcal{S}
\end{tikzcd}$$

All together, the parallel composite matrix category $\mathcal{R} \otimes \mathcal{S} : \mathbb{A} \,\|\, \mathbb{C}$ is the following codescent object.

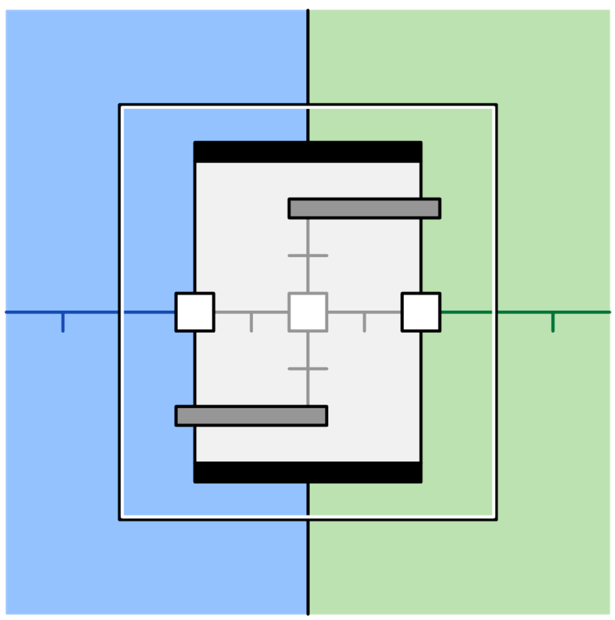
$$\mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle * \mathcal{S} \;\substack{\xrightarrow{\odot * \langle\mathbb{B}\rangle * \mathcal{S}} \\ \xrightarrow{\mathcal{R} * \circ * \mathcal{S}} \\ \xrightarrow[\mathcal{R} * \langle\mathbb{B}\rangle * \odot]{}}\; \mathcal{R} * \langle\mathbb{B}\rangle * \mathcal{S} \;\substack{\xrightarrow{\odot * \mathcal{S}} \\ \xleftarrow{\mathcal{R} * \mathrm{id} * \mathcal{S}} \\ \xrightarrow[\mathcal{R} * \odot]{}}\; \mathcal{R} * \mathcal{S} \overset{co.desc}{\rightsquigarrow} \mathcal{R} \otimes \mathcal{S}$$

We denote the codescent object by the following "arrow sum" notation, dual to 2.2.1.

$$(\mathcal{R} \otimes \mathcal{S})(\mathrm{A}, \mathrm{C}) \quad \equiv \quad \vec{\Sigma}\mathrm{B} : \mathbb{B}.\ \mathcal{R}(\mathrm{A}, \mathrm{B}) \times \mathcal{S}(\mathrm{B}, \mathrm{C})$$

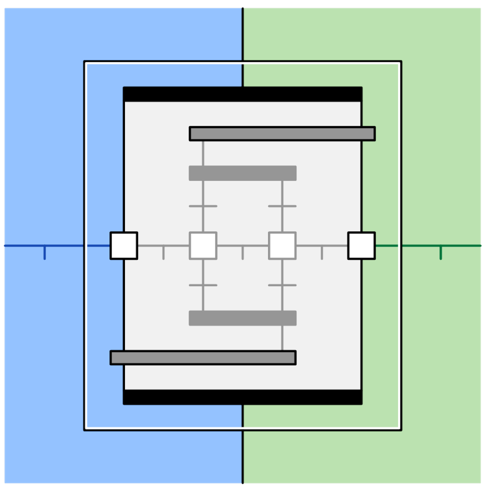
So, the parallel composite $\mathcal{R} \otimes \mathcal{S} : \mathbb{A} \,\|\, \mathbb{C}$ consists of pairs $\mathrm{b}.(r, s) : \mathrm{B}_0.(R_0, S_0) \to \mathrm{B}_1.(R_1, S_1)$, plus a coherent associator $\alpha_{\mathcal{RS}} : \mathrm{B}_0.(R, \bar{\mathrm{b}} \odot S) \cong \mathrm{B}_1.(R \odot \bar{\mathrm{b}}, S)$.

The iso-coinserter which constructs the associator is drawn in string diagrams as follows: the black bead is the colimiting span functor from $(\mathcal{R} * \mathcal{S})$ to $(\mathcal{R} * \mathcal{S})_\alpha$, and the inner face is the associator isomorphism.



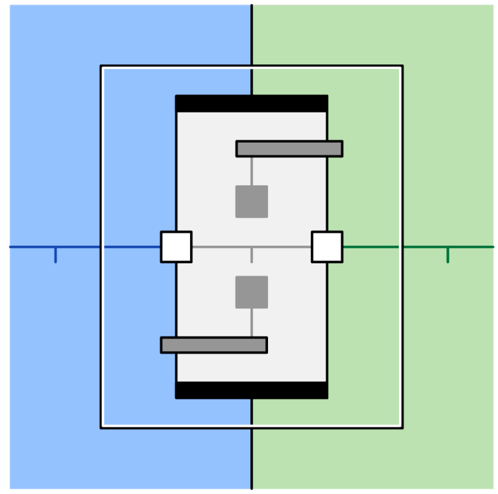
$\alpha_{\mathcal{RS}} : \mathrm{B}_0.(R, \bar{\mathrm{b}} \odot S) \;\cong\; \mathrm{B}_1.(R \odot \bar{\mathrm{b}}, S)$

Each coequifier on the associator can be drawn as the cube which it makes well-defined.



associator coherence

$(R, \bar{\mathrm{b}}_1 \odot (\bar{\mathrm{b}}_2 \odot S)) \rightrightarrows ((R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2), S)$



unitor coherence

$(R, \bar{\mathrm{id}}.\mathrm{B} \odot S) \rightrightarrows (R \odot \bar{\mathrm{id}}.\mathrm{B}, S)$

Matrix profunctors compose similarly; we need only impose one equation, for naturality of the adjoined associators.

Definition 50. Let $m(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$ and $n(g,h) : \mathcal{S}(\mathbb{Y},\mathbb{Z}) \mid \mathcal{T}(\mathbb{B},\mathbb{C})$ be matrix profunctors.

$$\begin{tikzcd}
\mathbb{X} \arrow[dd, "f"', "\shortmid" marking] & \mathcal{Q} \arrow[l] \arrow[r] \arrow[d, no head] & \mathbb{Y} \arrow[d, no head] & \mathcal{S} \arrow[l] \arrow[r] \arrow[d, no head] & \mathbb{Z} \arrow[dd, "h", "\shortmid" marking] \\
{} & m \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d] & g \arrow[d] & n \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d] & {} \\
\mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{B} & \mathcal{T} \arrow[l] \arrow[r] & \mathbb{C}
\end{tikzcd}$$

The **composite** matrix profunctor $m \otimes n : \mathcal{Q} \otimes \mathcal{S} \mid \mathcal{R} \otimes \mathcal{T}$ is defined as the following coequalizer.

$$\begin{tikzcd}[column sep=small]
 & & (\mathcal{R} * \langle\mathbb{B}\rangle) * \mathcal{T} \arrow[rrr] \arrow[ddd] & & & \mathcal{R} * \mathcal{T} \arrow[drrr, "\iota"] & & & & & \\
 & & & & & & & & \mathcal{R} \otimes \mathcal{T} \arrow[rr, equal] & & \mathcal{R} \otimes \mathcal{T} \\
 & (m*\langle g\rangle)*n \arrow[uur] \arrow[ddl, no head] \arrow[rrrr, Rightarrow] \arrow[dddd, Rightarrow, "\cong"] & & \arrow[uu, phantom, "\cong"] & m*n \arrow[uur] \arrow[dddl, no head] \arrow[ddrrr, Rightarrow] & & & & & & \\
 & & \mathcal{R} * (\langle\mathbb{B}\rangle * \mathcal{T}) \arrow[rrr] & & & \mathcal{R} * \mathcal{T} \arrow[uurrr, "\iota"] & & & & & \\
 & & & & & & & \iota_!(m*n) \arrow[uuur] \arrow[dddl, no head] \arrow[rr, Rightarrow, "co.equ"] & & m \otimes n \arrow[uuur] \arrow[dddl, no head] & \\
(\mathcal{Q} * \langle\mathbb{Y}\rangle) * \mathcal{S} \arrow[rrr] \arrow[ddd] & & & \mathcal{Q} * \mathcal{S} \arrow[ddrrr, "\iota"] & & & & & & & \\
 & m*(\langle g\rangle*n) \arrow[uuur] \arrow[ddl, no head] \arrow[rrrr, Rightarrow] \arrow[dr, phantom, "\cong"] & & & m*n \arrow[uuur] \arrow[ddl, no head] \arrow[uurrr, Rightarrow] & & & & & & \\
 & & {} & & & & \mathcal{Q} \otimes \mathcal{S} \arrow[rr, equal] & & \mathcal{Q} \otimes \mathcal{S} & & \\
\mathcal{Q} * (\langle\mathbb{Y}\rangle * \mathcal{S}) \arrow[rrr] & & & \mathcal{Q} * \mathcal{S} \arrow[urrr, "\iota"] & & & & & & &
\end{tikzcd}$$

The profunctor $\iota_!(m * n)$ forms all composites of elements $\mathrm{g}.(m,n)$ and the morphisms of $\mathcal{Q}\otimes\mathcal{S}$ and $\mathcal{R}\otimes\mathcal{T}$. Then, the coequalizer imposes that the associators are natural with respect to the elements.

So the elements of the composite $(m\otimes n)(f,h) : (\mathcal{Q}\otimes\mathcal{S})(\mathbb{X},\mathbb{Z}) \mid (\mathcal{R}\otimes\mathcal{T})(\mathbb{A},\mathbb{C})$ are composites of:

$$\begin{array}{rrl} \text{morphisms} & \mathrm{y}.(q,s): & (\mathcal{Q}\otimes\mathcal{S})(\mathrm{Y}_0.(Q_0,S_0),\mathrm{Y}_1.(Q_1,S_1)) \\ \text{associators} & \alpha_{\mathcal{Q}\mathcal{S}}: & (\mathcal{Q}\otimes\mathcal{S})(\mathrm{Y}_0.(Q,\bar{\mathrm{y}}\odot S),\mathrm{Y}_1.(Q\odot\bar{\mathrm{y}},S)) \\ \text{elements} & \mathrm{g}.(m,n): & (m*n)(\mathrm{Y}.(Q,S),\mathrm{B}.(R,T)) \\ \text{associators} & \alpha_{\mathcal{R}\mathcal{T}}: & (\mathcal{R}\otimes\mathcal{T})(\mathrm{B}_0.(R,\bar{\mathrm{b}}\odot T),\mathrm{B}_1.(R\odot\bar{\mathrm{b}},T)) \\ \text{morphisms} & \mathrm{b}.(r,t): & (\mathcal{R}\otimes\mathcal{T})(\mathrm{B}_0.(R_0,T_0),\mathrm{B}_1.(R_1,T_1)) \end{array}$$

such that for any $[\mathrm{g}_0,\mathrm{g}_1] : \langle g\rangle(\bar{\mathrm{y}},\bar{\mathrm{b}})$ and $m : m(\mathrm{f},\mathrm{g}_0)$, $n : n(\mathrm{g}_1,\mathrm{h})$ the following commutes.

$$\begin{tikzcd} \mathrm{Y}_0.(Q,\bar{\mathrm{y}}\odot S) \arrow[r, "\alpha_{\mathcal{Q}\mathcal{S}}"] \arrow[d, "\mathrm{g}_0.(m,[\mathrm{g}_0,\mathrm{g}_1]\odot n)"'] & \mathrm{Y}_1.(Q\odot\bar{\mathrm{y}},S) \arrow[d, "\mathrm{g}_1.(m\odot[\mathrm{g}_0,\mathrm{g}_1],n)"] \\ \mathrm{B}_0.(R,\bar{\mathrm{b}}\odot T) \arrow[r, "\alpha_{\mathcal{R}\mathcal{T}}"'] & \mathrm{B}_1.(R\odot\bar{\mathrm{b}},T) \end{tikzcd}$$

We denote the composite by the same "arrow sum" notation as for matrix categories.

$$(m\otimes n)(\mathrm{f},\mathrm{h}) \quad\equiv\quad \vec{\Sigma}\mathrm{g} : g.\ m(\mathrm{f},\mathrm{g})\times n(\mathrm{g},\mathrm{h})$$

We now show that parallel composition defines a span of span functors $\mathrm{Mat}\mathbb{C}\mathrm{at} * \mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathrm{Mat}\mathbb{C}\mathrm{at}$ — but not a span of double functors.

Proposition 51. Parallel composition of matrix categories defines a span functor

$$\otimes : \mathrm{Mat}\mathbb{C}\mathrm{at} * \mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathrm{Mat}\mathbb{C}\mathrm{at}.$$

Proof. As composition is defined by colimit, it is canonically functorial. Let $[\![\mathcal{R}]\!] : \mathcal{R}_0(\mathbb{A}_0,\mathbb{B}_0) \to \mathcal{R}_1(\mathbb{A}_1,\mathbb{B}_1)$ and $[\![\mathcal{S}]\!]([\![\mathbb{B}]\!],[\![\mathbb{C}]\!]) : \mathcal{S}_0(\mathbb{B}_0,\mathbb{C}_0) \to \mathcal{S}_1(\mathbb{B}_1,\mathbb{C}_1)$ be matrix functors. The composite

$$([\![\mathcal{R}]\!]\otimes[\![\mathcal{S}]\!]) : (\mathcal{R}_0\otimes\mathcal{S}_0)(\mathbb{A}_0,\mathbb{C}_0) \to (\mathcal{R}_1\otimes\mathcal{S}_1)(\mathbb{A}_1,\mathbb{C}_1)$$

is defined by applying the functors $[\![\mathcal{R}]\!]$ and $[\![\mathcal{S}]\!]$ in parallel

$$([\![\mathcal{R}]\!] \otimes [\![\mathcal{S}]\!])(\mathrm{B}_0.(R_0, S_0)) = [\![\mathrm{B}_0]\!].([\![R_0]\!], [\![S_0]\!])$$

and mapping the "inner associator" of $\mathcal{R}_0 \otimes \mathcal{S}_0$ to that of $\mathcal{R}_1 \otimes \mathcal{S}_1$.

$$([\![\mathcal{R}]\!] \otimes [\![\mathcal{S}]\!])(\alpha(\mathrm{b}_0.(R_0, S_0))) = \alpha([\![\mathrm{b}_0]\!].([\![R_0]\!], [\![S_0]\!]))$$

The joins of this matrix functor are inherited from those of $[\![\mathcal{R}]\!]$ and $[\![\mathcal{S}]\!]$.

$$[\![\mathrm{a}_0]\!] \odot ([\![\mathrm{B}_0]\!].([\![R_0]\!], [\![S_0]\!])) \odot [\![\mathrm{c}_0]\!] \;=\; [\![\mathrm{B}_0]\!].([\![\mathrm{a}_0]\!] \odot [\![R_0]\!], [\![S_0]\!] \odot [\![\mathrm{c}_0]\!]) \;\cong\; [\![\mathrm{B}_0]\!].([\![\mathrm{a}_0 \odot R_0]\!], [\![S_0 \odot \mathrm{c}_0]\!])$$

Finally, $- \otimes -$ clearly preserves matrix functor composition and identity. Hence it defines a span functor $\mathrm{MatCat} * \mathrm{MatCat} \to \mathrm{MatCat}$. $\square$

Proposition 52. Parallel composition of matrix profunctors defines a span functor

$$\otimes : \mathrm{MatProf} * \mathrm{MatProf} \to \mathrm{MatProf}.$$

Proof. Let $m(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A}, \mathbb{B})$ and $n(g, h) : \mathcal{S}(\mathbb{Y}, \mathbb{Z}) \,|\, \mathcal{T}(\mathbb{B}, \mathbb{C})$ be matrix profunctors with subscripts $0, 1$.

Let $[\![m]\!]([\![f]\!], [\![g]\!]) : m_0(f_0, g_0) \Rightarrow m_1(f_1, g_1)$ and $[\![n]\!]([\![g]\!], [\![h]\!]) : n_0(g_0, h_0) \Rightarrow n_1(g_1, h_1)$ be matrix transformations.

$$\begin{tikzcd}
\mathbb{X}_1 \arrow[dddd, "f_1"'] & & \mathcal{Q}_1 \arrow[ll] \arrow[r] & \mathbb{Y}_1 & \mathcal{S}_1 \arrow[l] \arrow[rr] & & \mathbb{Z}_1 \arrow[dddd, "h_1"] \\
& \mathbb{X}_0 \arrow[ul] \arrow[dd, "f_0"'] & \mathcal{Q}_0 \arrow[l] \arrow[r] \arrow[u] \arrow[dd, "m_0"] & \mathbb{Y}_0 \arrow[u] \arrow[dd, "g_0"] & \mathcal{S}_0 \arrow[l] \arrow[r] \arrow[u] \arrow[dd, "n_0"] & \mathbb{Z}_0 \arrow[ur] \arrow[dd, "h_0"] & \\
\arrow[r, phantom, "\Longleftarrow"] & \arrow[r, phantom, "\Longleftarrow"] & \arrow[r, phantom, "\Longrightarrow"] & \arrow[r, phantom, "\Longleftarrow"] & \arrow[r, phantom, "\Longrightarrow"] & \arrow[r, phantom, "\Longrightarrow"] & \\
& \mathbb{A}_0 \arrow[dl] & \mathcal{R}_0 \arrow[l] \arrow[r] \arrow[d] & \mathbb{B}_0 \arrow[d] & \mathcal{T}_0 \arrow[l] \arrow[r] \arrow[d] & \mathbb{C}_0 \arrow[dr] & \\
\mathbb{A}_1 & & \mathcal{R}_1 \arrow[ll] \arrow[r] & \mathbb{B}_1 & \mathcal{T}_1 \arrow[l] \arrow[rr] & & \mathbb{C}_1
\end{tikzcd}$$

Then the composite matrix transformation

$$(\llbracket m \rrbracket \otimes \llbracket n \rrbracket) : (m_0 \otimes n_0)(f_0, h_0) \Rightarrow (m_1 \otimes n_1)(f_1, h_1)$$

is defined by applying the transformations $\llbracket m \rrbracket$ and $\llbracket n \rrbracket$ in parallel.

$$(\llbracket m \rrbracket \otimes \llbracket n \rrbracket)(\mathrm{g}_0.(\mathrm{f}_0, \mathrm{h}_0)) = \llbracket \mathrm{g}_0 \rrbracket.(\llbracket \mathrm{f}_0 \rrbracket, \llbracket \mathrm{h}_0 \rrbracket)$$

The coherence of $\llbracket m \rrbracket \otimes \llbracket n \rrbracket$ with the joins of $\llbracket \mathcal{Q} \rrbracket \otimes \llbracket \mathcal{S} \rrbracket$ and $\llbracket \mathcal{R} \rrbracket \otimes \llbracket \mathcal{T} \rrbracket$ follows from that of $\llbracket m \rrbracket$ with $\llbracket \mathcal{Q} \rrbracket$ and $\llbracket \mathcal{R} \rrbracket$, and $\llbracket n \rrbracket$ with $\llbracket \mathcal{S} \rrbracket$ and $\llbracket \mathcal{T} \rrbracket$.

Finally, $- \otimes -$ clearly preserves matrix transformation composition and identity. Hence it defines a span functor $\mathrm{MatProf} * \mathrm{MatProf} \to \mathrm{MatProf}$. □

We have defined parallel composition of matrix categories, and matrix profunctors.

Now: is parallel composition a *double functor*? The answer is in fact *no*: parallel composition does not preserve sequential composition of matrix profunctors — in fact, it is neither lax nor colax.

$$(i \otimes m) \diamond (j \otimes n) \qquad \not\leftrightarrow \qquad (i \diamond j) \otimes (m \diamond n)$$

The reason has to do with the combination of *strict* and *weak* colimits: weak-to-strict (lax, left-to-right above) is not total, while strict-to-weak (colax, right-to-left above) is not well-defined.

Sequential composition is given by coequalizer, while parallel composition is given by codescent object. The former *equates* elements, while the latter *creates* an associator isomorphism.

So the sequence-of-parallel composite $(i \otimes m) \diamond (j \otimes n)$ contains composites with associators, which cannot be expressed as a parallel-of-sequence composite $(i \diamond j) \otimes (m \diamond n)$.

$$
\begin{tikzcd}
\mathrm{U} \arrow[rr, "Q"] \arrow[d, "\mathrm{d}"] \arrow[drr, phantom, "\Downarrow i"] & & \mathrm{X} \arrow[rr, "R"] \arrow[d, "\mathrm{f}"] \arrow[drr, phantom, "\Downarrow m"] & & \mathrm{A} \arrow[d, "\mathrm{k}"] \\
\mathrm{V} \arrow[r, "P"'] \arrow[d, equal] & \mathrm{Y}_0 \arrow[r, "\mathrm{y}"'] & \mathrm{Y}_1 \arrow[rr, "S"'] \arrow[d, phantom, "\cong"] & & \mathrm{B} \arrow[d, equal] \\
\mathrm{V} \arrow[rr, "P"] \arrow[d, "\mathrm{e}"] \arrow[drr, phantom, "\Downarrow j"] & & \mathrm{Y}_0 \arrow[r, "\mathrm{y}"] \arrow[d, "\mathrm{g}"] \arrow[drr, phantom, "\Downarrow n"] & \mathrm{Y}_1 \arrow[r, "S"] & \mathrm{B} \arrow[d, "\mathrm{l}"] \\
\mathrm{W} \arrow[rr, "Q"'] & & \mathrm{Z} \arrow[rr, "T"'] & & \mathrm{C}
\end{tikzcd}
$$

Hence there is no transformation $(i \otimes m) \diamond (j \otimes n) \Rightarrow (i \diamond j) \otimes (m \diamond n)$.

Yet in the other direction, there is a dual obstruction. To define sequential composition, each associativity zig-zag $(\bar{\mathrm{y}}_i) : (\mathrm{f}_0, \mathrm{g}_0) = (\mathrm{f}_1, \mathrm{g}_1)$ in $f \diamond g$ is given by squares in $\langle f \rangle$ and $\langle g \rangle$; yet elements of $(i \diamond j) \otimes (m \diamond n)$ are "parallel-composable pairs" along an equality $(\mathrm{f}_0, \mathrm{g}_0) = (\mathrm{f}_1, \mathrm{g}_1)$, without a specific choice of zig-zag.

$$
\begin{tikzcd}
\mathrm{U} \arrow[r, "Q"] \arrow[d, "\mathrm{d}"] \arrow[dr, phantom, "\Downarrow i"] & \mathrm{X} \arrow[r, equal] \arrow[d, "\mathrm{f}_0"] & \mathrm{X} \arrow[r, "R"] \arrow[d, "\mathrm{f}_1"] \arrow[dr, phantom, "\Downarrow m"] & \mathrm{A} \arrow[d, "\mathrm{k}"] \\
\mathrm{V} \arrow[r, "P"] \arrow[d, "\mathrm{e}"] \arrow[dr, phantom, "\Downarrow j"] & \mathrm{Y}_0 \arrow[d, "\mathrm{g}_0"] & \mathrm{Y}_1 \arrow[r, "S"] \arrow[d, "\mathrm{g}_1"] \arrow[dr, phantom, "\Downarrow n"] & \mathrm{B} \arrow[d, "\mathrm{l}"] \\
\mathrm{W} \arrow[r, "Q"] & \mathrm{Z} \arrow[r, equal] & \mathrm{Z} \arrow[r, "T"] & \mathrm{C}
\end{tikzcd}
$$

So a transformation $(i \diamond j) \otimes (m \diamond n) \Rightarrow (i \otimes m) \diamond (j \diamond n)$ would have to be independent of the choice of zig-zag. Yet there is no canonical choice; there are many distinct zig-zags which reassociate from $(\mathrm{f}_0, \mathrm{g}_0)$ to $(\mathrm{f}_1, \mathrm{g}_1)$, and they each give distinct actions on the parallel pairs.

Thus, parallel composition is *neither* lax nor colax with respect to sequential composition; there is simply no interchange transformation between the two operations. Recall also that that the weave construction $\langle - \rangle$ is not lax nor colax 2.1.2. So while $\mathbb{C}\mathrm{at}$ and $\mathrm{Mat}\mathbb{C}\mathrm{at}$ are double categories, parallel composition of $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \rightarrow \mathbb{C}\mathrm{at}$ is a structure on *span categories*.

We define a *metalogic* to be a fibered logic $\mathbb{C} \leftarrow \mathbb{M} \rightarrow \mathbb{C}$ with the structure of a "2-weak category" in the tricategory of span categories. The structure is a "triple category without interchange", and its weakness of parallel composition and unit is like that of a tricategory [8].

Lastly, what ensures that this weak parallel composition has coherent associator and unitors? Matrix categories and matrix profunctors are each *exponentiable*, meaning composition has a right adjoint, and hence preserves the colimits which define parallel composition.

It is known that two-sided fibrations are exponentiable [20], and so matrix categories are as well. We showed in Theorem 38 that matrix profunctors are exponentiable.

Definition 53. A **metalogic** is a logic $\mathbb{C}$ and a fibered logic $\mathbb{M} \to \mathbb{C} \times \mathbb{C}$, with the structure of a 2-weak category in the tricategory of span categories.

Theorem 54. $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ forms a metalogic.

Proof. As we showed, $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a fibered span of logics

$$\begin{tikzcd}
\mathbb{C} & \mathbb{MC} \arrow[l] \arrow[r] & \mathbb{C} \\
\mathbb{P} \arrow[u] \arrow[d] & \mathbb{MP} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathbb{P} \arrow[u] \arrow[d] \\
\mathbb{C} & \mathbb{MC} \arrow[l] \arrow[r] & \mathbb{C}
\end{tikzcd}$$

equipped with span functors, for composition and identity

$$\begin{tikzcd}
\mathbb{MC} *_h \mathbb{MC} \arrow[r, "\otimes"] & \mathbb{MC} \\
\mathbb{MP} *_h \mathbb{MP} \arrow[r, "\otimes"] \arrow[u] \arrow[d] & \mathbb{MP} \arrow[u] \arrow[d] \\
\mathbb{MC} *_h \mathbb{MC} \arrow[r, "\otimes"] & \mathbb{MC}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\langle - \rangle"] & \mathbb{MC} \\
\mathbb{P} \arrow[r, "\langle - \rangle"] \arrow[u] \arrow[d] & \mathbb{MP} \arrow[u] \arrow[d] \\
\mathbb{C} \arrow[r, "\langle - \rangle"] & \mathbb{MC}
\end{tikzcd}$$

with invertible span transformations for associativity,

$$\begin{tikzcd}
\mathbb{MC}_0 *_h \mathbb{MC}_0 \arrow[d, "\otimes"] & \mathbb{MC}_0 *_h \mathbb{MC}_0 *_h \mathbb{MC}_0 \arrow[l, "\mathbb{MC}_0 * \otimes"'] \arrow[r, "\otimes * \mathbb{MC}_0"] \arrow[d, "\alpha"] & \mathbb{MC}_0 *_h \mathbb{MC}_0 \arrow[d, "\otimes"] \\
\mathbb{MC}_0 & \mathbb{MC}_1 \arrow[l] \arrow[r] & \mathbb{MC}_0
\end{tikzcd}$$

$$\alpha : \mathcal{R} \otimes (\mathcal{S} \otimes \mathcal{T}) \cong (\mathcal{R} \otimes \mathcal{S}) \otimes \mathcal{T}$$

and span transformations for left and right unitality

$$\begin{tikzcd}
\mathbb{MC}_0 \arrow[r, equal] \arrow[d, equal] & \mathbb{MC}_0 \arrow[r, "\langle - \rangle * \mathbb{MC}_0"] \arrow[d, "\lambda^\circ"] & \mathbb{MC}_0 *_h \mathbb{MC}_0 \arrow[d, "\otimes"] \\
\mathbb{MC}_0 & \mathbb{MC}_1 \arrow[l] \arrow[r] & \mathbb{MC}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{MC}_0 \arrow[r, equal] \arrow[d, equal] & \mathbb{MC}_0 \arrow[r, "\mathbb{MC}_0 * \langle - \rangle"] \arrow[d, "\rho^\circ"] & \mathbb{MC}_0 *_h \mathbb{MC}_0 \arrow[d, "\otimes"] \\
\mathbb{MC}_0 & \mathbb{MC}_1 \arrow[l] \arrow[r] & \mathbb{MC}_0
\end{tikzcd}$$

$$\lambda^\circ = R.(\mathrm{id}.\mathrm{A}, R) : \mathcal{R} \to \langle \mathbb{A} \rangle \otimes \mathcal{R} \qquad \rho^\circ = R.(R, \mathrm{id}.\mathrm{B}) : \mathcal{R} \to \mathcal{R} \otimes \langle \mathbb{B} \rangle$$

$$\begin{tikzcd}
\mathbb{MC}_0 *_h \mathbb{MC}_0 \arrow[d, "\otimes"] & \mathbb{MC}_0 \arrow[l, "\langle - \rangle * \mathbb{MC}_0"'] \arrow[r, equal] \arrow[d, "\lambda^\bullet"] & \mathbb{MC}_0 \arrow[d, equal] \\
\mathbb{MC}_0 & \mathbb{MC}_1 \arrow[l] \arrow[r] & \mathbb{MC}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{MC}_0 *_h \mathbb{MC}_0 \arrow[d, "\otimes"] & \mathbb{MC}_0 \arrow[l, "\mathbb{MC}_0 * \langle - \rangle"'] \arrow[r, equal] \arrow[d, "\rho^\bullet"] & \mathbb{MC}_0 \arrow[d, equal] \\
\mathbb{MC}_0 & \mathbb{MC}_1 \arrow[l] \arrow[r] & \mathbb{MC}_0
\end{tikzcd}$$

$$\lambda^\bullet = \odot_{\mathbb{A}} : \langle \mathbb{A} \rangle \otimes \mathcal{R} \to \mathcal{R} \qquad \rho^\bullet = \odot_{\mathbb{B}} : \mathcal{R} \otimes \langle \mathbb{B} \rangle \to \mathcal{R}$$

so that $(\lambda^\circ, \lambda^\bullet)$ and $(\rho^\circ, \rho^\bullet)$ form adjoint equivalences.

$$\begin{tikzcd}
\mathbb{MC}_0 \arrow[r, equal] \arrow[d, "\mathrm{id}"] & \mathbb{MC}_0 \arrow[r, "(\lambda^\circ, \lambda^\bullet)"] \arrow[d, "\eta_\lambda"] & \mathbb{MC}_1 *_t \mathbb{MC}_1 \arrow[d, "\cdot"] \\
\mathbb{MC}_1 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{MC}_0 \arrow[r, equal] \arrow[d, "\mathrm{id}"] & \mathbb{MC}_0 \arrow[r, "(\rho^\circ, \rho^\bullet)"] \arrow[d, "\eta_\rho"] & \mathbb{MC}_1 *_t \mathbb{MC}_1 \arrow[d, "\cdot"] \\
\mathbb{MC}_1 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{R} \arrow[rr, equal] \arrow[dr, "\lambda^\circ"'] & \arrow[d, phantom, "\Downarrow \eta_\lambda"] & \mathcal{R} \\
& \langle \mathbb{A} \rangle \otimes \mathcal{R} \arrow[ur, "\lambda^\bullet"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \arrow[rr, equal] \arrow[dr, "\rho^\circ"'] & \arrow[d, phantom, "\Downarrow \eta_\rho"] & \mathcal{R} \\
& \mathcal{R} \otimes \langle \mathbb{B} \rangle \arrow[ur, "\rho^\bullet"'] &
\end{tikzcd}$$

$$\eta_\lambda = \upsilon_{\mathbb{A}} : R \cong \overline{\mathrm{id}}.\mathrm{A} \odot R \qquad \eta_\rho = \upsilon_{\mathbb{B}} : R \cong R \odot \overline{\mathrm{id}}.\mathrm{B}$$

$$
\begin{tikzcd}
\mathbb{MC}_1 *_t \mathbb{MC}_1 \arrow[d, dotted, "\cdot"] & \mathbb{MC}_0 \arrow[l, "(\lambda^\bullet{,}\lambda^\circ)"'] \arrow[r, equal] \arrow[d, "\varepsilon_\lambda"] & \mathbb{MC}_0 \arrow[d, equal] \\
\mathbb{MC}_1 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{MC}_1 *_t \mathbb{MC}_1 \arrow[d, dotted, "\cdot"] & \mathbb{MC}_0 \arrow[l, "(\rho^\bullet{,}\rho^\circ)"'] \arrow[r, equal] \arrow[d, "\varepsilon_\rho"] & \mathbb{MC}_0 \arrow[d, equal] \\
\mathbb{MC}_1 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{R} \arrow[dr, "\lambda^\circ"] \arrow[d, phantom, "\Downarrow \varepsilon_\lambda"] & \\
\langle \mathbb{A} \rangle \otimes \mathcal{R} \arrow[ur, "\lambda^\bullet"] \arrow[rr, equal] & & \langle \mathbb{A} \rangle \otimes \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{R} \arrow[dr, "\rho^\circ"] \arrow[d, phantom, "\Downarrow \varepsilon_\rho"] & \\
\mathcal{R} \otimes \langle \mathbb{B} \rangle \arrow[ur, "\rho^\bullet"] \arrow[rr, equal] & & \mathcal{R} \otimes \langle \mathbb{B} \rangle
\end{tikzcd}
$$

$$\varepsilon_\lambda = \alpha_{\mathbb{A}} : (\mathrm{id.A}_0, \bar{\mathrm{a}} \odot R) \cong (\bar{\mathrm{a}}, R) \qquad \varepsilon_\rho = \alpha_{\mathbb{B}} : (R \odot \bar{\mathrm{b}}, \mathrm{id.B}_1) \cong (R, \bar{\mathrm{b}})$$

Similarly, for each matrix profunctor there are span transformations

$$
\begin{tikzcd}
\mathbb{MP}_0 \arrow[r, equal] \arrow[d, equal] & \mathbb{MP}_0 \arrow[r, "\langle - \rangle * \mathbb{MP}_0"] \arrow[d, "\lambda^\circ"] & \mathbb{MP}_0 *_h \mathbb{MP}_0 \arrow[d, "\otimes"] \\
\mathbb{MP}_0 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MP}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{MP}_0 \arrow[r, equal] \arrow[d, equal] & \mathbb{MP}_0 \arrow[r, "\mathbb{MP}_0 * \langle - \rangle"] \arrow[d, "\rho^\circ"] & \mathbb{MP}_0 *_h \mathbb{MP}_0 \arrow[d, "\otimes"] \\
\mathbb{MP}_0 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MP}_0
\end{tikzcd}
$$

$$\lambda^\circ = i.(\mathrm{id.f}, i) : i \Rightarrow \langle f \rangle \otimes i \qquad \rho^\circ = i.(i, \mathrm{id.g}) : i \Rightarrow i \otimes \langle g \rangle$$

$$
\begin{tikzcd}
\mathbb{MP}_0 *_h \mathbb{MP}_0 \arrow[d, "\otimes"] & \mathbb{MP}_0 \arrow[l, "\langle - \rangle * \mathbb{MP}_0"'] \arrow[r, equal] \arrow[d, "\lambda^\bullet"] & \mathbb{MP}_0 \arrow[d, equal] \\
\mathbb{MP}_0 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MP}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{MP}_0 *_h \mathbb{MP}_0 \arrow[d, "\otimes"] & \mathbb{MP}_0 \arrow[l, "\mathbb{MP}_0 * \langle - \rangle"'] \arrow[r, equal] \arrow[d, "\rho^\bullet"] & \mathbb{MP}_0 \arrow[d, equal] \\
\mathbb{MP}_0 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MP}_0
\end{tikzcd}
$$

$$\lambda^\bullet = \odot_f : i \otimes \langle f \rangle \Rightarrow i \qquad \rho^\bullet = \odot_g : i \otimes \langle g \rangle \Rightarrow i$$

so that the unitor isomorphisms cohere with these transformations, as in a modification:

$$\begin{tikzcd}
\mathbb{MC}_0 \arrow[r, equal] \arrow[dd, "\mathrm{id}"] & \mathbb{MC}_0 \arrow[r, equal] \arrow[dd, "\eta_\lambda"] & \mathbb{MC}_0 \arrow[d, "(\lambda^\circ_Q{,}\lambda^\bullet_Q)"] & \mathbb{MP}_0 \arrow[l] \arrow[r] \arrow[d, "(\lambda^\circ_i{,}\lambda^\bullet_i)"] & \mathbb{MC}_0 \arrow[r, equal] \arrow[d, "(\lambda^\circ_R{,}\lambda^\bullet_R)"] & \mathbb{MC}_0 \arrow[r, equal] \arrow[dd, "\varepsilon_\lambda"] & \mathbb{MC}_0 \arrow[dd, "\mathrm{id}"] \\
 & & \mathbb{MC}_1 *_t \mathbb{MC}_1 \arrow[d, "\cdot"] & \mathbb{MP}_1 *_t \mathbb{MP}_1 \arrow[l] \arrow[r] \arrow[d, "\cdot"] & \mathbb{MC}_1 *_t \mathbb{MC}_1 \arrow[d, "\cdot"] & & \\
\mathbb{MC}_1 \arrow[d, equal] & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1 \arrow[d, equal] \\
\mathbb{MC}_1 \arrow[d, equal] & & & \mathbb{MP}_1 *_v \mathbb{MP}_1 *_v \mathbb{MP}_1 \arrow[lll] \arrow[rrr] \arrow[d, "\diamond"] & & & \mathbb{MC}_1 \arrow[d, equal] \\
\mathbb{MC}_1 & & & \mathbb{MP}_1 \arrow[lll] \arrow[rrr] & & & \mathbb{MC}_1
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
\mathbb{MC}_0 \arrow[d, "\mathrm{id}"] & \mathbb{MP}_0 \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbb{MC}_0 \arrow[d, "\mathrm{id}"] \\
\mathbb{MC}_1 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1
\end{tikzcd}$$

and this is given by the naturality of the unitors with respect to matrix profunctor elements.

$$\begin{tikzcd}
 & \mathcal{Q} \arrow[r, "i"] \arrow[d] \arrow[dl, equal] \arrow[dr, phantom, "\Vert\ \lambda_i\ \Downarrow"] & \mathcal{R} \arrow[d] \arrow[dr, equal] & \\
\mathcal{Q} \arrow[r, phantom, "\cong"] \arrow[dr, equal] & \langle \mathbb{X} \rangle \otimes \mathcal{Q} \arrow[r, "\langle f \rangle \otimes i"] \arrow[d] \arrow[dr, phantom, "\Vert\ \lambda^\bullet_i\ \Downarrow"] & \langle \mathbb{A} \rangle \otimes \mathcal{R} \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{R} \arrow[dl, equal] \\
 & \mathcal{Q} \arrow[r, "i"'] & \mathcal{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{Q} \arrow[r, "i"] \arrow[d] \arrow[dl, equal] \arrow[dr, phantom, "\Vert\ \rho^\bullet_i\ \Downarrow"] & \mathcal{R} \arrow[d] \arrow[dr, equal] & \\
\mathcal{Q} \arrow[r, phantom, "\cong"] \arrow[dr, equal] & \mathcal{Q} \otimes \langle \mathbb{Y} \rangle \arrow[r, "i \otimes \langle g \rangle"] \arrow[d] \arrow[dr, phantom, "\Vert\ \rho_i\ \Downarrow"] & \mathcal{R} \otimes \langle \mathbb{B} \rangle \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{R} \arrow[dl, equal] \\
 & \mathcal{Q} \arrow[r, "i"'] & \mathcal{R} &
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\mathrm{id}}.\mathrm{X} \odot Q \arrow[r, "\upsilon_{\mathbb{X}}"] \arrow[d, "\mathrm{id.f} \circ i"] & Q \arrow[d, "i"] \\
\overline{\mathrm{id}}.\mathrm{A} \odot R \arrow[r, "\upsilon_{\mathbb{A}}"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \odot \overline{\mathrm{id}}.\mathrm{Y} \arrow[r, "\upsilon_{\mathbb{Y}}"] \arrow[d, "i \circ \mathrm{id.g}"] & Q \arrow[d, "i"] \\
R \odot \overline{\mathrm{id}}.\mathrm{B} \arrow[r, "\upsilon_{\mathbb{B}}"'] & R
\end{tikzcd}$$

The analogous coherence holds for the right unitor ρ.

The "pentagon identity" for reassociating a composite is replaced by a "pentagonator".

$$\begin{tikzcd}
\mathcal{Q}\otimes((\mathcal{R}\otimes\mathcal{S})\otimes\mathcal{T}) \arrow[rr] & & (\mathcal{Q}\otimes(\mathcal{R}\otimes\mathcal{S}))\otimes\mathcal{T} \arrow[d] \\
\mathcal{Q}\otimes(\mathcal{R}\otimes(\mathcal{S}\otimes\mathcal{T})) \arrow[u] \arrow[dr] & \Downarrow\pi & ((\mathcal{Q}\otimes\mathcal{R})\otimes\mathcal{S})\otimes\mathcal{T} \\
& (\mathcal{Q}\otimes\mathcal{R})\otimes(\mathcal{S}\otimes\mathcal{T}) \arrow[ur] &
\end{tikzcd}$$

In our case, this isomorphism is an *equality*, because the associator simply moves parentheses. Hence it satisfies the coherence equation, which can be found in the definition of tricategory [8].

Last, the unitors respect parallel composition by the "triangulator" invertible transformation:

$$\begin{tikzcd}
\mathbb{MC}_0 \arrow[d, "\mathrm{id}"] & \mathbb{MC}_0 *_h \mathbb{MC}_0 \arrow[l, "\otimes"'] \arrow[r, "(\rho^\circ\otimes\mathrm{id},\alpha,\mathrm{id}\otimes\lambda^\bullet)"] \arrow[d, "\tau"] & \mathbb{MC}_1 *_t \mathbb{MC}_1 *_t \mathbb{MC}_1 \arrow[d, "\cdot"] \\
\mathbb{MC}_1 & \mathbb{MP}_1 \arrow[l] \arrow[r] & \mathbb{MC}_1
\end{tikzcd}$$

which is given by the unitor

$$\begin{tikzcd}
\mathcal{R}\otimes\mathcal{S} \arrow[rr, equal] \arrow[d, "\mathcal{R}\otimes\rho^\circ"'] & \Downarrow\tau & \mathcal{R}\otimes\mathcal{S} \\
\mathcal{R}\otimes(\langle\mathbb{B}\rangle\otimes\mathcal{S}) \arrow[rr, "\alpha"] & & (\mathcal{R}\otimes\langle\mathbb{B}\rangle)\otimes\mathcal{S} \arrow[u, "\lambda^\bullet\otimes\mathcal{S}"']
\end{tikzcd}$$

$$\tau = \upsilon_{\mathbb{B}} : (R, S) \cong (R \odot \mathrm{id}.\mathrm{B}, S)$$

and which coheres with matrix profunctors, as in a modification.

$$\begin{tikzcd}
\mathcal{Q}\otimes\mathcal{S} \arrow[rrr, equal] \arrow[dr, "\rho^\circ\otimes\mathcal{S}"] \arrow[ddd, "m\otimes n"'] & & & \mathcal{Q}\otimes\mathcal{S} \arrow[ddd, "m\otimes n"] \\
& (\mathcal{Q}\otimes\langle\mathbb{Y}\rangle)\otimes\mathcal{S} \arrow[r, "\alpha"] \arrow[d, "(m\otimes\langle g\rangle)\otimes n"] & \mathcal{Q}\otimes(\langle\mathbb{Y}\rangle\otimes\mathcal{S}) \arrow[ur, "\mathcal{Q}\otimes\lambda^\bullet"] \arrow[d, "m\otimes(\langle g\rangle\otimes n)"'] & \\
& (\mathcal{R}\otimes\langle\mathbb{B}\rangle)\otimes\mathcal{T} \arrow[r, "\alpha"] & \mathcal{R}\otimes(\langle\mathbb{B}\rangle\otimes\mathcal{T}) \arrow[dr, "\mathcal{R}\otimes\lambda^\bullet"] & \\
\mathcal{R}\otimes\mathcal{T} \arrow[ur, "\rho^\circ\otimes\mathcal{T}"] \arrow[rrr, equal] & & & \mathcal{R}\otimes\mathcal{T}
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
\mathcal{Q}\otimes\mathcal{S} \arrow[r, equal] \arrow[d, "m\otimes n"', "\shortmid" marking] \arrow[dr, phantom, "="] & \mathcal{Q}\otimes\mathcal{S} \arrow[d, "m\otimes n", "\shortmid" marking] \\
\mathcal{R}\otimes\mathcal{T} \arrow[r, equal] & \mathcal{R}\otimes\mathcal{T}
\end{tikzcd}$$

For its coherence, the two ways to transform the top composite to the associator are equal:

$$\begin{tikzcd}
\mathcal{R}\otimes(\mathcal{S}\otimes(\langle\mathbb{C}\rangle\otimes\mathcal{T})) \arrow[rr] & & \mathcal{R}\otimes((\mathcal{S}\otimes\langle\mathbb{C}\rangle)\otimes\mathcal{T}) \arrow[d] \\
\mathcal{R}\otimes(\mathcal{S}\otimes\mathcal{T}) \arrow[u] \arrow[dr] & \Downarrow\;\Downarrow & \mathcal{R}\otimes(\mathcal{S}\otimes\mathcal{T}) \arrow[dl] \\
& (\mathcal{R}\otimes\mathcal{S})\otimes\mathcal{T} &
\end{tikzcd}$$

meaning that applying the triangulator commutes with reassociating.

This holds by the naturality of the unitor with respect to the associator.

$$\begin{tikzcd}
(R,(S,T)) \arrow[r, mapsto, "\lambda^{\circ}_{\mathcal{T}}\cdot\alpha_{\mathbb{C}}\cdot\rho^{\bullet}_{\mathcal{S}}\cdot\alpha_{\mathcal{S}}"] \arrow[d, equal] & ((R,S\odot\mathrm{id.C}),T) \arrow[r, "\alpha"] \arrow[d, "{(\tau_{\mathcal{RS}},T)}"] & (R,(S\odot\mathrm{id.C},T)) \arrow[d, "{(R,\tau_{\mathcal{ST}})}"] \\
(R,(S,T)) \arrow[r, mapsto, "\alpha"'] & ((R,S),T) & (R,(S,T)) \arrow[l, "\alpha^{-1}"']
\end{tikzcd}$$

The analogous coherence holds for applying the triangulator on the other side of the associator.

This completes the exposition of $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \rightarrow \mathbb{C}\mathrm{at}$ as a metalogic.

□

Our final result is the duality of composition-by-codescent (2.5) and hom-by-descent (2.2.1).

Theorem 55. For every pair of matrix categories $\mathcal{R} : \mathbb{A} \parallel \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \parallel \mathbb{C}$ and matrix category $\mathcal{T} : \mathbb{A} \parallel \mathbb{C}$, there is a natural equivalence of categories of matrix functors.

$$\mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{R} \otimes \mathcal{S}, \mathcal{T}) \simeq \mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{R}, [\mathcal{S}, \mathcal{T}])$$

Proof. The composite $\mathcal{R} \otimes \mathcal{S}$ is a coequifier of an iso-coinserter, while the hom $[(\mathcal{R} \otimes \mathcal{S}), \mathcal{T}]$ is an equifier of an iso-inserter. These are constructed pointwise in $\mathbb{C}\mathrm{at}$; the first coordinate of $\mathbb{C}\mathrm{at}(-, -)$ converts 2-colimits into 2-limits, while the second preserves 2-limits [10]. The Fubini equivalence is given in [3].

Hence we have the following equivalence.

$$\begin{aligned}
\mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{R} \otimes \mathcal{S}, \mathcal{T}) &= \vec{\Pi} \mathrm{A}, \mathrm{C} \quad \mathbb{C}\mathrm{at}((\mathcal{R} \otimes \mathcal{S})(\mathrm{A}, \mathrm{C}), \mathcal{T}(\mathrm{A}, \mathrm{C})) \\
&= \vec{\Pi} \mathrm{A}, \mathrm{C} \quad \mathbb{C}\mathrm{at}(\vec{\Sigma} \mathrm{B} \;\; \mathcal{R}(\mathrm{A}, \mathrm{B}) \times \mathcal{S}(\mathrm{B}, \mathrm{C}), \mathcal{T}(\mathrm{A}, \mathrm{C})) \\
&\simeq \vec{\Pi} \mathrm{A}, \mathrm{C} \;\; \vec{\Pi} \mathrm{B} \quad \mathbb{C}\mathrm{at}(\mathcal{R}(\mathrm{A}, \mathrm{B}) \times \mathcal{S}(\mathrm{B}, \mathrm{C}), \mathcal{T}(\mathrm{A}, \mathrm{C})) \\
&\simeq \vec{\Pi} \mathrm{A}, \mathrm{B}, \mathrm{C} \quad \mathbb{C}\mathrm{at}(\mathcal{R}(\mathrm{A}, \mathrm{B}), [\mathcal{S}(\mathrm{B}, \mathrm{C}) \to \mathcal{T}(\mathrm{A}, \mathrm{C})]) \\
&\simeq \vec{\Pi} \mathrm{A}, \mathrm{B} \quad \mathbb{C}\mathrm{at}(\mathcal{R}(\mathrm{A}, \mathrm{B}), \vec{\Pi} \mathrm{C} \;\; [\mathcal{S}(\mathrm{B}, \mathrm{C}) \to \mathcal{T}(\mathrm{A}, \mathrm{C})]) \quad = \quad \mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{R}, [\mathcal{S}, \mathcal{T}])
\end{aligned}$$

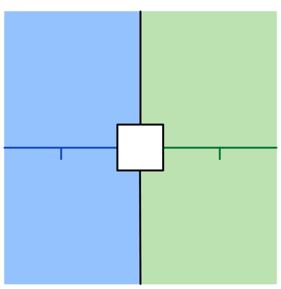
$\square$

Chapter 3

The metalogic of logics

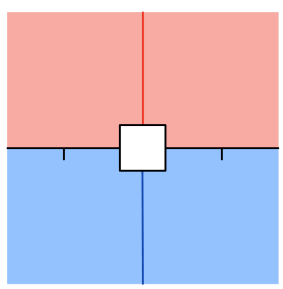
Now we can define a *logic*, or *bifibrant double category*: a matrix category $\mathbb{A} : \underline{\mathbb{A}} \,\|\, \underline{\mathbb{A}}$ with composition $\circ : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ and unit $\mathrm{id} : \underline{\mathbb{A}} \to \mathbb{A}$, with coherent associator and unitors — a *pseudomonad* in $\mathrm{Mat}\mathbb{C}\mathrm{at}$.

Since we have developed all the necessary infrastructure, we can define the whole "multiverse" of logics. Because a logic is two-dimensional, there are *two* kinds of relations between logics: a *vertical profunctor* consists of processes between logics, and a *horizontal profunctor* consists of relations between logics. Two pairs are connected by a *double profunctor*, which consists of inferences between relations, along processes.



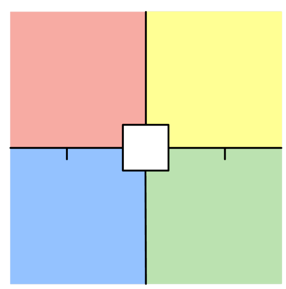
meta relation

[horiz. profunctor]



meta process

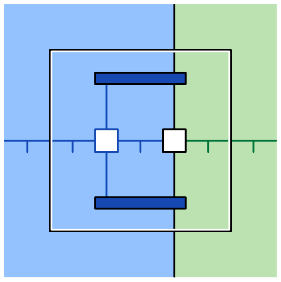
[vert. profunctor]



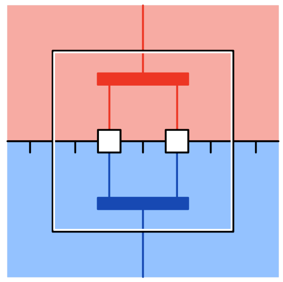
meta inference

[dbl. profunctor]

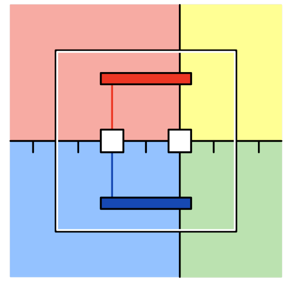
Because $\mathrm{Mat}\mathbb{C}\mathrm{at}$ consists of categories and profunctors, the above profunctors already have sequential composition; so we only need to add the structure of *parallel* composition. For horizontal profunctors, this is a familiar *bimodule* action. But as vertical profunctors are orthogonal, parallel composition defines a *monad* structure, and double profunctors are bimodules thereof.



H-prof. composition

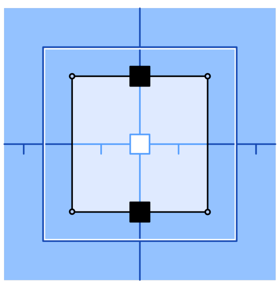


V-prof. composition

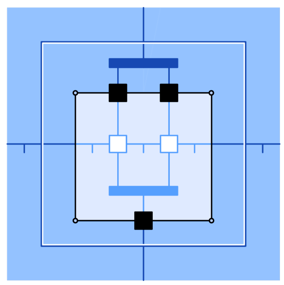


D-prof. composition

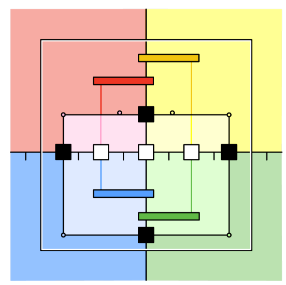
So logics have two kinds of “relations”, and one kind of “function”: a *double functor* $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ maps squares of $\mathbb{A}_0$ to squares of $\mathbb{A}_1$, preserving relation composition and unit up to coherent isomorphism. This generalizes to transformations of vertical, horizontal, and double profunctors; all four are defined by mapping squares in a way that coheres with parallel composition.



double functor



preserves composition;



double transformation

All together, logics form a metalogic: the three kinds of 1-morphism are profunctor, matrix category, and functor; the three kinds of 2-morphism are double profunctor, vertical transformation, and horizontal transformation; and the 3-morphism is a double transformation.

	$\mathrm{Mat}\mathbb{C}\mathrm{at}$	$\mathrm{H.PsMnd}(-)$	$bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$	Logic
0	category	(H)-pseudomonad	bifibrant double category	logic
V	profunctor	(H)-vertical monad	vertical profunctor	meta process
H	matrix category	(H)-pseudobimodule	horizontal profunctor	meta relation
VH	matrix profunctor	(H)-vertical bimodule	double profunctor	meta inference
T	functor	ps. mnd. morphism	double functor	flow type
TV	transformation	v. mnd. morphism	vertical transformation	flow process
TH	matrix functor	ps. bim. morphism	horizontal transformation	flow relation
TVH	matrix transformation	v. bim. morphism	double transformation	flow inference

We construct the double category $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ of bifibrant double categories and double functors, vertical profunctors and vertical transformations.

We construct the double category $bf.\mathrm{DblProf}$ of horizontal profunctors and horizontal transformations, double profunctors and double transformations.

Finally, we define parallel composition of horizontal profunctors. As for matrix categories in 2.5, the composite is constructed by a *codescent object*, which adjoins a coherent associator for the middle action. We show that this defines the structure of a metalogic.

3.1 Logic [Bifibrant double category]

Definition 56. A **logic** $\mathbb{A}$, a.k.a. *bifibrant double category*, is a pseudomonad in $\mathrm{Mat}\mathbb{C}\mathrm{at}$.

Hence a logic is a category $\underline{\mathbb{A}}$ with a matrix category $\mathbb{A} : \underline{\mathbb{A}} \parallel \underline{\mathbb{A}}$

$$\begin{tikzcd}
\underline{\mathbb{A}} \arrow[d, "\shortmid"{marking}] & \mathbb{A} \arrow[l] \arrow[r] \arrow[d, "\shortmid"{marking}] \arrow[l, phantom, shift right=6, "\Leftarrow"] \arrow[r, phantom, shift right=6, "\Rightarrow"] & \underline{\mathbb{A}} \arrow[d, "\shortmid"{marking}] \\
\underline{\mathbb{A}} & \mathbb{A} \arrow[l] \arrow[r] & \underline{\mathbb{A}}
\end{tikzcd}$$

with matrix functors $\circ : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ for composition and $\mathrm{id} : \underline{\mathbb{A}} \to \mathbb{A}$ for unit

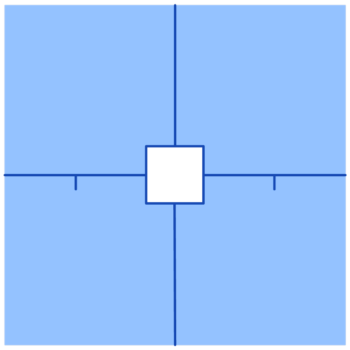
$$\begin{tikzcd}
\underline{\mathbb{A}} \arrow[d, equal] & \mathbb{A} \otimes \mathbb{A} \arrow[l] \arrow[r] \arrow[d, "\circ"] & \underline{\mathbb{A}} \arrow[d, equal] \\
\underline{\mathbb{A}} & \mathbb{A} \arrow[l] \arrow[r] & \underline{\mathbb{A}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbb{A}} \arrow[d, equal] & \underline{\mathbb{A}} \arrow[l, equal] \arrow[r, equal] \arrow[d, "\mathrm{id}"] & \underline{\mathbb{A}} \arrow[d, equal] \\
\underline{\mathbb{A}} & \mathbb{A} \arrow[l] \arrow[r] & \underline{\mathbb{A}}
\end{tikzcd}$$

and invertible matrix transformations for associativity and unit

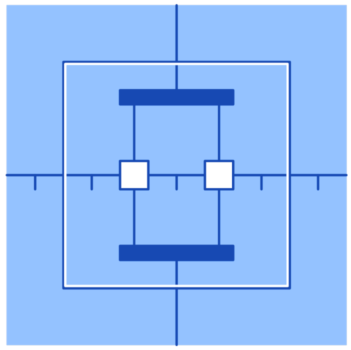
$$\begin{tikzcd}
\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A} \arrow[r, "\circ\otimes\mathbb{A}"] \arrow[d, "\mathbb{A}\otimes\circ"'] & \mathbb{A} \otimes \mathbb{A} \arrow[d, "\circ"] \\
\mathbb{A} \otimes \mathbb{A} \arrow[r, "\circ"'] \arrow[ur, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[r, "\mathbb{A}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathbb{A}"'] \arrow[dr, equal] & \mathbb{A} \otimes \mathbb{A} \arrow[d, "\circ"] \\
\mathbb{A} \otimes \mathbb{A} \arrow[r, "\circ"'] & \mathbb{A}
\end{tikzcd}
\quad (\lambda, \rho)$$

which satisfy the associator and unitor coherence.

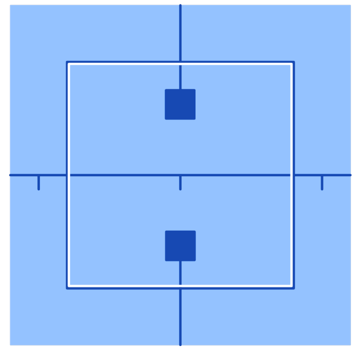
$$\begin{tikzcd}
\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A} \arrow[r, bend left, "A_1\circ(A_2\circ(A_3\circ A_4))"] \arrow[r, bend right, "((A_1\circ A_2)\circ A_3)\circ A_4"'] \arrow[r, phantom, "\Downarrow\;\Downarrow"] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[r, bend left, "A_1\circ(\mathrm{id}\circ A_2)"] \arrow[r, bend right, "(A_1\circ\mathrm{id})\circ A_2"'] \arrow[r, phantom, "\Downarrow\;\Downarrow"] & \mathbb{A}
\end{tikzcd}$$



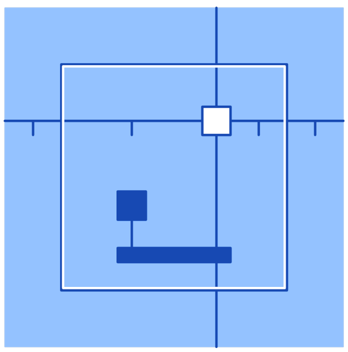
bifibrant double category
matrix category
$\mathbb{A} : \underline{\mathbb{A}} \parallel \underline{\mathbb{A}}$



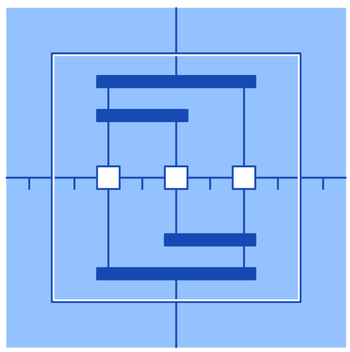
composition
matrix functor
$\circ : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$



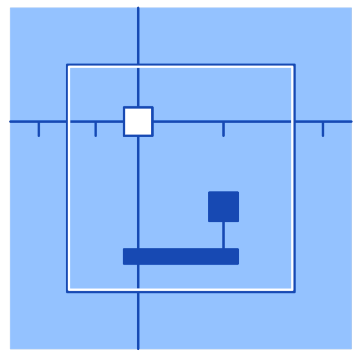
unit
matrix functor
$\mathrm{id} : \underline{\mathbb{A}} \to \mathbb{A}$



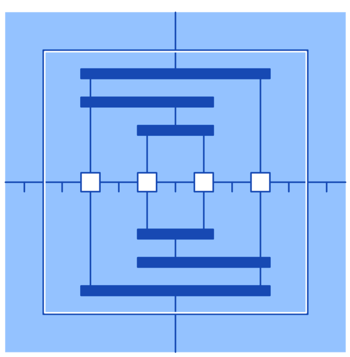
left unitor
matrix transformation
$\lambda : \mathbb{A} \cong \mathrm{id} \circ \mathbb{A}$



associator
matrix transformation
$\alpha : (\mathbb{A} \circ \mathbb{A}) \circ \mathbb{A} \cong \mathbb{A} \circ (\mathbb{A} \circ \mathbb{A})$

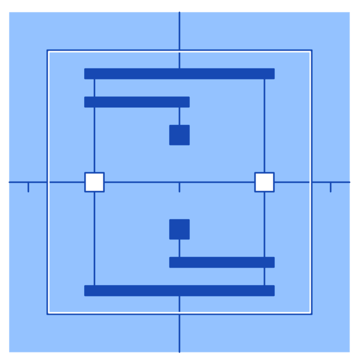


right unitor
matrix transformation
$\rho : \mathbb{A} \cong \mathbb{A} \circ \mathrm{id}$



associator coherence

$((\mathrm{A}_0 \circ \mathrm{A}_1) \circ \mathrm{A}_2) \circ \mathrm{A}_3 \rightrightarrows \mathrm{A}_0 \circ (\mathrm{A}_1 \circ (\mathrm{A}_2 \circ \mathrm{A}_3))$



unitor coherence

$(\mathrm{A}_1 \circ \mathrm{id}) \circ \mathrm{A}_2 \rightrightarrows \mathrm{A}_1 \circ (\mathrm{id} \circ \mathrm{A}_2)$

3.2 Relations [Double profunctor]

Definition 57. Let $\mathbb{X}, \mathbb{A}$ be bifibrant double categories. A **vertical profunctor** $f : \mathbb{X} \mid \mathbb{A}$, i.e. **meta process**, is a *vertical monad* between pseudomonads in $\mathbb{M}\mathrm{at}\mathbb{C}\mathrm{at}$.

Hence it is a profunctor $\underline{f} : \underline{\mathbb{X}} \mid \underline{\mathbb{A}}$ and a matrix profunctor $f(\underline{f}, \underline{f}) : \mathbb{X}(\underline{\mathbb{X}}, \underline{\mathbb{X}}) \mid \mathbb{A}(\underline{\mathbb{A}}, \underline{\mathbb{A}})$

$$\begin{tikzcd}
\underline{\mathbb{X}} \arrow[d, "\underline{f}"'] & \mathbb{X} \arrow[l] \arrow[r] \arrow[d, "f" description] \arrow[l, phantom, shift right=6, "\Leftarrow"] \arrow[r, phantom, shift right=6, "\Rightarrow"] & \underline{\mathbb{X}} \arrow[d, "\underline{f}"] \\
\underline{\mathbb{A}} & \mathbb{A} \arrow[l] \arrow[r] & \underline{\mathbb{A}}
\end{tikzcd}$$

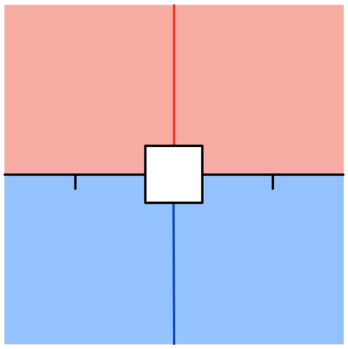
with matrix transformations $\circ : f * f \Rightarrow f$ for composition and $\mathrm{id} : \underline{f} \Rightarrow f$ for unit

$$\begin{tikzcd}
\underline{f} \arrow[d, equal] & f \otimes f \arrow[l] \arrow[r] \arrow[d, "\circ" description] & \underline{f} \arrow[d, equal] \\
\underline{f} & f \arrow[l] \arrow[r] & \underline{f}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{f} \arrow[d, equal] & \underline{f} \arrow[l, equal] \arrow[r, equal] \arrow[d, "\mathrm{id}" description] & \underline{f} \arrow[d, equal] \\
\underline{f} & f \arrow[l] \arrow[r] & \underline{f}
\end{tikzcd}$$

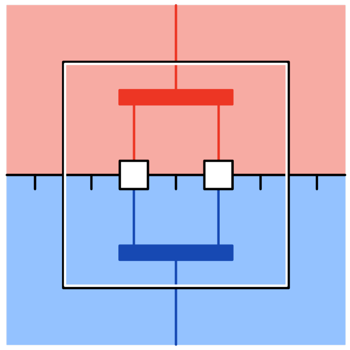
which cohere with the associators and unitors of $\mathbb{X}$ and $\mathbb{A}$.

$$\begin{tikzcd}
\mathbb{X}\otimes\mathbb{X} \arrow[rrr] \arrow[ddd, "f\otimes f"'] & & & \mathbb{X} \arrow[ddd, "f"] \\
& \mathbb{X}\otimes\mathbb{X}\otimes\mathbb{X} \arrow[ul] \arrow[r] \arrow[d, "f\otimes f\otimes f"] \arrow[r, phantom, shift left=8, "\cong"] \arrow[l, phantom, shift right=6, "\Leftarrow f\otimes\circ"] & \mathbb{X}\otimes\mathbb{X} \arrow[ur] \arrow[d, "f\otimes f"] \arrow[l, phantom, shift right=6, "\Rightarrow \circ\otimes f"] \arrow[r, phantom, shift right=6, "\Rightarrow \circ"] & \\
& \mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A} \arrow[dl] \arrow[r] \arrow[r, phantom, shift right=8, "\cong"] \arrow[r, phantom, shift left=4, "\Rrightarrow \circ"] & \mathbb{A}\otimes\mathbb{A} \arrow[dr] & \\
\mathbb{A}\otimes\mathbb{A} \arrow[rrr] & & & \mathbb{A}
\end{tikzcd}$$

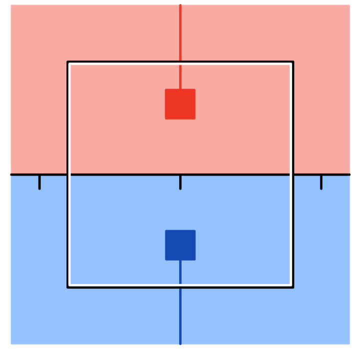
$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{id}\otimes\mathbb{X}"] \arrow[d, "f"'] \arrow[rr, bend left, equal] \arrow[r, phantom, shift right=6, "\Rightarrow \mathrm{id}\otimes f"] & \mathbb{X}\otimes\mathbb{X} \arrow[r, "\circ"] \arrow[d, "f\otimes f"] \arrow[r, phantom, shift left=6, "\cong"] \arrow[r, phantom, shift right=6, "\Rightarrow \circ"] & \mathbb{X} \arrow[d, "f"] \\
\mathbb{A} \arrow[r, "\mathrm{id}\otimes\mathbb{A}"'] \arrow[rr, bend right, equal] & \mathbb{A}\otimes\mathbb{A} \arrow[r, "\circ"'] \arrow[r, phantom, shift right=6, "\cong"] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathbb{X}\otimes\mathrm{id}"] \arrow[d, "f"'] \arrow[rr, bend left, equal] \arrow[r, phantom, shift right=6, "\Rightarrow f\otimes\mathrm{id}"] & \mathbb{X}\otimes\mathbb{X} \arrow[r, "\circ"] \arrow[d, "f\otimes f"] \arrow[r, phantom, shift left=6, "\cong"] \arrow[r, phantom, shift right=6, "\Rightarrow \circ"] & \mathbb{X} \arrow[d, "f"] \\
\mathbb{A} \arrow[r, "\mathbb{A}\otimes\mathrm{id}"'] \arrow[rr, bend right, equal] & \mathbb{A}\otimes\mathbb{A} \arrow[r, "\circ"'] \arrow[r, phantom, shift right=6, "\cong"] & \mathbb{A}
\end{tikzcd}$$



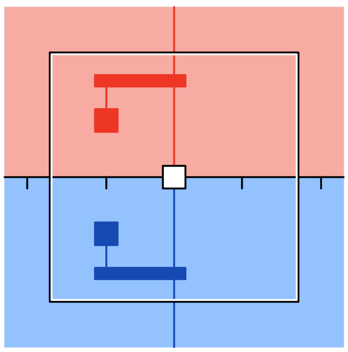
vertical profunctor
matrix profunctor
$f(\underline{f}, \underline{f}) : \mathbb{X}(\underline{\mathbb{X}}, \underline{\mathbb{X}}) \mid \mathbb{A}(\underline{\mathbb{A}}, \underline{\mathbb{A}})$



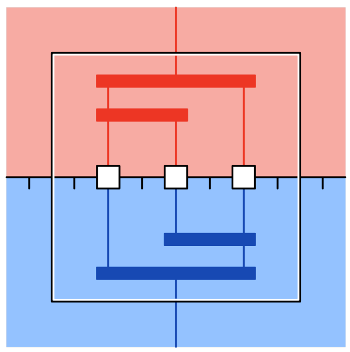
composition
matrix transformation
$\circ : f * f \to f$



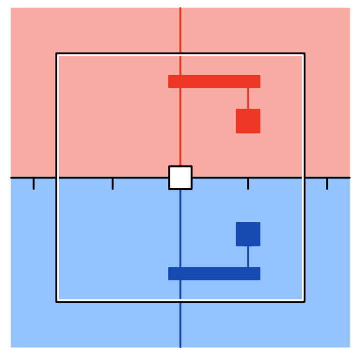
unit
matrix transformation
$\mathrm{id} : \underline{f} \to f$



left unit coherence
$\mathrm{id}.\mathbb{X} \circ \mathbb{X} \rightrightarrows \mathrm{id}.\mathbb{A} \circ \mathbb{A}$



assoc coherence
$(\mathbb{X} \circ \mathbb{X}) \circ \mathbb{X} \rightrightarrows \mathbb{A} \circ (\mathbb{A} \circ \mathbb{A})$



right unit coherence
$\mathbb{X} \circ \mathrm{id}.\mathbb{X} \rightrightarrows \mathbb{A} \circ \mathrm{id}.\mathbb{A}$

Definition 58. Let $\mathbb{A}$ and $\mathbb{B}$ be bifibrant double categories. A **horizontal profunctor** $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$, i.e. **meta relation**, is a matrix category which forms a bimodule of pseudomonads.

Hence it is a matrix category $\mathcal{R} : \underline{\mathbb{A}} \,\|\, \underline{\mathbb{B}}$, with action matrix functors $\mathbb{A} \otimes \mathcal{R} \to \mathcal{R}$ and $\mathcal{R} \otimes \mathbb{B} \to \mathcal{R}$, and invertible matrix transformations for associators and unitors

$$\begin{tikzcd}
\mathbb{A} \otimes \mathbb{A} \otimes \mathcal{R} \arrow[r, "\mathbb{A}\otimes\circ"] \arrow[d, "\circ\otimes\mathcal{R}"'] & \mathbb{A} \otimes \mathcal{R} \arrow[d, "\circ"] \\
\mathbb{A} \otimes \mathcal{R} \arrow[r, "\circ"'] \arrow[ur, phantom, "\Nearrow \alpha_{\mathbb{A}}"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \otimes \mathcal{R} \otimes \mathbb{B} \arrow[r, "\circ\otimes\mathbb{B}"] \arrow[d, "\mathbb{A}\otimes\circ"'] & \mathcal{R} \otimes \mathbb{B} \arrow[d, "\circ"] \\
\mathbb{A} \otimes \mathcal{R} \arrow[r, "\circ"'] \arrow[ur, phantom, "\Nearrow \alpha_{\mathcal{R}}"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \otimes \mathbb{B} \otimes \mathbb{B} \arrow[r, "\circ\otimes\mathbb{B}"] \arrow[d, "\mathcal{R}\otimes\circ"'] & \mathcal{R} \otimes \mathbb{B} \arrow[d, "\circ"] \\
\mathcal{R} \otimes \mathbb{B} \arrow[r, "\circ"'] \arrow[ur, phantom, "\Nearrow \alpha_{\mathbb{B}}"] & \mathcal{R}
\end{tikzcd}$$

$$\alpha_{\mathbb{A}} : (A_1 \circ A_2) \circ R \cong A_1 \circ (A_2 \circ R) \alpha_{\mathcal{R}} : A \circ (R \circ B) \cong (A \circ R) \circ B \alpha_{\mathbb{B}} : R \circ (B_1 \circ B_2) \cong (R \circ B_1) \circ B_2$$

$$\begin{tikzcd}
\mathcal{R} \arrow[d, "\mathcal{R}\otimes\mathrm{id}"'] \arrow[dr, equal] & \\
\mathbb{A} \otimes \mathcal{R} \arrow[r, "\circ"'] \arrow[ur, phantom, "\Swarrow \lambda"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \arrow[d, "\mathcal{R}\otimes\mathrm{id}"'] \arrow[dr, equal] & \\
\mathcal{R} \otimes \mathbb{B} \arrow[r, "\circ"'] \arrow[ur, phantom, "\Swarrow \rho"] & \mathcal{R}
\end{tikzcd}$$

$$\upsilon_{\mathbb{A}} : R \cong \mathrm{id.A} \circ R \qquad \upsilon_{\mathbb{B}} : R \cong R \circ \mathrm{id.B}$$

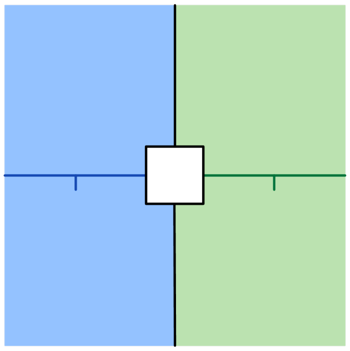
satisfying the associator coherence

$$\begin{tikzcd}
\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A} \otimes \mathcal{R} \arrow[r, shift left=3, "\mathbb{A}\circ(\mathbb{A}\circ(\mathbb{A}\circ\mathcal{R}))"] \arrow[r, shift right=3, "((\mathbb{A}\circ\mathbb{A})\circ\mathbb{A})\circ\mathcal{R}"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \otimes \mathcal{R} \otimes \mathbb{B} \otimes \mathbb{B} \arrow[r, shift left=3, "\mathbb{A}\circ(\mathcal{R}\circ(\mathbb{B}\circ\mathbb{B}))"] \arrow[r, shift right=3, "((\mathbb{A}\circ\mathcal{R})\circ\mathbb{B})\circ\mathbb{B}"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathcal{R}
\end{tikzcd}$$

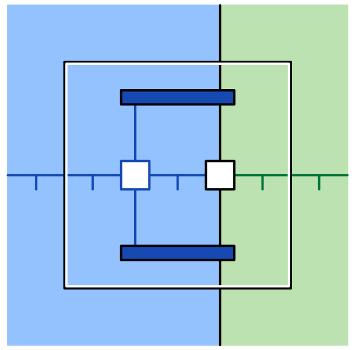
$$\begin{tikzcd}
\mathbb{A} \otimes \mathbb{A} \otimes \mathcal{R} \otimes \mathbb{B} \arrow[r, shift left=3, "(\mathbb{A}\circ\mathbb{A})\circ(\mathcal{R}\circ\mathbb{B})"] \arrow[r, shift right=3, "(\mathbb{A}\circ(\mathbb{A}\circ\mathcal{R}))\circ\mathbb{B}"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \otimes \mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B} \arrow[r, shift left=3, "\mathcal{R}\circ(\mathbb{B}\circ(\mathbb{B}\circ\mathbb{B}))"] \arrow[r, shift right=3, "((\mathcal{R}\circ\mathbb{B})\circ\mathbb{B})\circ\mathbb{B}"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathcal{R}
\end{tikzcd}$$

and unitor coherence.

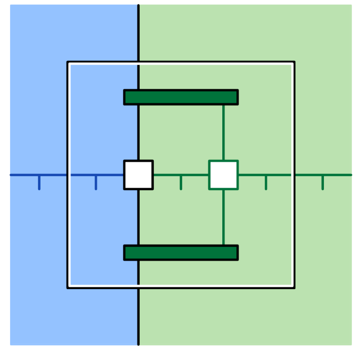
$$\begin{tikzcd}
\mathbb{A} \otimes \mathcal{R} \arrow[r, shift left=3, "\mathbb{A}\circ(\mathrm{id}\circ\mathcal{R})"] \arrow[r, shift right=3, "(\mathbb{A}\circ\mathrm{id})\circ\mathcal{R}"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \otimes \mathbb{B} \arrow[r, shift left=3, "\mathcal{R}\circ(\mathrm{id}\circ\mathbb{B})"] \arrow[r, shift right=3, "(\mathcal{R}\circ\mathrm{id})\circ\mathbb{B}"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & \mathcal{R}
\end{tikzcd}$$



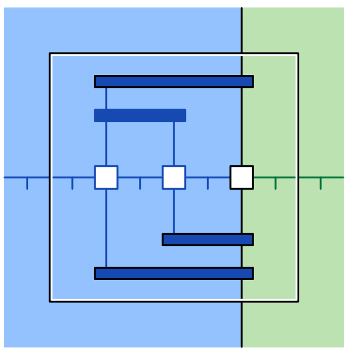
horizontal profunctor
matrix category
$\mathcal{R} : \underline{\mathbb{A}} \,\|\, \underline{\mathbb{B}}$



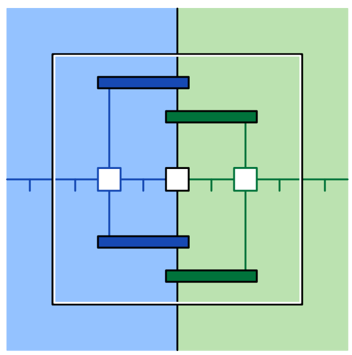
left composition
matrix functor
$\circ : \mathbb{A} * \mathcal{R} \to \mathcal{R}$



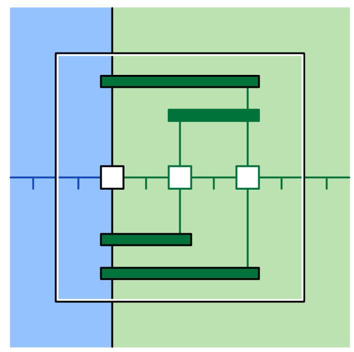
right composition
matrix functor
$\circ : \mathcal{R} * \mathbb{B} \to \mathcal{R}$



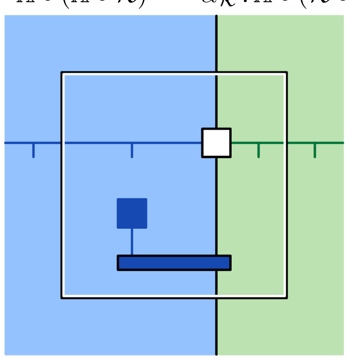
left associator
matrix transformation
$\alpha_{\mathbb{A}} : (\mathbb{A} \circ \mathbb{A}) \circ \mathcal{R} \cong \mathbb{A} \circ (\mathbb{A} \circ \mathcal{R})$



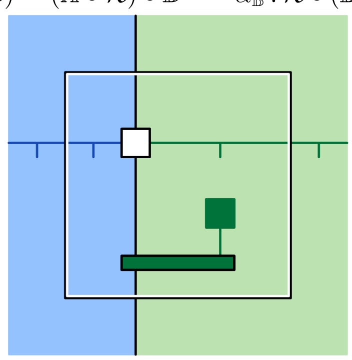
center associator
matrix transformation
$\alpha_{\mathcal{R}} : \mathbb{A} \circ (\mathcal{R} \circ \mathbb{B}) \cong (\mathbb{A} \circ \mathcal{R}) \circ \mathbb{B}$



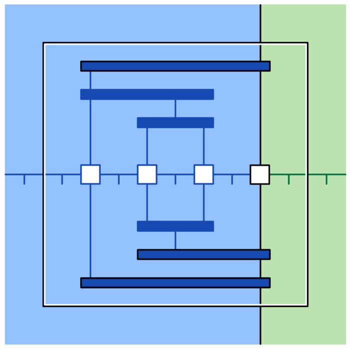
right associator
matrix transformation
$\alpha_{\mathbb{B}} : \mathcal{R} \circ (\mathbb{B} \circ \mathbb{B}) \cong (\mathcal{R} \circ \mathbb{B}) \circ \mathbb{B}$



left unitor
$\lambda : \mathcal{R} \cong \mathbb{A} \circ \mathcal{R}$

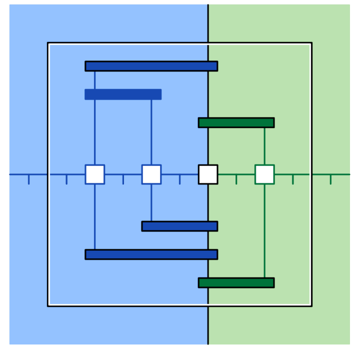


right unitor
$\rho : \mathcal{R} \cong \mathcal{R} \circ \mathbb{B}$



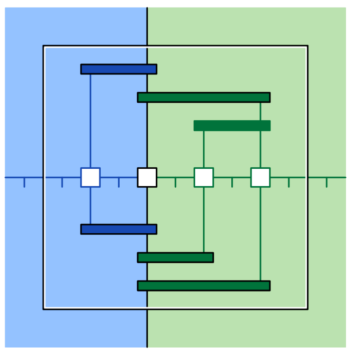
$\mathbb{A}$-assoc coherence

$((\mathbb{A} \circ \mathbb{A}) \circ \mathbb{A}) \circ \mathcal{R} \rightrightarrows \mathbb{A} \circ (\mathbb{A} \circ (\mathbb{A} \circ \mathcal{R}))$



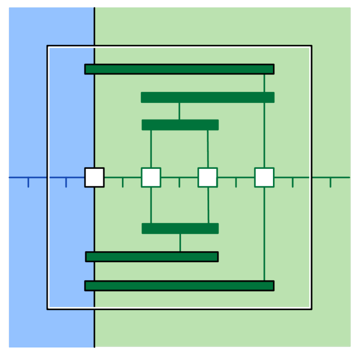
$\mathbb{A}\mathbb{A}\mathbb{B}$-assoc coherence

$(\mathbb{A} \circ \mathbb{A}) \circ (\mathcal{R} \circ \mathbb{B}) \rightrightarrows (\mathbb{A} \circ (\mathbb{A} \circ \mathcal{R})) \circ \mathbb{B}$



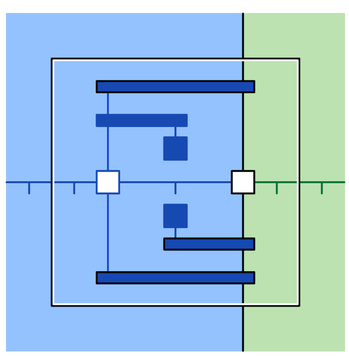
$\mathbb{A}\mathbb{B}\mathbb{B}$-assoc coherence

$\mathbb{A} \circ (\mathcal{R} \circ (\mathbb{B} \circ \mathbb{B})) \rightrightarrows ((\mathbb{A} \circ \mathcal{R}) \circ \mathbb{B}) \circ \mathbb{B}$



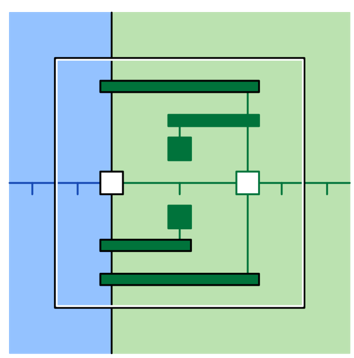
$\mathbb{B}$-assoc coherence

$\mathcal{R} \circ (\mathbb{B} \circ (\mathbb{B} \circ \mathbb{B})) \rightrightarrows ((\mathcal{R} \circ \mathbb{B}) \circ \mathbb{B}) \circ \mathbb{B}$



$\mathbb{A}$-unit coherence

$(\mathbb{A} \circ \mathrm{id}) \circ \mathcal{R} \rightrightarrows \mathbb{A} \circ (\mathrm{id} \circ \mathcal{R})$



$\mathbb{B}$-unit coherence

$\mathcal{R} \circ (\mathrm{id} \circ \mathbb{B}) \rightrightarrows (\mathcal{R} \circ \mathrm{id}) \circ \mathbb{B}$

Definition 59. Let $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$ be bifibrant double categories, let $\mathcal{Q} : \mathbb{X} \,\|\, \mathbb{Y}$ and $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ be horizontal profunctors, and let $f : \mathbb{X} \mid \mathbb{A}$ and $g : \mathbb{Y} \mid \mathbb{B}$ be vertical profunctors.

A **double profunctor**, i.e. **meta inference**, $i(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \mid \mathcal{R}(\mathbb{A}, \mathbb{B})$ is a matrix profunctor which forms a "vertical bimodule" of weak bimodules. Hence it is equipped with action matrix transformations $\circ : f \otimes i \Rightarrow i$ and $\circ : i \otimes g \Rightarrow i$ which cohere with the associators of $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$

$$\begin{tikzcd}
\mathbb{X} \otimes \mathcal{Q} \arrow[rrr] \arrow[ddd, "f \otimes i"'] & & & \mathcal{Q} \arrow[ddd, "i"] \\
& \mathbb{X} \otimes \mathcal{Q} \otimes \mathbb{Y} \arrow[ul] \arrow[r] \arrow[d, "f \otimes i \otimes g"'] \arrow[u, phantom, "\cong"] & \mathcal{Q} \otimes \mathbb{Y} \arrow[ur] \arrow[d, "i \otimes g"] & \\
& \mathcal{R} \otimes \mathbb{A} \otimes \mathbb{B} \arrow[dl] \arrow[r] \arrow[d, phantom, "\cong"] \arrow[u, phantom, "\Leftarrow \circ \otimes g"] & \mathcal{R} \otimes \mathbb{B} \arrow[dr] \arrow[u, phantom, "\Rightarrow \circ"] & \\
\mathbb{A} \otimes \mathcal{R} \arrow[rrr] & & & \mathcal{R}
\end{tikzcd}$$

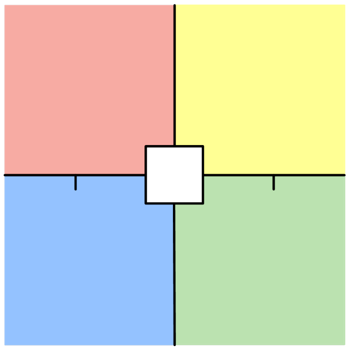
$$\begin{tikzcd}
\mathbb{X} \otimes \mathcal{Q} \arrow[rrr] \arrow[ddd, "f \otimes i"'] & & & \mathcal{Q} \arrow[ddd, "i"] \\
& \mathbb{X} \otimes \mathbb{X} \otimes \mathcal{Q} \arrow[ul] \arrow[r] \arrow[d, "f \otimes f \otimes i"'] \arrow[u, phantom, "\cong"] & \mathbb{X} \otimes \mathcal{Q} \arrow[ur] \arrow[d, "f \otimes i"] & \\
& \mathbb{A} \otimes \mathbb{A} \otimes \mathcal{R} \arrow[dl] \arrow[r] \arrow[d, phantom, "\cong"] \arrow[u, phantom, "\Leftarrow \circ \otimes i"] & \mathbb{A} \otimes \mathcal{R} \arrow[dr] \arrow[u, phantom, "\Rightarrow \circ"] & \\
\mathbb{A} \otimes \mathcal{R} \arrow[rrr] & & & \mathcal{R}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{Q} \otimes \mathbb{Y} \arrow[rrr] \arrow[ddd, "i \otimes g"'] & & & \mathcal{Q} \arrow[ddd, "i"] \\
& \mathcal{Q} \otimes \mathbb{Y} \otimes \mathbb{Y} \arrow[ul] \arrow[r] \arrow[d, "i \otimes g \otimes g"'] \arrow[u, phantom, "\cong"] & \mathcal{Q} \otimes \mathbb{Y} \arrow[ur] \arrow[d, "i \otimes g"] & \\
& \mathcal{R} \otimes \mathbb{B} \otimes \mathbb{B} \arrow[dl] \arrow[r] \arrow[d, phantom, "\cong"] \arrow[u, phantom, "\Leftarrow \circ \otimes g"] & \mathcal{R} \otimes \mathbb{B} \arrow[dr] \arrow[u, phantom, "\Rightarrow \circ"] & \\
\mathcal{R} \otimes \mathbb{B} \arrow[rrr] & & & \mathcal{R}
\end{tikzcd}$$

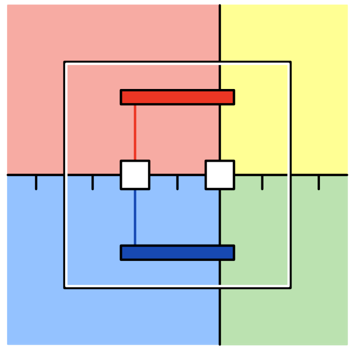
and cohere with the unitors of $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$.

$$
\begin{tikzcd}
 & \mathcal{Q} \arrow[rr, "i"] \arrow[d] \arrow[ddl, equal] \arrow[dl, phantom, "\cong"] \arrow[drr, phantom, "\Downarrow \mathrm{id}\otimes i"] & & \mathcal{R} \arrow[d] \arrow[ddr, equal] \arrow[dr, phantom, "\cong"] & \\
 & \mathbb{X}\otimes\mathcal{Q} \arrow[rr, "f\otimes i"] \arrow[dl] \arrow[drr, phantom, "\Downarrow \circ"] & & \mathbb{A}\otimes\mathcal{R} \arrow[dr] & \\
\mathcal{Q} \arrow[rrrr, "i"'] & & & & \mathcal{R}
\end{tikzcd}
$$

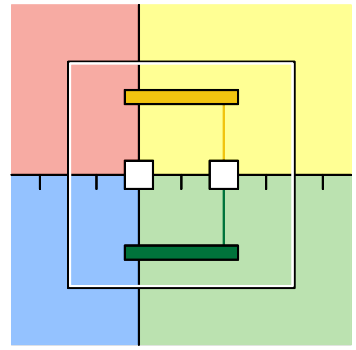
$$
\begin{tikzcd}
 & \mathcal{Q} \arrow[rr, "i"] \arrow[d] \arrow[ddl, equal] \arrow[dl, phantom, "\cong"] \arrow[drr, phantom, "\Downarrow i\otimes\mathrm{id}"] & & \mathcal{R} \arrow[d] \arrow[ddr, equal] \arrow[dr, phantom, "\cong"] & \\
 & \mathcal{Q}\otimes\mathbb{Y} \arrow[rr, "i\otimes g"] \arrow[dl] \arrow[drr, phantom, "\Downarrow \circ"] & & \mathcal{R}\otimes\mathbb{B} \arrow[dr] & \\
\mathcal{Q} \arrow[rrrr, "i"'] & & & & \mathcal{R}
\end{tikzcd}
$$



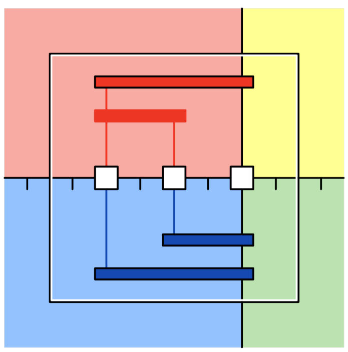
double profunctor
matrix profunctor
$i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$



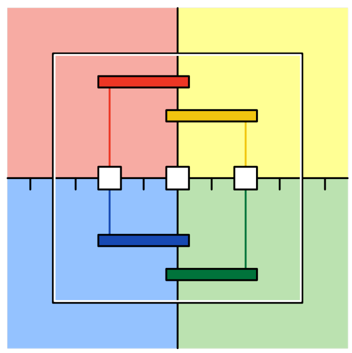
left composition
matrix transformation
$\circ : f \otimes i \to i$



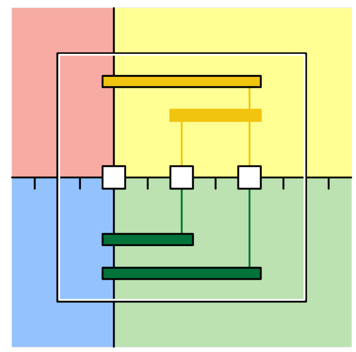
right composition
matrix transformation
$\circ : i \otimes g \to i$



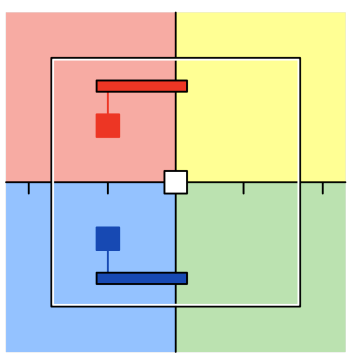
l-assoc coherence
$(\mathbb{X} \circ \mathbb{X}) \circ \mathcal{Q} \rightrightarrows \mathbb{A} \circ (\mathbb{A} \circ \mathcal{R})$



c-assoc coherence
$(\mathbb{X} \circ \mathcal{Q}) \circ \mathbb{Y} \rightrightarrows \mathbb{A} \circ (\mathcal{R} \circ \mathbb{B})$

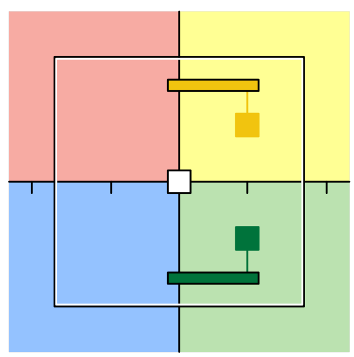


r-assoc coherence
$(\mathcal{Q} \circ \mathbb{Y}) \circ \mathbb{Y} \rightrightarrows \mathcal{R} \circ (\mathbb{B} \circ \mathbb{B})$



l-unit coherence

$\mathrm{id}.\mathbb{X} \circ \mathcal{Q} \rightrightarrows \mathrm{id}.\mathbb{A} \circ \mathcal{R}$



r-unit coherence

$\mathcal{Q} \circ \mathrm{id}.\mathbb{Y} \rightrightarrows \mathcal{R} \circ \mathrm{id}.\mathbb{B}$

3.3 Morphisms [Double transformation]

Definition 60. Let $\mathbb{A}_0, \mathbb{A}_1$ be bifibrant double categories. A **double functor**, i.e. **flow type**, is a morphism of pseudomonads. Hence it is a matrix functor $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ with invertible matrix transformations called the **join** and **unit**

$$\begin{tikzcd}
\mathbb{A}_0 \otimes \mathbb{A}_0 \arrow[r, "{[\![\mathbb{A}]\!] \otimes [\![\mathbb{A}]\!]}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\Downarrow {[\![\circ]\!]}"] & \mathbb{A}_1 \otimes \mathbb{A}_1 \arrow[d, "\circ"] \\
\mathbb{A}_0 \arrow[r, "{[\![\mathbb{A}]\!]}"'] & \mathbb{A}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbb{A}}_0 \arrow[r, "\underline{[\![\mathbb{A}]\!]}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow {[\![\mathrm{id}]\!]}"] & \underline{\mathbb{A}}_1 \arrow[d, "\mathrm{id}"] \\
\mathbb{A}_0 \arrow[r, "{[\![\mathbb{A}]\!]}"'] & \mathbb{A}_1
\end{tikzcd}$$

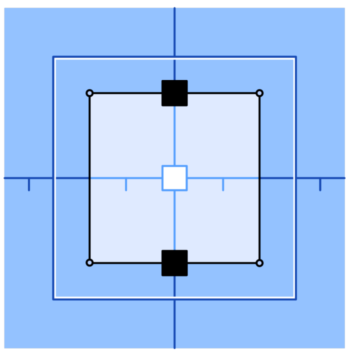
which cohere with the associators of $\mathbb{A}_0, \mathbb{A}_1$

$$\begin{tikzcd}
\mathbb{A}_0 \otimes \mathbb{A}_0 \arrow[rrrr] \arrow[dddd] & & & & \mathbb{A}_1 \otimes \mathbb{A}_1 \arrow[dddd] \\
& \mathbb{A}_0 \otimes \mathbb{A}_0 \otimes \mathbb{A}_0 \arrow[ul] \arrow[rr] \arrow[dd] & \arrow[u, phantom, "\Uparrow {[\![\mathbb{A}]\!] \otimes [\![\circ]\!]}"] & \mathbb{A}_1 \otimes \mathbb{A}_1 \otimes \mathbb{A}_1 \arrow[ur] \arrow[dd] & \\
\arrow[r, phantom, "\cong"] & & \arrow[u, phantom, "\Downarrow {[\![\circ]\!] \otimes [\![\mathbb{A}]\!]}"] \arrow[r, phantom, "\Downarrow {[\![\circ]\!]}"] & & \arrow[l, phantom, "\cong"] \\
& \mathbb{A}_0 \otimes \mathbb{A}_0 \arrow[dl] \arrow[rr] & \arrow[d, phantom, "\Downarrow {[\![\circ]\!]}"] & \mathbb{A}_1 \otimes \mathbb{A}_1 \arrow[dr] & \\
\mathbb{A}_0 \arrow[rrrr] & & & & \mathbb{A}_1
\end{tikzcd}$$

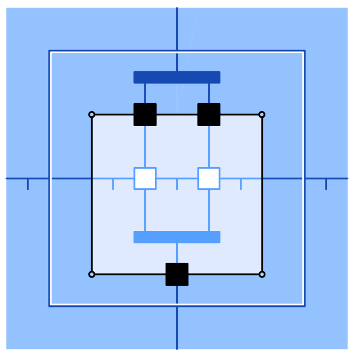
and the unitors of $\mathbb{A}_0, \mathbb{A}_1$.

$$\begin{tikzcd}
& \mathbb{A}_0 \arrow[rr, "{[\![\mathbb{A}]\!]}"] \arrow[d, "\mathrm{id} \otimes \mathbb{A}_0"] \arrow[dl, equal] \arrow[dl, phantom, bend left, "\cong"] & \arrow[d, phantom, "\Downarrow {[\![\mathrm{id}]\!] \otimes [\![\mathbb{A}]\!]}"] & \mathbb{A}_1 \arrow[d, "\mathrm{id} \otimes \mathbb{A}_1"'] \arrow[dr, equal] \arrow[dr, phantom, bend right, "\cong"'] & \\
& \mathbb{A}_0 \otimes \mathbb{A}_0 \arrow[rr, "{[\![\mathbb{A}]\!] \otimes [\![\mathbb{A}]\!]}"] \arrow[dl, "\circ"] & \arrow[d, phantom, "\Downarrow {[\![\circ]\!]}"] & \mathbb{A}_1 \otimes \mathbb{A}_1 \arrow[dr, "\circ"'] & \\
\mathbb{A}_0 \arrow[rrrr, "{[\![\mathbb{A}]\!]}"'] & & & & \mathbb{A}_1
\end{tikzcd}$$

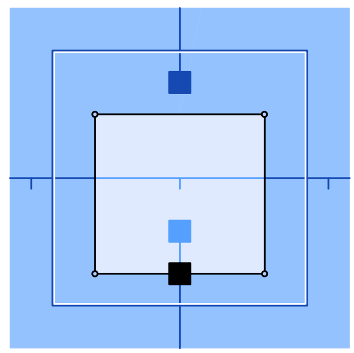
$$
\begin{tikzcd}
 & \mathbb{A}_0 \arrow[rr, "\llbracket\mathbb{A}\rrbracket"] \arrow[d, "\mathbb{A}_0\otimes\mathrm{id}"] \arrow[ddl, equal, "\cong"'] & \arrow[d, phantom, "\Downarrow \llbracket\mathbb{A}\rrbracket\otimes\llbracket\mathrm{id}\rrbracket"] & \mathbb{A}_1 \arrow[d, "\mathbb{A}_1\otimes\mathrm{id}"] \arrow[ddr, equal, "\cong"] & \\
 & \mathbb{A}_0\otimes\mathbb{A}_0 \arrow[rr, "\llbracket\mathbb{A}\rrbracket\otimes\llbracket\mathbb{A}\rrbracket"] \arrow[dl, "\circ"] & \arrow[d, phantom, "\Downarrow \llbracket\circ\rrbracket"] & \mathbb{A}_1\otimes\mathbb{A}_1 \arrow[dr, "\circ"] & \\
\mathbb{A}_0 \arrow[rrrr, "\llbracket\mathbb{A}\rrbracket"'] & & & & \mathbb{A}_1
\end{tikzcd}
$$



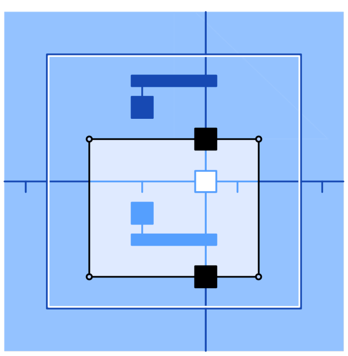
double functor
$\llbracket\mathbb{A}\rrbracket : \mathbb{A}_0 \to \mathbb{A}_1$



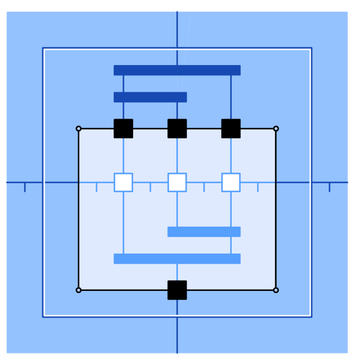
join
$\llbracket\circ\rrbracket : \llbracket\mathbb{A}\rrbracket \circ \llbracket\mathbb{A}\rrbracket \cong \llbracket\mathbb{A} \circ \mathbb{A}\rrbracket$



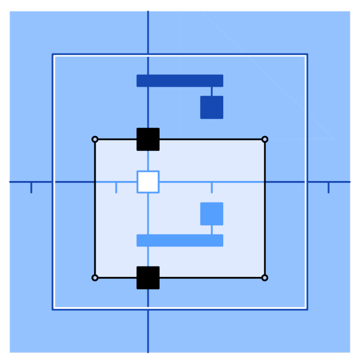
unit
$\llbracket\mathrm{id}\rrbracket : \mathrm{id}.\llbracket\mathbb{A}\rrbracket \cong \llbracket\mathrm{id}.\mathbb{A}\rrbracket$



left unit coherence
$\mathrm{id} \circ \llbracket\mathbb{A}\rrbracket \rightrightarrows \llbracket\mathrm{id} \circ \mathbb{A}\rrbracket$



associator coherence
$(\llbracket\mathbb{A}\rrbracket \circ \llbracket\mathbb{A}\rrbracket) \circ \llbracket\mathbb{A}\rrbracket \rightrightarrows \llbracket\mathbb{A} \circ (\mathbb{A} \circ \mathbb{A})\rrbracket$



right unit coherence
$\llbracket\mathbb{A}\rrbracket \circ \mathrm{id} \rightrightarrows \llbracket\mathbb{A} \circ \mathrm{id}\rrbracket$

Definition 61. Let $\mathbb{X}_0, \mathbb{X}_1, \mathbb{A}_0, \mathbb{A}_1$ be bifibrant double categories, let $\llbracket \mathbb{X} \rrbracket : \mathbb{X}_0 \to \mathbb{X}_1$ and $\llbracket \mathbb{A} \rrbracket : \mathbb{A}_0 \to \mathbb{A}_1$ be double functors, and let $f_0 : \mathbb{X}_0 \mid \mathbb{A}_0$ and $f_1 : \mathbb{X}_1 \mid \mathbb{A}_1$ be vertical profunctors.

A **vertical transformation**, i.e. **flow process**, $\llbracket f \rrbracket(\llbracket \mathbb{X} \rrbracket, \llbracket \mathbb{A} \rrbracket) : f_0(\mathbb{X}_0, \mathbb{A}_0) \Rightarrow f_1(\mathbb{X}_1, \mathbb{A}_1)$ is a transformation of vertical modules.

Hence it is a transformation $\llbracket \underline{f} \rrbracket : \underline{f}_0 \Rightarrow \underline{f}_1$ and a matrix transformation $\llbracket f \rrbracket : f_0 \Rightarrow f_1$

$$
\begin{tikzcd}
\underline{f}_0 \arrow[d, "\llbracket \underline{f} \rrbracket"'] & f_0 \arrow[l] \arrow[r] \arrow[d, "\llbracket f \rrbracket"] & \underline{f}_0 \arrow[d, "\llbracket \underline{f} \rrbracket"] \\
\underline{f}_1 & f_1 \arrow[l] \arrow[r] & \underline{f}_1
\end{tikzcd}
$$

which coheres with the joins of $\llbracket \mathbb{X} \rrbracket$ and $\llbracket \mathbb{A} \rrbracket$.

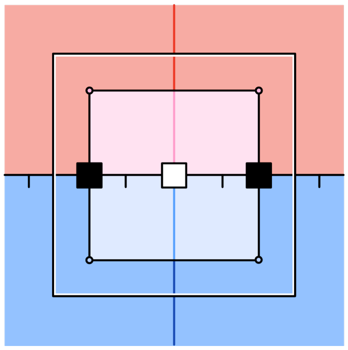
$$
\begin{tikzcd}
\mathbb{X}_0 \arrow[rrrr] \arrow[dddd, "f_0"' description] & & & & \mathbb{X}_1 \arrow[dddd, "f_1" description] \\
& \mathbb{X}_0 * \mathbb{X}_0 \arrow[ul] \arrow[rr] \arrow[dd, "f_0 * f_0" description] & \Uparrow \mu & \mathbb{X}_1 * \mathbb{X}_1 \arrow[ur] \arrow[dd, "f_1 * f_1" description] & \\
\Leftarrow \circ = & & = \llbracket f \rrbracket * \llbracket f \rrbracket \Rightarrow & & = \circ \Rightarrow \\
& \mathbb{A}_0 * \mathbb{A}_0 \arrow[dl] \arrow[rr] & \Downarrow \mu & \mathbb{A}_1 * \mathbb{A}_1 \arrow[dr] & \\
\mathbb{A}_0 \arrow[rrrr] & & & & \mathbb{A}_1
\end{tikzcd}
$$

$$\llbracket f \rrbracket$$

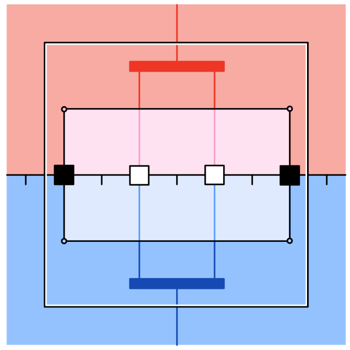
and the units of $\llbracket \mathbb{X} \rrbracket$ and $\llbracket \mathbb{A} \rrbracket$.

$$
\begin{tikzcd}
\mathbb{X}_0 \arrow[rrrr] \arrow[dddd, "f_0"' description] & & & & \mathbb{X}_1 \arrow[dddd, "f_1" description] \\
& \underline{\mathbb{X}}_0 \arrow[ul] \arrow[rr] \arrow[dd, "\underline{f}_0" description] & \Uparrow \eta & \underline{\mathbb{X}}_1 \arrow[ur] \arrow[dd, "\underline{f}_1" description] & \\
\Leftarrow \mathrm{id} = & & = \llbracket \underline{f} \rrbracket \Rightarrow & & = \mathrm{id} \Rightarrow \\
& \underline{\mathbb{A}}_0 \arrow[dl] \arrow[rr] & \Downarrow \eta & \underline{\mathbb{A}}_1 \arrow[dr] & \\
\mathbb{A}_0 \arrow[rrrr] & & & & \mathbb{A}_1
\end{tikzcd}
$$

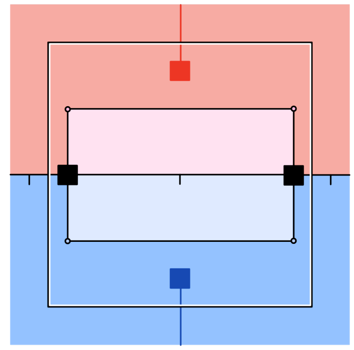
$$\llbracket f \rrbracket$$



vertical transformation
$[\![f]\!] : f_0 \Rightarrow f_1$



join coherence
$[\![\mathbb{X}]\!] \circ [\![\mathbb{X}]\!] \rightrightarrows [\![\mathbb{A} \circ \mathbb{A}]\!]$



unit coherence
$\mathrm{id}.[\![\mathbb{X}]\!] \rightrightarrows [\![\mathrm{id}.\mathbb{A}]\!]$

Definition 62. Let $\mathbb{A}_0, \mathbb{B}_0, \mathbb{A}_1, \mathbb{B}_1$ be bifibrant double categories, let $\llbracket \mathbb{A} \rrbracket : \mathbb{A}_0 \to \mathbb{A}_1$ and $\llbracket \mathbb{B} \rrbracket : \mathbb{B}_0 \to \mathbb{B}_1$ be double functors, and let $\mathcal{R}_0 : \mathbb{A}_0 \parallel \mathbb{B}_0$ and $\mathcal{R}_1 : \mathbb{A}_1 \parallel \mathbb{B}_1$ be horizontal profunctors.

A **horizontal transformation**, i.e. **flow relation**, $\llbracket \mathcal{R} \rrbracket(\llbracket \mathbb{A} \rrbracket, \llbracket \mathbb{B} \rrbracket) : \mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0) \to \mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ is a transformation of weak bimodules. Hence it is a matrix functor $\llbracket \mathcal{R} \rrbracket : \mathcal{R}_0 \to \mathcal{R}_1$ with invertible matrix transformations called **left and right join**

$$
\begin{tikzcd}
\mathbb{A}_0 \otimes \mathcal{R}_0 \arrow[r, "\llbracket \mathbb{A} \rrbracket \otimes \llbracket \mathcal{R} \rrbracket"] \arrow[d, "\circ^0_{\mathbb{A}}"'] \arrow[dr, phantom, "\overset{\parallel}{\llbracket \circ_{\mathbb{A}} \rrbracket} \Downarrow"] & \mathbb{A}_1 \otimes \mathcal{R}_1 \arrow[d, "\circ^1_{\mathbb{A}}"] \\
\mathcal{R}_0 \arrow[r, "\llbracket \mathcal{R} \rrbracket"'] & \mathcal{R}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R}_0 \otimes \mathbb{B}_0 \arrow[r, "\llbracket \mathcal{R} \rrbracket \otimes \llbracket \mathbb{B} \rrbracket"] \arrow[d, "\circ^0_{\mathbb{B}}"'] \arrow[dr, phantom, "\overset{\parallel}{\llbracket \circ_{\mathbb{B}} \rrbracket} \Downarrow"] & \mathcal{R}_1 \otimes \mathbb{B}_1 \arrow[d, "\circ^1_{\mathbb{B}}"] \\
\mathcal{R}_0 \arrow[r, "\llbracket \mathcal{R} \rrbracket"'] & \mathcal{R}_1
\end{tikzcd}
$$

$$
\llbracket \circ_{\mathbb{A}} \rrbracket : \llbracket A_0 \rrbracket \circ_1 \llbracket R_0 \rrbracket \cong \llbracket A_0 \circ_0 R_0 \rrbracket
\qquad
\llbracket \circ_{\mathbb{B}} \rrbracket : \llbracket R_0 \rrbracket \circ_1 \llbracket B_0 \rrbracket \cong \llbracket R_0 \circ_0 B_0 \rrbracket
$$

which coheres with the joins of $\llbracket \mathbb{A} \rrbracket$ and $\llbracket \mathbb{B} \rrbracket$, along the associators of $\mathcal{R}_0$ and $\mathcal{R}_1$

$$
\begin{tikzcd}
\mathbb{A}_0 \otimes \mathcal{R}_0 \arrow[rrr] \arrow[ddd] & & & \mathbb{A}_1 \otimes \mathcal{R}_1 \arrow[ddd] \\
& \mathbb{A}_0 \otimes \mathbb{A}_0 \otimes \mathcal{R}_0 \arrow[ul] \arrow[r] \arrow[d] & \mathbb{A}_1 \otimes \mathbb{A}_1 \otimes \mathcal{R}_1 \arrow[ur] \arrow[d] & \\
& \mathbb{A}_0 \otimes \mathcal{R}_0 \arrow[dl] \arrow[r] & \mathbb{A}_1 \otimes \mathcal{R}_1 \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrr] & & & \mathcal{R}_1
\arrow[from=1-2, to=2-3, phantom, "\llbracket \circ_{\mathbb{A}} \rrbracket \otimes \llbracket \mathcal{R} \rrbracket \Uparrow"]
\arrow[from=2-2, to=3-3, phantom, "\llbracket \mathbb{A} \rrbracket \otimes \llbracket \circ_{\mathbb{A}} \rrbracket \Downarrow \ \llbracket \circ_{\mathbb{A}} \rrbracket \Downarrow"]
\arrow[from=3-2, to=4-3, phantom, "\overset{\parallel}{\llbracket \circ_{\mathbb{A}} \rrbracket} \Downarrow"]
\arrow[from=2-1, to=3-2, phantom, "\cong"]
\arrow[from=2-3, to=3-4, phantom, "\cong"]
\end{tikzcd}
$$

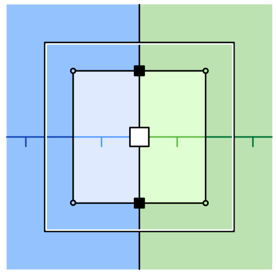
$$
\begin{tikzcd}
\mathbb{A}_0 \otimes \mathcal{R}_0 \arrow[rrr] \arrow[ddd] & & & \mathbb{A}_1 \otimes \mathcal{R}_1 \arrow[ddd] \\
& \mathbb{A}_0 \otimes \mathcal{R}_0 \otimes \mathbb{B}_0 \arrow[ul] \arrow[r] \arrow[d] & \mathbb{A}_1 \otimes \mathcal{R}_1 \otimes \mathbb{B}_1 \arrow[ur] \arrow[d] & \\
& \mathcal{R}_0 \otimes \mathbb{B}_0 \arrow[dl] \arrow[r] & \mathcal{R}_1 \otimes \mathbb{B}_1 \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrr] & & & \mathcal{R}_1
\arrow[from=1-2, to=2-3, phantom, "\llbracket \circ_{\mathbb{A}} \rrbracket \otimes \llbracket \mathbb{B} \rrbracket \Uparrow"]
\arrow[from=2-2, to=3-3, phantom, "\llbracket \mathbb{A} \rrbracket \otimes \llbracket \circ_{\mathbb{B}} \rrbracket \Downarrow \ \llbracket \circ_{\mathbb{A}} \rrbracket \Downarrow"]
\arrow[from=3-2, to=4-3, phantom, "\overset{\parallel}{\llbracket \circ_{\mathbb{B}} \rrbracket} \Downarrow"]
\arrow[from=2-1, to=3-2, phantom, "\cong"]
\arrow[from=2-3, to=3-4, phantom, "\cong"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{R}_0 \otimes \mathbb{B}_0 \arrow[rrrr] \arrow[ddd] & & \arrow[d, phantom, "\Uparrow \llbracket \circ_{\mathbb{B}} \rrbracket \otimes \llbracket \mathbb{B} \rrbracket"] \arrow[ddd, phantom, "\Downarrow \llbracket \circ_{\mathbb{B}} \rrbracket"'] & & \mathcal{R}_1 \otimes \mathbb{B}_1 \arrow[ddd] \\
\arrow[d, phantom, "\cong"] & \mathcal{R}_0 \otimes \mathbb{B}_0 \otimes \mathbb{B}_0 \arrow[ul] \arrow[rr] \arrow[d] & \arrow[d, phantom, "\Downarrow \llbracket \mathcal{R} \rrbracket \otimes \llbracket \circ_{\mathbb{B}} \rrbracket"] & \mathcal{R}_1 \otimes \mathbb{B}_1 \otimes \mathbb{B}_1 \arrow[ur] \arrow[d] & \arrow[d, phantom, "\cong"] \\
 & \mathcal{R}_0 \otimes \mathbb{B}_0 \arrow[rr] \arrow[dl] & \arrow[d, phantom, "\Downarrow \llbracket \circ_{\mathbb{B}} \rrbracket"] & \mathcal{R}_1 \otimes \mathbb{B}_1 \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrrr] & & & & \mathcal{R}_1
\end{tikzcd}
$$

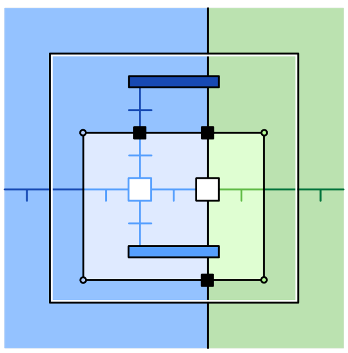
and the units of $\llbracket \mathbb{A} \rrbracket$ and $\llbracket \mathbb{B} \rrbracket$.

$$
\begin{tikzcd}
 & \mathcal{R}_0 \arrow[r, "\llbracket \mathcal{R} \rrbracket"] \arrow[d] \arrow[ddl, equal, "\cong"'] \arrow[dr, phantom, "\Downarrow \llbracket \mathrm{id} \rrbracket \otimes \llbracket \mathcal{R} \rrbracket"] & \mathcal{R}_1 \arrow[d] \arrow[ddr, equal, "\cong"] & \\
 & \mathbb{A}_0 \otimes \mathcal{R}_0 \arrow[r, "\llbracket \mathbb{A} \rrbracket \otimes \llbracket \mathcal{R} \rrbracket"] \arrow[dl] \arrow[dr, phantom, "\Downarrow \llbracket \circ_{\mathbb{A}} \rrbracket"] & \mathbb{A}_1 \otimes \mathcal{R}_1 \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrr, "\llbracket \mathcal{R} \rrbracket"] & & & \mathcal{R}_1
\end{tikzcd}
$$

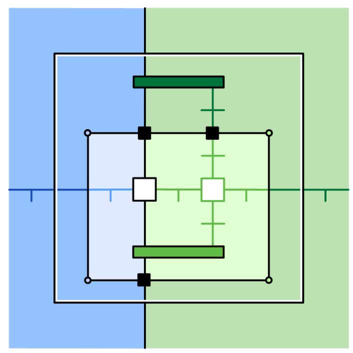
$$
\begin{tikzcd}
 & \mathcal{R}_0 \arrow[r, "\llbracket \mathcal{R} \rrbracket"] \arrow[d] \arrow[ddl, equal, "\cong"'] \arrow[dr, phantom, "\Downarrow \llbracket \mathcal{R} \rrbracket \otimes \llbracket \mathrm{id} \rrbracket"] & \mathcal{R}_1 \arrow[d] \arrow[ddr, equal, "\cong"] & \\
 & \mathcal{R}_0 \otimes \mathbb{B}_0 \arrow[r, "\llbracket \mathcal{R} \rrbracket \otimes \llbracket \mathbb{B} \rrbracket"] \arrow[dl] \arrow[dr, phantom, "\Downarrow \llbracket \circ_{\mathbb{B}} \rrbracket"] & \mathcal{R}_1 \otimes \mathbb{B}_1 \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrr, "\llbracket \mathcal{R} \rrbracket"] & & & \mathcal{R}_1
\end{tikzcd}
$$



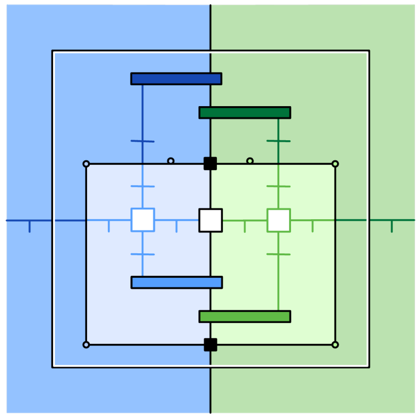
horizontal transformation
matrix functor
$[\![\mathcal{R}]\!]([\![\mathbb{A}]\!],[\![\mathbb{B}]\!]) : \mathcal{R}_0(\mathbb{A}_0,\mathbb{B}_0) \to \mathcal{R}_1(\mathbb{A}_1,\mathbb{B}_1)$



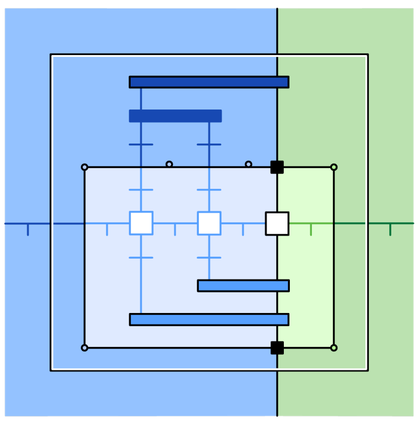
left join
matrix transformation
$[\![\circ_{\mathbb{A}}]\!] : [\![\mathbb{A}]\!] \circ [\![\mathcal{R}]\!] \cong [\![\mathbb{A} \circ \mathcal{R}]\!]$



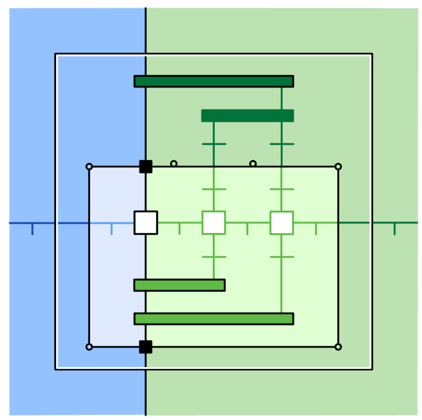
right join
matrix transformation
$[\![\circ_{\mathbb{B}}]\!] : [\![\mathcal{R}]\!] \circ [\![\mathbb{B}]\!] \cong [\![\mathcal{R} \circ \mathbb{B}]\!]$



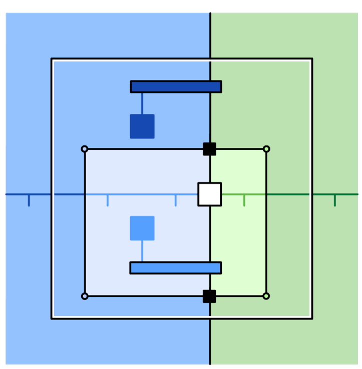
center assoc. coherence
equality
$[\![A]\!] \circ ([\![R]\!] \circ [\![B]\!]) \rightrightarrows [\![(A \circ R) \circ B]\!]$



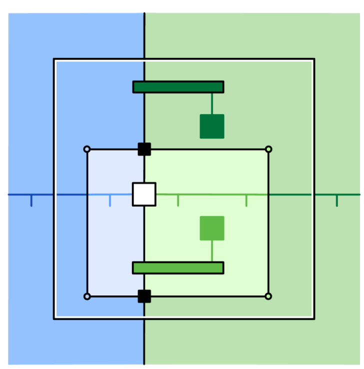
left assoc. coherence
equality
$([\![A_1]\!] \circ [\![A_2]\!]) \circ [\![R]\!] \rightrightarrows [\![A_1 \circ (A_2 \circ R)]\!]$



right assoc coherence
equality
$[\![R]\!] \circ ([\![B_1]\!] \circ [\![B_2]\!]) \rightrightarrows [\![(R \circ B_1) \circ B_2]\!]$



left unit coherence
equality
$\mathrm{id}.[\![\mathrm{A}]\!] \circ [\![R]\!] \rightrightarrows [\![\mathrm{id}.\mathrm{A} \circ R]\!]$



right unit coherence
equality
$[\![R]\!] \circ \mathrm{id}.[\![\mathrm{B}]\!] \rightrightarrows [\![R \circ \mathrm{id}.\mathrm{B}]\!]$

Definition 63. Let $i_0(f_0, g_0) : \mathcal{Q}_0(\mathbb{X}_0, \mathbb{Y}_0) \mid \mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$ and $i_1(f_1, g_1) : \mathcal{Q}_1(\mathbb{X}_1, \mathbb{Y}_1) \mid \mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ be matrix profunctors. Let $\llbracket \mathbb{X} \rrbracket : \mathbb{X}_0 \to \mathbb{X}_1$ etc. be double functors, $\llbracket f \rrbracket(\llbracket \mathbb{X} \rrbracket, \llbracket \mathbb{A} \rrbracket) : f_0 \Rightarrow f_1$, $\llbracket g \rrbracket(\llbracket \mathbb{Y} \rrbracket, \llbracket \mathbb{B} \rrbracket) : g_0 \Rightarrow g_1$ be vertical transformations, and $\llbracket \mathcal{Q} \rrbracket(\llbracket \mathbb{X} \rrbracket, \llbracket \mathbb{Y} \rrbracket) : \mathcal{Q}_0 \to \mathcal{Q}_1$, $\llbracket \mathcal{R} \rrbracket(\llbracket \mathbb{A} \rrbracket, \llbracket \mathbb{B} \rrbracket) : \mathcal{R}_0 \to \mathcal{R}_1$ be horizontal transformations.

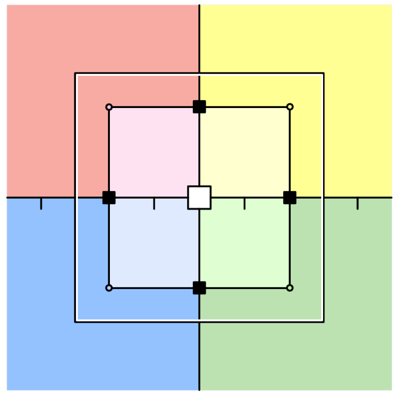
A **double transformation**, i.e. **flow inference** or simply **flow**, $\llbracket i \rrbracket(\llbracket f \rrbracket, \llbracket g \rrbracket) : i_0(f_0, g_0) \Rightarrow i_1(f_1, g_1)$ is a transformation of vertical bimodules of weak bimodules. Hence it is a matrix transformation

$$
\begin{tikzcd}
f_0 \arrow[d, "\llbracket f \rrbracket"] & i_0 \arrow[l] \arrow[r] \arrow[d, "\llbracket i \rrbracket"] & g_0 \arrow[d, "\llbracket g \rrbracket"] \\
f_1 & i_1 \arrow[l] \arrow[r] & g_1
\end{tikzcd}
$$

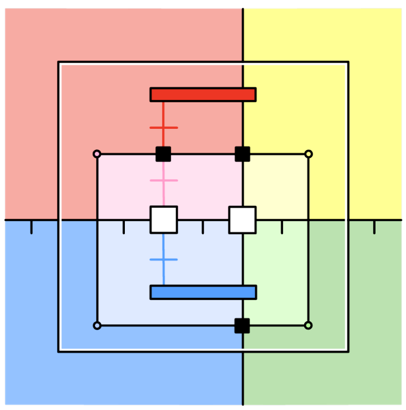
which coheres with the left and right joins of the horizontal transformations.

$$
\begin{tikzcd}
\mathcal{Q}_0 \arrow[rrrr, "\llbracket \mathcal{Q} \rrbracket"] \arrow[dddd, "i_0"'] & & & & \mathcal{Q}_1 \arrow[dddd, "i_1"] \\
& \mathbb{X}_0 \otimes \mathcal{Q}_0 \arrow[ul] \arrow[rr, "\llbracket \mathbb{X} \rrbracket \otimes \llbracket \mathcal{Q} \rrbracket"] \arrow[dd, "f_0 \otimes i_0"] & \arrow[u, Rightarrow, "\llbracket \circ_{\mathbb{X}} \rrbracket"] & \mathbb{X}_1 \otimes Q_1 \arrow[ur] \arrow[dd, "f_1 \otimes i_1"] & \\
\arrow[r, phantom, "\Leftarrow \circ_0 ="] & \arrow[rr, phantom, "\llbracket f \rrbracket \otimes \llbracket i \rrbracket \Rightarrow"] & & \arrow[r, phantom, "= \circ_1 \Rightarrow"] & \\
& \arrow[rr, Rightarrow, "\llbracket i \rrbracket"] \mathbb{A}_0 \otimes \mathcal{R}_0 \arrow[dl] \arrow[rr, "\llbracket \mathbb{A} \rrbracket \otimes \llbracket \mathcal{R} \rrbracket"'] & \arrow[d, Rightarrow, "\llbracket \circ_{\mathbb{A}} \rrbracket"] & \mathbb{A}_1 \otimes \mathcal{R}_1 \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrrr, "\llbracket \mathcal{R} \rrbracket"'] & & & & \mathcal{R}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\llbracket X \rrbracket \circ \llbracket Q \rrbracket \arrow[r, "\llbracket \circ_{\mathbb{X}} \rrbracket"] \arrow[d, "\llbracket f \rrbracket \circ \llbracket i \rrbracket"] & \llbracket X \circ Q \rrbracket \arrow[d, "\llbracket f \circ i \rrbracket"] \\
\llbracket A \rrbracket \circ \llbracket R \rrbracket \arrow[r, "\llbracket \circ_{\mathbb{A}} \rrbracket"] & \llbracket A \circ R \rrbracket
\end{tikzcd}
$$

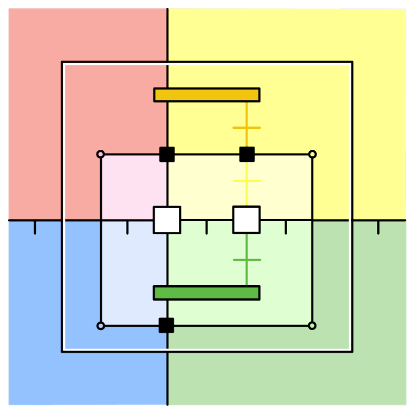
$$
\begin{tikzcd}
\mathcal{Q}_0 \arrow[rrrr, "\llbracket \mathcal{Q} \rrbracket"] \arrow[dddd, "i_0"'] & & & & \mathcal{Q}_1 \arrow[dddd, "i_1"] \\
& \mathcal{Q}_0 \otimes \mathbb{Y}_0 \arrow[ul] \arrow[rr, "\llbracket \mathcal{Q} \rrbracket \otimes \llbracket \mathbb{Y} \rrbracket"] \arrow[dd, "i_0 \otimes g_0"] & \arrow[u, Rightarrow, "\llbracket \circ_{\mathbb{Y}} \rrbracket"] & \mathcal{Q}_1 \otimes \mathbb{Y}_1 \arrow[ur] \arrow[dd, "i_1 \otimes g_1"] & \\
\arrow[r, phantom, "\Leftarrow \circ_0 ="] & \arrow[rr, phantom, "\llbracket i \rrbracket \otimes \llbracket g \rrbracket \Rightarrow"] & & \arrow[r, phantom, "= \circ_1 \Rightarrow"] & \\
& \arrow[rr, Rightarrow, "\llbracket i \rrbracket"] \mathcal{R}_0 \otimes \mathbb{B}_0 \arrow[dl] \arrow[rr, "\llbracket \mathcal{R} \rrbracket \otimes \llbracket \mathbb{B} \rrbracket"'] & \arrow[d, Rightarrow, "\llbracket \circ_{\mathbb{B}} \rrbracket"] & \mathcal{R}_1 \otimes \mathbb{B}_1 \arrow[dr] & \\
\mathcal{R}_0 \arrow[rrrr, "\llbracket \mathcal{R} \rrbracket"'] & & & & \mathcal{R}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\llbracket Q \rrbracket \circ \llbracket Y \rrbracket \arrow[r, "\llbracket \circ_{\mathbb{Y}} \rrbracket"] \arrow[d, "\llbracket i \rrbracket \circ \llbracket g \rrbracket"] & \llbracket Q \circ Y \rrbracket \arrow[d, "\llbracket i \circ g \rrbracket"] \\
\llbracket R \rrbracket \circ \llbracket B \rrbracket \arrow[r, "\llbracket \circ_{\mathbb{B}} \rrbracket"] & \llbracket R \circ B \rrbracket
\end{tikzcd}
$$



double transformation
matrix transformation
$[\![i]\!]([\![f]\!],[\![g]\!]) : i_0(f_0,g_0) \Rightarrow i_1(f_1,g_1)$



left join coherence
equality
$[\![X]\!] \circ [\![Q]\!] \Rrightarrow [\![A \circ R]\!]$



right join coherence
equality
$[\![Q]\!] \circ [\![Y]\!] \Rrightarrow [\![R \circ B]\!]$

3.4 The metalogic of logics

Proposition 64. Bifibrant double categories and functors, vertical profunctors and transformations form a double category, which we call $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$.

Proof. Given double functors $[\![\mathbb{A}]\!]_1 : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{A}]\!]_2 : \mathbb{A}_1 \to \mathbb{A}_2$, the composite $[\![[\![\mathbb{A}]\!]_1]\!]_2 : \mathbb{A}_0 \to \mathbb{A}_2$ is a double functor, with structure given by $[\![\circ]\!]_1 \circ [\![\circ]\!]_2$ and $[\![\mathrm{id}]\!]_1 \circ [\![\mathrm{id}]\!]_2$; these satisfy the coherence by composing equations. Composition of double functors is clearly associative and unital.

$$
\begin{tikzcd}
\mathbb{A}_0 \otimes \mathbb{A}_0 \arrow[r, "{[\![\mathbb{A}_0]\!] \otimes [\![\mathbb{A}_0]\!]}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{[\![\circ]\!]_1}}"] & \mathbb{A}_1 \otimes \mathbb{A}_1 \arrow[r, "{[\![\mathbb{A}_1]\!] \otimes [\![\mathbb{A}_1]\!]}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{[\![\circ]\!]_2}}"] & \mathbb{A}_2 \otimes \mathbb{A}_2 \arrow[d, "\circ"'] \\
\mathbb{A}_0 \arrow[r, "{[\![\mathbb{A}_0]\!]}"'] & \mathbb{A}_1 \arrow[r, "{[\![\mathbb{A}_1]\!]}"'] & \mathbb{A}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbb{A}}_0 \arrow[r, "{[\![\underline{\mathbb{A}}_0]\!]}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{[\![\mathrm{id}]\!]_1}}"] & \underline{\mathbb{A}}_1 \arrow[r, "{[\![\underline{\mathbb{A}}_1]\!]}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{[\![\mathrm{id}]\!]_2}}"] & \underline{\mathbb{A}}_2 \arrow[d, "\mathrm{id}"'] \\
\mathbb{A}_0 \arrow[r, "{[\![\mathbb{A}_0]\!]}"'] & \mathbb{A}_1 \arrow[r, "{[\![\mathbb{A}_1]\!]}"'] & \mathbb{A}_2
\end{tikzcd}
$$

Composition of vertical transformations is given in the same way.

$$
\begin{tikzcd}
\mathbb{X}_0 \arrow[rr, "{[\![\mathbb{X}_0]\!]}"] \arrow[dddd, "f_0"'] & & \mathbb{X}_1 \arrow[rr, "{[\![\mathbb{X}_1]\!]}"] & & \mathbb{X}_2 \arrow[dddd, "f_2"] \\
& & \arrow[ul, phantom, "\Uparrow {[\![\circ]\!]_0}"] \arrow[ur, phantom, "\Uparrow {[\![\circ]\!]_1}"] & & \\
& \mathbb{X}_0 \otimes \mathbb{X}_0 \arrow[uul, "\circ"] \arrow[r, "{[\![\mathbb{X}_0]\!] \otimes [\![\mathbb{X}_0]\!]}"] \arrow[d, "f_0 \otimes f_0"'] & \mathbb{X}_1 \otimes \mathbb{X}_1 \arrow[uu, "\circ"'] \arrow[r, "{[\![\mathbb{X}_1]\!] \otimes [\![\mathbb{X}_1]\!]}"] \arrow[d, "f_1 \otimes f_1"] & \mathbb{X}_2 \otimes \mathbb{X}_2 \arrow[uur, "\circ"'] \arrow[d, "f_2 \otimes f_2"] & \\
& \mathbb{A}_0 \otimes \mathbb{A}_0 \arrow[ddl, "\circ"'] \arrow[r, "{[\![\mathbb{A}_0]\!] \otimes [\![\mathbb{A}_0]\!]}"'] & \mathbb{A}_1 \otimes \mathbb{A}_1 \arrow[dd, "\circ"] \arrow[r, "{[\![\mathbb{A}_1]\!] \otimes [\![\mathbb{A}_1]\!]}"'] & \mathbb{A}_2 \otimes \mathbb{A}_2 \arrow[ddr, "\circ"] & \\
\mathbb{A}_0 & & \mathbb{A}_1 \arrow[ll, "{[\![\mathbb{A}_0]\!]}"] \arrow[rr, "{[\![\mathbb{A}_1]\!]}"'] & & \mathbb{A}_2
\end{tikzcd}
$$

$$\Leftarrow \circ = f_0 \otimes f_0 = [\![f_0]\!] \otimes [\![f_0]\!] \Rightarrow f_1 \otimes f_1 = [\![f_1]\!] \otimes [\![f_1]\!] \Rightarrow f_2 \otimes f_2 = \circ \Rightarrow \qquad = [\![f_0]\!] \cdot [\![f_1]\!] \Rightarrow \qquad \Downarrow [\![\circ]\!]_0 \qquad \Downarrow [\![\circ]\!]_1$$

So it remains to define sequential composition of vertical profunctors, and verify that it is functorial, i.e. preserves composition of vertical transformations.

Consider the following sequential composite of vertical profunctors.

$$
\begin{tikzcd}
\underline{\mathbb{X}} \arrow[d, "\underline{f}"', "\shortmid" marking] \arrow[dr, phantom, "\Longleftarrow"] & \mathbb{X} \arrow[l] \arrow[r] \arrow[d, "f" description] \arrow[dr, phantom, "\Longrightarrow"] & \underline{\mathbb{X}} \arrow[d, "\underline{f}", "\shortmid" marking] \\
\underline{\mathbb{Y}} \arrow[d, "\underline{g}"', "\shortmid" marking] \arrow[dr, phantom, "\Longleftarrow"] & \mathbb{Y} \arrow[l] \arrow[r] \arrow[d, "g" description] \arrow[dr, phantom, "\Longrightarrow"] & \underline{\mathbb{Y}} \arrow[d, "\underline{g}", "\shortmid" marking] \\
\underline{\mathbb{Z}} & \mathbb{Z} \arrow[l] \arrow[r] & \underline{\mathbb{Z}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbb{X}} \arrow[d, "\underline{f\circ g}"', "\shortmid" marking] \arrow[dr, phantom, "\Longleftarrow"] & \mathbb{X} \arrow[l] \arrow[r] \arrow[d, "f\circ g" description] \arrow[dr, phantom, "\Longrightarrow"] & \underline{\mathbb{X}} \arrow[d, "\underline{f\circ g}", "\shortmid" marking] \\
\underline{\mathbb{Z}} & \mathbb{Z} \arrow[l] \arrow[r] & \underline{\mathbb{Z}}
\end{tikzcd}
$$

In the same way as for matrix profunctors, the equalities adjoined by the quotient are represented by squares in $\mathbb{Y}$. The sequential composite matrix profunctor $f \diamond g : \mathbb{X} \mid \mathbb{Z}$ is a vertical profunctor, with composition and unit given by sequentially composing that of f and g.

$$
\begin{tikzcd}
\mathbb{X} \otimes \mathbb{X} \arrow[r, "\circ" description] \arrow[d, "f\otimes f"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\circ}{\Longrightarrow}"] & \mathbb{X} \arrow[d, "f", "\shortmid" marking] \\
\mathbb{Y} \otimes \mathbb{Y} \arrow[r, "\circ" description] \arrow[d, "g\otimes g"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\circ}{\Longrightarrow}"] & \mathbb{Y} \arrow[d, "g", "\shortmid" marking] \\
\mathbb{Z} \otimes \mathbb{Z} \arrow[r, "\circ" description] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbb{X}} \arrow[r, "\mathrm{id}" description] \arrow[d, "\underline{f}"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Longrightarrow}"] & \mathbb{X} \arrow[d, "f", "\shortmid" marking] \\
\underline{\mathbb{Y}} \arrow[r, "\mathrm{id}" description] \arrow[d, "\underline{g}"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Longrightarrow}"] & \mathbb{Y} \arrow[d, "g", "\shortmid" marking] \\
\underline{\mathbb{Z}} \arrow[r, "\mathrm{id}" description] & \mathbb{Z}
\end{tikzcd}
$$

Again, these satisfy the coherence simply by composing equations.

Sequential composition of vertical profunctors is functorial: let $[\![f]\!] : f_0 \Rightarrow f_1$ and $[\![g]\!] : g_0 \Rightarrow g_1$ be vertical transformations; then $([\![f]\!] \diamond [\![g]\!]) : (f_0 \diamond g_0) \Rightarrow (f_1 \diamond g_1)$ is defined by sequential composition.

$$
\begin{tikzcd}
\mathbb{X}_0 \arrow[rrrr] \arrow[ddd, "f_0"'] & & \Uparrow\mu & & \mathbb{X}_1 \arrow[ddd, "f_1"] \\
& \mathbb{X}_0 \otimes \mathbb{X}_0 \arrow[ul] \arrow[rr] \arrow[dd, "f_0 \otimes f_0"'] \arrow[rr, phantom, shift right=4, "{[\![f]\!]}"] & & \mathbb{X}_1 \otimes \mathbb{X}_1 \arrow[ur] \arrow[dd, "f_1 \otimes f_1"] & \\
\Leftarrow \circ & & {[\![f]\!] \otimes [\![f]\!]} \Longrightarrow & & \circ \Rightarrow \\
\mathbb{Y}_0 \arrow[ddd, "g_0"'] & \mathbb{Y}_0 \otimes \mathbb{Y}_0 \arrow[l] \arrow[rr] \arrow[dd, "g_0 \otimes g_0"'] & & \mathbb{Y}_1 \otimes \mathbb{Y}_1 \arrow[r] \arrow[dd, "g_1 \otimes g_1"] & \mathbb{Y}_1 \arrow[ddd, "g_1"] \\
\Leftarrow \circ & & {[\![g]\!] \otimes [\![g]\!]} \Longrightarrow & & \circ \Rightarrow \\
& \mathbb{Z}_0 \otimes \mathbb{Z}_0 \arrow[dl] \arrow[rr] \arrow[rr, phantom, shift left=4, "{[\![g]\!]}"] & & \mathbb{Z}_1 \otimes \mathbb{Z}_1 \arrow[dr] & \\
\mathbb{Z}_0 \arrow[rrrr] & & \Downarrow\mu & & \mathbb{Z}_1
\end{tikzcd}
$$

This preserves composition of transformations: picture two of the above such cubes, composed from left to right. So, sequential composition is functorial.

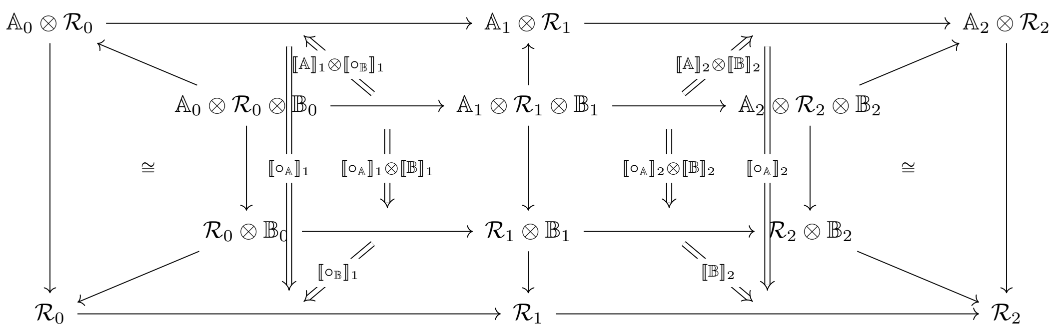
Hence bifibrant double categories and double functors, vertical profunctors and vertical transformations form a double category $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$. $\square$

Proposition 65. Horizontal profunctors and transformations, double profunctors and transformations form a double category, which we call $bf.\mathrm{DblProf}$.

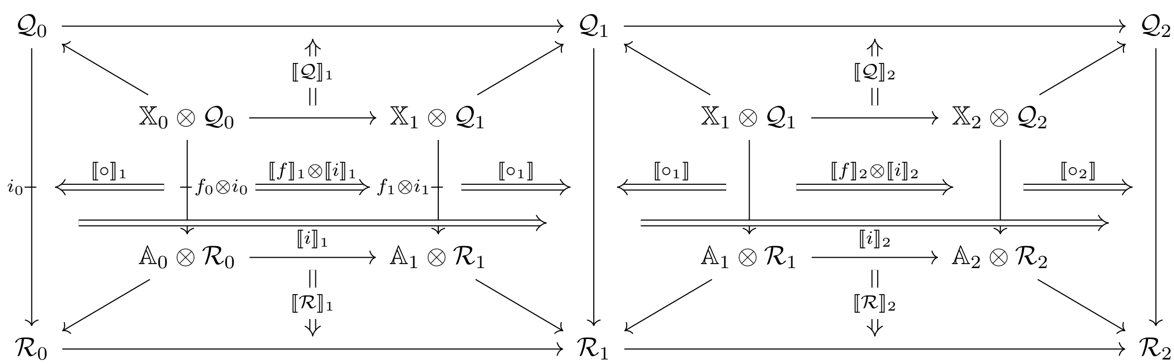
Proof. Composition of horizontal transformations $[\![\mathcal{R}]\!]_1 : \mathcal{R}_0 \to \mathcal{R}_1$ and $[\![\mathcal{R}]\!]_2 : \mathcal{R}_1 \to \mathcal{R}_2$ is defined by that of matrix functors, and that of the joins.

$$
\begin{tikzcd}
\mathbb{A}_0 \otimes \mathcal{R}_0 \arrow[r, "{[\![\mathbb{A}]\!]_1 \otimes [\![\mathcal{R}]\!]_1}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "{\Downarrow [\![\circ_{\mathbb{A}}]\!]_1}"] & \mathbb{A}_1 \otimes \mathcal{R}_1 \arrow[r, "{[\![\mathbb{A}]\!]_2 \otimes [\![\mathcal{R}]\!]_2}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "{\Downarrow [\![\circ_{\mathbb{A}}]\!]_2}"] & \mathbb{A}_2 \otimes \mathcal{R}_2 \arrow[d, "\circ"] \\
\mathcal{R}_0 \arrow[r, "{[\![\mathcal{R}]\!]_1}"'] & \mathcal{R}_1 \arrow[r, "{[\![\mathcal{R}]\!]_2}"'] & \mathcal{R}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R}_0 \otimes \mathbb{B}_0 \arrow[r, "{[\![\mathcal{R}]\!]_1 \otimes [\![\mathbb{B}]\!]_1}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "{\Downarrow [\![\circ_{\mathbb{B}}]\!]_1}"] & \mathcal{R}_1 \otimes \mathbb{B}_1 \arrow[r, "{[\![\mathcal{R}]\!]_2 \otimes [\![\mathbb{B}]\!]_2}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "{\Downarrow [\![\circ_{\mathbb{B}}]\!]_2}"] & \mathcal{R}_2 \otimes \mathbb{B}_2 \arrow[d, "\circ"] \\
\mathcal{R}_0 \arrow[r, "{[\![\mathcal{R}]\!]_1}"'] & \mathcal{R}_1 \arrow[r, "{[\![\mathcal{R}]\!]_2}"'] & \mathcal{R}_2
\end{tikzcd}
$$

This coheres with the associators, simply by composing equations.



Composition of double transformations $[\![i]\!]_1 \cdot [\![i]\!]_2 : i_0 \Rightarrow i_2$ is defined by that of matrix transformations, and again this coheres with the joins of $[\![[\![\mathcal{Q}]\!]_1]\!]_2$ and $[\![[\![\mathcal{R}]\!]_1]\!]_2$ by composing equations.



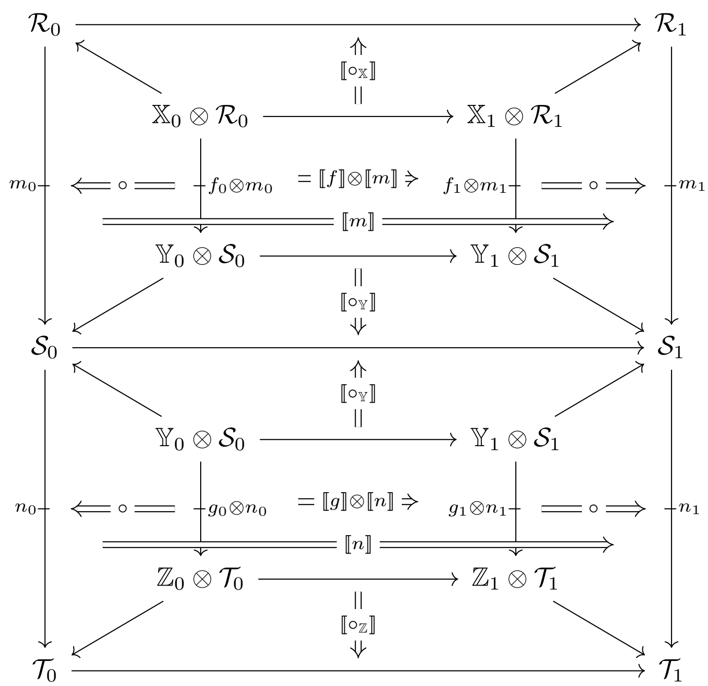
So it remains to define sequential composition of double profunctors, and verify that it is functorial. Just as composition of matrix profunctors is defined by that of span profunctors (Def. 43), composition of double profunctors is defined by that of matrix profunctors.

$$
\begin{tikzcd}
\mathbb{X} \arrow[d, "f"'] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "m"] \arrow[l, phantom, "\Longleftarrow"] \arrow[r, phantom, "\Longrightarrow"] & \mathbb{A} \arrow[d, "k"] \\
\mathbb{Y} \arrow[d, "g"'] & \mathcal{S} \arrow[l] \arrow[r] \arrow[d, "n"] \arrow[l, phantom, "\Longleftarrow"] \arrow[r, phantom, "\Longrightarrow"] & \mathbb{B} \arrow[d, "\ell"] \\
\mathbb{Z} & \mathcal{T} \arrow[l] \arrow[r] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[d, "f \circ g"'] & \mathcal{R} \arrow[l] \arrow[r] \arrow[d, "m \circ n"] \arrow[l, phantom, "\Longleftarrow"] \arrow[r, phantom, "\Longrightarrow"] & \mathbb{A} \arrow[d, "k \circ \ell"] \\
\mathbb{Z} & \mathcal{T} \arrow[l] \arrow[r] & \mathbb{C}
\end{tikzcd}
$$

So the sequential composite double transformation is given by the composite matrix transformation.

$$\begin{tikzcd}
\mathcal{R}_0 \arrow[r, "m_0", "\shortmid"{marking}] \arrow[d, "{[\![\mathcal{R}]\!]}"'] \arrow[dr, phantom, "{\Downarrow [\![m]\!]}"] & \mathcal{S}_0 \arrow[r, "n_0", "\shortmid"{marking}] \arrow[d, "{[\![\mathcal{S}]\!]}"'] \arrow[dr, phantom, "{\Downarrow [\![n]\!]}"] & \mathcal{T}_0 \arrow[d, "{[\![\mathcal{T}]\!]}"] \\
\mathcal{R}_1 \arrow[r, "m_1", "\shortmid"{marking}] & \mathcal{S}_1 \arrow[r, "n_1", "\shortmid"{marking}] & \mathcal{T}_1
\end{tikzcd}$$

The coherence with the joins is given by composing equations.



Sequential composition of double transformations preserves transformation composition, because that of matrix transformations does. Thus, horizontal profunctors and transformations, double profunctors and transformations form a double category $bf.\mathrm{DblProf}$.

$$\begin{tikzcd}
\mathcal{R}_0 \arrow[r, "m_0", "\shortmid"{marking}] \arrow[d, "{[\![\mathcal{R}]\!]}"'] \arrow[dr, phantom, "{\Downarrow [\![m]\!]}"] & \mathcal{S}_0 \arrow[d, "{[\![\mathcal{S}]\!]}"] \\
\mathcal{R}_1 \arrow[r, "m_1"', "\shortmid"{marking}] & \mathcal{S}_1
\end{tikzcd}$$

$\square$

Proposition 66. $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at} \leftarrow bf.\mathrm{DblProf} \rightarrow bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ is a fibered logic.

Proof. Substitution of double functors in a horizontal profunctor, and vertical transformations in a double profunctor, are defined in the same way as that of functors in matrix categories and transformations in matrix profunctors, by pullback. Sequential composition of vertical profunctors preserves this substitution, in the same way as for matrix profunctors. □

Parallel composition

We now define parallel composition of horizontal profunctors, and show that this forms the metalogic of bifibrant double categories.

Composition is defined in the same way as for matrix categories, in Section 2.5: by a codescent object, which adjoins a coherent associator for the middle action — in fact, all the proofs are essentially the same. The only difference is now $\mathbb{B}$ is a general bifibrant double category, rather than an weave double category $\langle\mathbb{B}\rangle$, so the action of $\mathbb{B}$ is composition by its horizontal morphisms, i.e. relations.

The construction gives a well-defined composition of a metalogic, because composition along a matrix category is pullback along a fibration, which preserves colimits [20, Prop 4.3].

Definition 67. Let $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \,\|\, \mathbb{C}$ be horizontal profunctors. The **parallel composite** $\mathcal{R} \otimes S : \mathbb{A} \,\|\, \mathbb{C}$ is defined as follows. First, to the composite matrix category $\mathcal{R} \otimes_{\mathbb{M}} \S : \mathbb{A} \,\|\, \mathbb{C}$ we adjoin for every horizontal morphism $B : \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$ an associator $\mathrm{B}_0.(R, B \circ S) \cong \mathrm{B}_1.(R \circ B, S)$, by forming the following iso-coinserter.

$$\begin{tikzcd}
\mathcal{R} \otimes_{\mathbb{M}} (\mathbb{B} \otimes_{\mathbb{M}} \mathcal{S}) \arrow[r, "\mathcal{R}\otimes_{\mathbb{M}}\circ_{\mathbb{B}}"] \arrow[dd, "\cong"] \arrow[ddr, phantom, "\Downarrow \alpha_{\mathcal{R}\mathcal{S}}"] & \mathcal{R} \otimes_{\mathbb{M}} \mathcal{S} \arrow[dr, squiggly, "\iota"] & \\
& & (\mathcal{R} \otimes \mathcal{S})_{\alpha} \\
(\mathcal{R} \otimes_{\mathbb{M}} \mathbb{B}) \otimes_{\mathbb{M}} \mathcal{S} \arrow[r, "\circ_{\mathbb{B}}\otimes_{\mathbb{M}}\mathcal{S}"'] & \mathcal{R} \otimes_{\mathbb{M}} \mathcal{S} \arrow[ur, squiggly, "\iota"'] &
\end{tikzcd}$$

This associator is natural by its universal construction, so for every square $b : \mathbb{B}(B_0, B_1)$ and $r : \mathcal{R}(R_0, R_1)$,

$s : \mathcal{S}(S_0, S_1)$ the following commutes.

$$\begin{tikzcd}
(R_0, B_0 \circ S_0) \arrow[r, "\alpha_{\mathcal{R}\mathcal{S}}"] \arrow[d, "(r, b\circ s)"] & (R_0 \circ B_0, S_0) \arrow[d, "(r\circ b, s)"] \\
(R_1, B_1 \circ S_1) \arrow[r, "\alpha_{\mathcal{R}\mathcal{S}}"] & (R_1 \circ B_1, S_1)
\end{tikzcd}$$

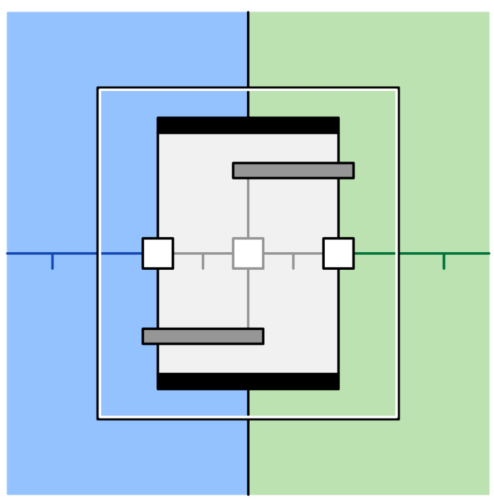
Then we form the following coequifier, for reassociating a composite and a unit.

$$\begin{tikzcd}
\mathcal{R} \otimes_{\mathbb{M}} \mathbb{B} \otimes_{\mathbb{M}} \mathbb{B} \otimes_{\mathbb{M}} \mathcal{S} \arrow[r, shift left, "\mathrm{B}_0.(R, B_1\circ(B_2\circ S))"] \arrow[r, shift right, "\mathrm{B}_2.((R\circ B_1)\circ B_2, S)"'] & (\mathcal{R} \otimes \mathcal{S})_\alpha \arrow[r, squiggly, "co.equif"] & (\mathcal{R} \otimes \mathcal{S})_\beta
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{R} \otimes_{\mathbb{M}} \mathcal{S} \arrow[r, shift left, "\mathrm{B}.(R, U_{\mathrm{B}}\circ S)"] \arrow[r, shift right, "\mathrm{B}.(R\circ U_{\mathrm{B}}, S)"'] & (\mathcal{R} \otimes \mathcal{S})_\beta \arrow[r, squiggly, "co.equif"] & \mathcal{R} \otimes \mathcal{S}
\end{tikzcd}$$

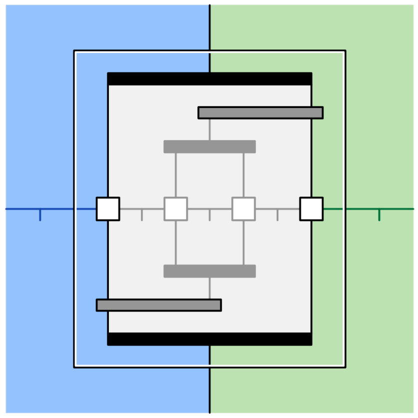
This defines the parallel composite horizontal profunctor $\mathcal{R} \otimes \mathcal{S} : bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}(\mathbb{A}, \mathbb{C})$.

The parallel composite consists of pairs of relations and pairs of inferences, plus a new associator.

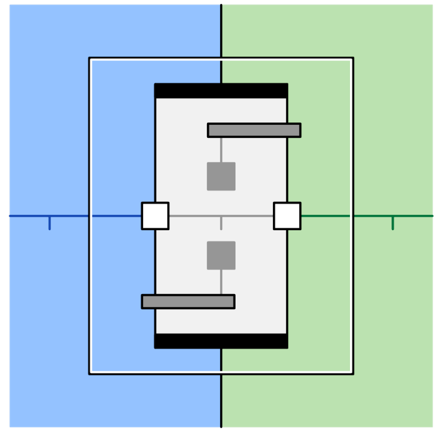


parallel composite metarelation

This associator is natural, and coherent with parallel composition and identity of $\mathbb{B}$.



parcomp associator coherence



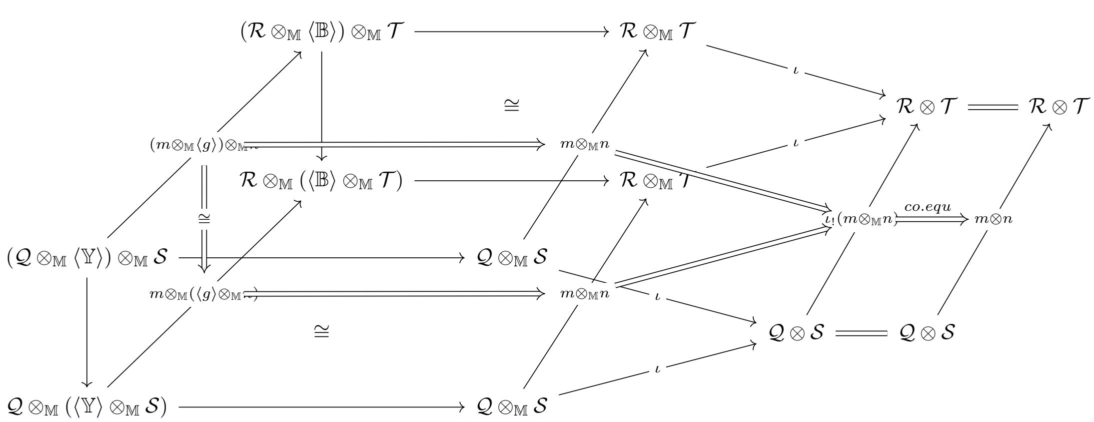
parcomp unitor coherence

Next, we define parallel composition of double profunctors along vertical profunctors.

Definition 68. Let $m(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$ and $n(g,h) : \mathcal{S}(\mathbb{Y},\mathbb{Z}) : \mathcal{T}(\mathbb{B},\mathbb{C})$ be double profunctors, composable along the vertical profunctor $g : \mathbb{Y} \mid \mathbb{B}$.

$$\begin{tikzcd}
\mathbb{X} \arrow[d, "f"', "|" marking] & \mathcal{Q} \arrow[l] \arrow[r] \arrow[d, "m"] \arrow[l, phantom, shift right=3ex, "\Leftarrow"] \arrow[r, phantom, shift right=3ex, "\Rightarrow"] & \mathbb{Y} \arrow[d, "g"] & \mathcal{S} \arrow[l] \arrow[r] \arrow[d, "n"] \arrow[l, phantom, shift right=3ex, "\Leftarrow"] \arrow[r, phantom, shift right=3ex, "\Rightarrow"] & \mathbb{Z} \arrow[d, "h", "|" marking] \\
\mathbb{A} & \mathcal{R} \arrow[l] \arrow[r] & \mathbb{B} & \mathcal{T} \arrow[l] \arrow[r] & \mathbb{C}
\end{tikzcd}$$

The **parallel composite** $(m \otimes n)(f,h) : (\mathcal{Q} \otimes \mathcal{S})(\mathbb{X},\mathbb{Z}) \mid (\mathcal{R} \otimes \mathcal{T})(\mathbb{A},\mathbb{C})$ is defined as the following coequalizer.



The profunctor $\iota_!(m \otimes_{\mathbb{M}} n)$ forms all composites of elements $\mathrm{g}.(m,n)$ and the morphisms of $\mathcal{Q} \otimes \mathcal{S}$ and $\mathcal{R} \otimes \mathcal{T}$. Then, the coequalizer imposes that the associators are natural with respect to the elements.

So the elements of the composite $(m \otimes n)(f, h) : (\mathcal{Q} \otimes \mathcal{S})(\mathbb{X}, \mathbb{Z}) \mid (\mathcal{R} \otimes \mathcal{T})(\mathbb{A}, \mathbb{C})$ are composites of:

morphisms	$\mathrm{y}.(q, s)$:	$(\mathcal{Q} \otimes \mathcal{S})(\mathrm{Y}_0.(Q_0, S_0), \mathrm{Y}_1.(Q_1, S_1))$
associators	$\alpha_{\mathcal{QS}}$:	$(\mathcal{Q} \otimes \mathcal{S})(\mathrm{Y}_0.(Q, Y \circ S), \mathrm{Y}_1.(Q \circ Y, S))$
elements	$\mathrm{g}.(m, n)$:	$(m \circ_{\mathbb{M}} n)(\mathrm{Y}.(Q, S), \mathrm{B}.(R, T))$
associators	$\alpha_{\mathcal{RT}}$:	$(\mathcal{R} \otimes \mathcal{T})(\mathrm{B}_0.(R, B \circ T), \mathrm{B}_1.(R \circ B, T))$
morphisms	$\mathrm{b}.(r, t)$:	$(\mathcal{R} \otimes \mathcal{T})(\mathrm{B}_0.(R_0, T_0), \mathrm{B}_1.(R_1, T_1))$

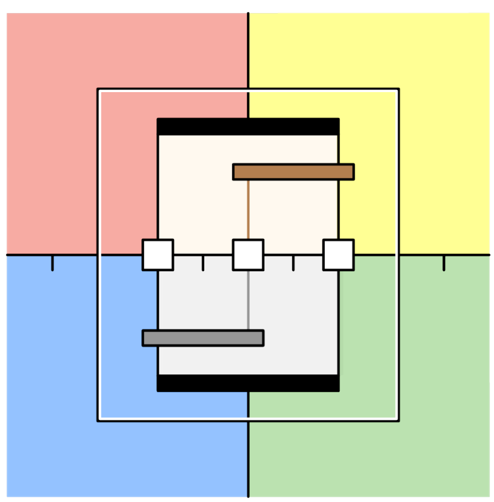
such that for any $g : g(\mathrm{g}_0, \mathrm{g}_1)(Y, B)$ and $m : m(\mathrm{f}, \mathrm{g}_0)$, $n : n(\mathrm{g}_1, \mathrm{h})$ the following commutes.

$$\begin{tikzcd}
\mathrm{Y}_0.(Q, Y \circ S) \arrow[r, "\alpha_{\mathcal{QS}}"] \arrow[d, "\mathrm{g}_0.(m, g \circ n)"'] & \mathrm{Y}_1.(Q \circ Y, S) \arrow[d, "\mathrm{g}_1.(m \circ g, n)"] \\
\mathrm{B}_0.(R, B \circ T) \arrow[r, "\alpha_{\mathcal{RT}}"'] & \mathrm{B}_1.(R \circ B, T)
\end{tikzcd}$$

We denote the composite by the same “arrow sum” notation as for horizontal profunctors.

$$(m \otimes n)(\mathrm{f}, \mathrm{h}) \quad \equiv \quad \vec{\Sigma}\mathrm{g} : g.\ m(\mathrm{f}, \mathrm{g}) \times n(\mathrm{g}, \mathrm{h})$$

The parallel composite matrix profunctor can be drawn as follows.



Parallel composition of horizontal profunctors and double profunctors is functorial in the same way as matrix categories and matrix profunctors, by functoriality of colimit.

Yet just as for matrix profunctors, parallel composition does not preserve sequential composition of horizontal profunctors. So following definition 54, bifibrant double categories form a metalogic.

Theorem 69. Bifibrant double categories form a metalogic.

Morphisms are double functors, vertical profunctors, and horizontal profunctors; squares are vertical transformations, horizontal transformations, and double profunctors; and cubes are double transformations.

$$bf.\mathrm{Dbl}\mathbb{C}\mathrm{at} \leftarrow bf.\mathrm{DblProf} \rightarrow bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$$

Proof. Let $\mathbb{DC}$ be the category of bifibrant double categories and double functors, and let $\mathbb{VP}$ be the category of vertical profunctors and vertical transformations; so $\mathbb{DC} \leftarrow \mathbb{VP} \rightarrow \mathbb{DC}$ is $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$.

Let $\mathbb{HP}$ be the category of horizontal profunctors and horizontal transformations, and let $\mathbb{DP}$ be the category of double profunctors and double transformations; so $\mathbb{HP} \leftarrow \mathbb{DP} \rightarrow \mathbb{HP}$ is $bf.\mathrm{DblProf}$.

As we showed, these form a fibered span of logics

$$\begin{tikzcd}
\mathbb{DC} & \mathbb{HP} \arrow[l] \arrow[r] & \mathbb{DC} \\
\mathbb{VP} \arrow[u] \arrow[d] & \mathbb{DP} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathbb{VP} \arrow[u] \arrow[d] \\
\mathbb{DC} & \mathbb{HP} \arrow[l] \arrow[r] & \mathbb{DC}
\end{tikzcd}$$

equipped with span functors for parallel composition and unit:

$$\begin{tikzcd}
\mathbb{HP} *_h \mathbb{HP} \arrow[r, "\otimes" description] & \mathbb{HP} \\
\mathbb{DP} *_h \mathbb{DP} \arrow[u] \arrow[d] \arrow[r, "\otimes" description] & \mathbb{DP} \arrow[u] \arrow[d] \\
\mathbb{HP} *_h \mathbb{HP} \arrow[r, "\otimes" description] & \mathbb{HP}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{DC} \arrow[r, "\mathbb{X}.\mathbb{X}(-,-)"] & \mathbb{HP} \\
\mathbb{VP} \arrow[u] \arrow[d] \arrow[r, "f.f(-,-)" description] & \mathbb{DP} \arrow[u] \arrow[d] \\
\mathbb{DC} \arrow[r, "\mathbb{A}.\mathbb{A}(-,-)"'] & \mathbb{HP}
\end{tikzcd}$$

with span transformations for left and right unitors, forming adjoint equivalences: for every horizontal profunctor $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$, its unitors and associators give the following horizontal transformations.

$$\begin{tikzcd}
\mathcal{R} \arrow[rr, equal] \arrow[dr, "\lambda^\circ"'] & \arrow[d, phantom, "\overset{\shortparallel}{\eta_\lambda}\Downarrow"] & \mathcal{R} \\
& \mathbb{A} \otimes \mathcal{R} \arrow[ur, "\lambda^\bullet"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \arrow[rr, equal] \arrow[dr, "\rho^\circ"'] & \arrow[d, phantom, "\overset{\shortparallel}{\eta_\rho}\Downarrow"] & \mathcal{R} \\
& \mathcal{R} \otimes \mathbb{B} \arrow[ur, "\rho^\bullet"'] &
\end{tikzcd}$$

$$\eta_\lambda = \upsilon_{\mathbb{A}} : R \cong U_{\mathrm{A}} \circ R \qquad \eta_\rho = \upsilon_{\mathbb{B}} : R \cong R \circ U_{\mathrm{B}}$$

$$\begin{tikzcd}
& \mathcal{R} \arrow[dr, "\lambda^\circ"] \arrow[d, phantom, "\overset{\shortparallel}{\varepsilon_\lambda}\Downarrow"] & \\
\mathbb{A} \otimes \mathcal{R} \arrow[ur, "\lambda^\bullet"] \arrow[rr, equal] & & \mathbb{A} \otimes \mathcal{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{R} \arrow[dr, "\rho^\circ"] \arrow[d, phantom, "\overset{\shortparallel}{\varepsilon_\rho}\Downarrow"] & \\
\mathcal{R} \otimes \mathbb{B} \arrow[ur, "\rho^\bullet"] \arrow[rr, equal] & & \mathcal{R} \otimes \mathbb{B}
\end{tikzcd}$$

$$\varepsilon_\lambda = \alpha_{\mathbb{A}} : (U_{\mathrm{A}_0}, A \circ R) \cong (A, R) \qquad \varepsilon_\rho = \alpha_{\mathbb{B}} : (R \circ B, U_{\mathrm{B}_1}) \cong (R, B)$$

Just as in $\mathrm{Mat}\mathbb{C}\mathrm{at}$, the naturality of unitors with respect to elements of double profunctors gives that the above transformations cohere with the unitor transformations for double profunctors, as in a modification.

$$\begin{tikzcd}
& \mathcal{Q} \arrow[r, "i", "\shortmid" marking] \arrow[d] \arrow[dl, equal] \arrow[dr, phantom, "\overset{\shortparallel}{\lambda_i}\Downarrow"] & \mathcal{R} \arrow[d] \arrow[dr, equal] & \\
\mathcal{Q} \arrow[r, phantom, "\cong"] & \mathbb{X} \otimes \mathcal{Q} \arrow[r, "f \otimes i", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\overset{\shortparallel}{\lambda_i^\bullet}\Downarrow"] & \mathbb{A} \otimes \mathcal{R} \arrow[d] \arrow[r, phantom, "\cong"] & \mathcal{R} \\
& \mathcal{Q} \arrow[r, "i"', "\shortmid" marking] \arrow[ul, equal] & \mathcal{R} \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{Q} \arrow[r, "i", "\shortmid" marking] \arrow[d] \arrow[dl, equal] \arrow[dr, phantom, "\overset{\shortparallel}{\rho_i^\bullet}\Downarrow"] & \mathcal{R} \arrow[d] \arrow[dr, equal] & \\
\mathcal{Q} \arrow[r, phantom, "\cong"] & \mathcal{Q} \otimes \mathbb{Y} \arrow[r, "i \otimes g", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\overset{\shortparallel}{\rho_i}\Downarrow"] & \mathcal{R} \otimes \mathbb{B} \arrow[d] \arrow[r, phantom, "\cong"] & \mathcal{R} \\
& \mathcal{Q} \arrow[r, "i"', "\shortmid" marking] \arrow[ul, equal] & \mathcal{R} \arrow[ur, equal] &
\end{tikzcd}$$

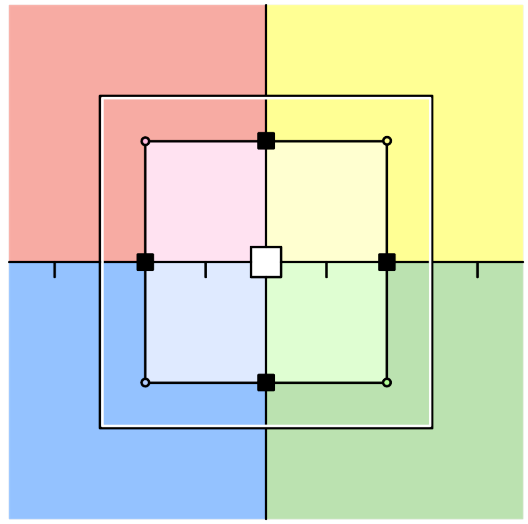
$$\begin{tikzcd}
U_{\mathrm{X}} \circ Q \arrow[r, "\upsilon_{\mathbb{X}}"] \arrow[d, "U_{\mathrm{f}} \circ i"'] & Q \arrow[d, "i"] \\
U_{\mathrm{A}} \circ R \arrow[r, "\upsilon_{\mathbb{A}}"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \circ U_{\mathrm{Y}} \arrow[r, "\upsilon_{\mathbb{Y}}"] \arrow[d, "i \circ U_{\mathrm{g}}"'] & Q \arrow[d, "i"] \\
R \circ U_{\mathrm{B}} \arrow[r, "\upsilon_{\mathbb{B}}"'] & R
\end{tikzcd}$$

The associator is an isomorphism $\mathcal{R} \otimes (\mathcal{S} \otimes \mathcal{T}) \cong (\mathcal{R} \otimes \mathcal{S}) \otimes \mathcal{T}$, with equality pentagonator.

The triangulator is given by the unitors, and its coherence follows from the naturality of the unitors with respect to the associator.

Hence $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ is a metalogic, whose cubes are drawn as follows.

$$
\begin{tikzcd}
\mathbb{X}_1 \arrow[dddd, "f_1"'] & & \mathcal{Q}_1 \arrow[ll] \arrow[rr] & & \mathbb{Y}_1 \arrow[dddd, "g_1"] \\
& \mathbb{X}_0 \arrow[ul, "{[\![\mathrm{X}]\!]}"] \arrow[dd, "f_0"'] & \mathcal{Q}_0 \arrow[l] \arrow[r] \arrow[u, "{[\![\mathcal{Q}]\!]}"] \arrow[dd, "i_0"] & \mathbb{Y}_0 \arrow[ur, "{[\![\mathrm{Y}]\!]}"'] \arrow[dd, "g_0"] & \\
\arrow[r, phantom, "\overset{[\![\mathrm{f}]\!]}{\Leftarrow}"] & \arrow[r, phantom, "\Leftarrow"] & \arrow[d, phantom, "{[\![i]\!]}\Downarrow"] \arrow[r, phantom, "\Rightarrow"] & \arrow[r, phantom, "\overset{[\![\mathrm{g}]\!]}{\Rightarrow}"] & \\
& \mathbb{A}_0 \arrow[dl, "{[\![\mathrm{A}]\!]}"'] & \mathcal{R}_0 \arrow[l] \arrow[r] \arrow[d, "{[\![\mathcal{R}]\!]}"] & \mathbb{B}_0 \arrow[dr, "{[\![\mathrm{B}]\!]}"] & \\
\mathbb{A}_1 & & \mathcal{R}_1 \arrow[ll] \arrow[rr] & & \mathbb{B}_1
\end{tikzcd}
$$



□

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