## Rotation with quaternions

Suppose you have a rotation through an angle $\theta$ about an axis whose unit vector is $\mathbf{n}$. (I will consider vectors like $\mathbf{n}$ to be certain quaternions.) The axis of rotation will make the angles $\alpha$, $\beta$, and $\gamma$ with the $x$, $y$, and $z$ axes respectively. Note that $(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1$. Thus, only 2 of those angles are independent. (More precisely, given 2 of them, there are at most 2 possibilities for the other, instead of infinitely many.) $\mathbf{n} = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma$. The rotation has the associated vector $\theta\mathbf{n}$. This is a quaternion, but it's not a *unit* quaternion, and we know that rotations are described by unit quaternions, which make up the group Spin(3). Now, just as the exponential of a purely imaginary complex number is a unit complex number, so the exponential of a purely vectorial quaternion is a unit quaternion. So you might think the unit quaternion we want is $\mathrm{e}^{\theta\mathbf{n}} = \cos\theta + \mathbf{n}\sin\theta$. But this is not right! If that were the case, a rotation of 360° would be the same as no rotation at all. But Spin(3) is a *double* cover of the rotation group SO(3), and the unit quaternion we want is $R := \mathrm{e}^{\theta\mathbf{n}/2} = \cos(\theta/2) + \mathbf{n}\sin(\theta/2) = \cos(\theta/2) + \mathbf{i}(\cos\alpha)\sin(\theta/2) + \mathbf{j}(\cos\beta)\sin(\theta/2) + \mathbf{k}(\cos\gamma)\sin(\theta/2)$.

How do vectors transform under rotations? Under the rotation $R = \mathrm{e}^{\theta\mathbf{n}/2}$, the vector $\mathbf{v}$ turns into $R\mathbf{v}R^{-1}$. If $\theta$ is changing with time, define the angular velocity $\boldsymbol{\omega}$ to be $\mathbf{n}\mathrm{d}\theta/\mathrm{d}t$, so $\mathrm{d}R/\mathrm{d}t = R\boldsymbol{\omega}/2$. If $\mathbf{v} = R\mathbf{v}_0R^{-1}$, then $\mathrm{d}\mathbf{v}/\mathrm{d}t = \mathrm{d}R\mathbf{v}_0R^{-1}/\mathrm{d}t = (\mathrm{d}R/\mathrm{d}t)\mathbf{v}_0R^{-1} + R\mathbf{v}_0(\mathrm{d}R^{-1}/\mathrm{d}t) = R\boldsymbol{\omega}\mathbf{v}_0R^{-1}/2 - R\mathbf{v}_0R^{-1}(\mathrm{d}R/\mathrm{d}t)R^{-1} = R(\boldsymbol{\omega}\mathbf{v}_0 - \mathbf{v}_0\boldsymbol{\omega})R^{-1}/2$. But $\boldsymbol{\omega}$, which points in the direction of the axis of rotation, is unchanged by the rotation; in other words, $\boldsymbol{\omega}$ commutes with $R$ (and $R^{-1}$). (You can calculate this algebraically by noting that both $\boldsymbol{\omega}$ and the vectorial part of $R$ are parallel to $\mathbf{n}$.) Therefore, $\mathrm{d}\mathbf{v}/\mathrm{d}t = (\boldsymbol{\omega}R\mathbf{v}_0R^{-1} - R\mathbf{v}_0R^{-1}\boldsymbol{\omega})/2 = (\boldsymbol{\omega}\mathbf{v} - \mathbf{v}\boldsymbol{\omega})/2 = \boldsymbol{\omega}\times\mathbf{v}$. Thus, as long as the axis of rotation ($\mathbf{n}$) remains constant, we get the well known cross product formula.

How do spinors transform under rotations? A spinor $S = \left(\begin{smallmatrix} u \\ d \end{smallmatrix}\right)$ with complex components $u$ and $d$ can be represented as the quaternion $\Im u + \mathbf{i}\Re d + \mathbf{j}\Im d + \mathbf{k}\Re u$, where '$\Re$' and '$\Im$' respectively indicate real and imaginary parts. $S$ becomes $RS$ under a rotation. Note that $S$ becomes $-S$ under a rotation of 360°. If $\theta$ is changing with time and $\boldsymbol{\omega} := \mathbf{n}\mathrm{d}\theta/\mathrm{d}t$, let $S$ be $RS_0$. Then $\mathrm{d}S/\mathrm{d}t = \mathrm{d}RS_0/\mathrm{d}t = (\mathrm{d}R/\mathrm{d}t)S_0 = R\boldsymbol{\omega}S_0/2 = \boldsymbol{\omega}RS_0/2 = \boldsymbol{\omega}S/2$. Thus, $\boldsymbol{\omega}S/2$ is the analogue of $\boldsymbol{\omega}\times\mathbf{v}$. The square of a spinor can be represented by a pair of vectors $\Re S^2 := \mathbf{i}\Re(dd - uu)/2 + \mathbf{j}\Im(dd + uu)/2 + \mathbf{k}\Re ud$ and $\Im S^2 := \mathbf{i}\Im(dd - uu)/2 - \mathbf{j}\Re(dd + uu)/2 + \mathbf{k}\Im ud$. (Or combine these into a single complex vector, a biquaternion.) If the spinor is normalised so that $|u|^2 + |d|^2 = 1$, then these vectors are orthonormal. Each of these vectors transforms under rotations like any other vector. As $S$ transforms to $-S$ under a rotation of 360°, so $S^2$ transforms to $(-S)^2 = S^2$ itself, like a vector should.