Quantum Quandaries:
A Category–Theoretic Perspective

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Les Treilles
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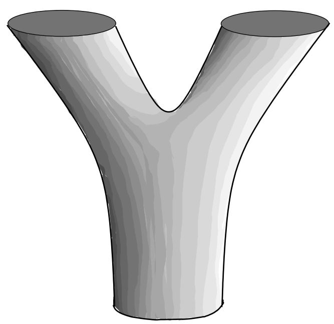
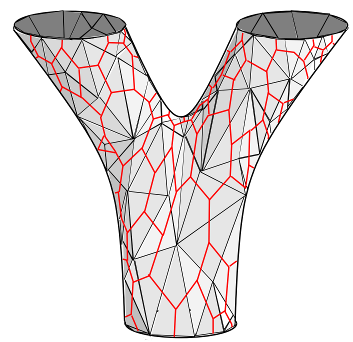


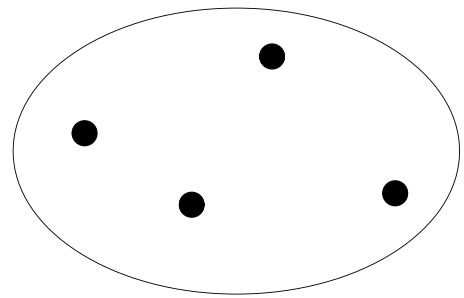


figures by Aaron Lauda

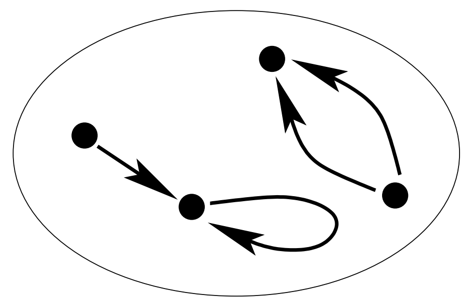
for more, see
http://math.ucr.edu/home/baez/quantum/

The Big Idea

Once upon a time, mathematics was all about *sets*:



In 1945, Eilenberg and Mac Lane introduced *categories*:



Category theory puts *processes* (morphisms):

$$\bullet \longrightarrow \bullet$$

on an equal footing with *things* (objects):

$$\bullet$$

An object of a category may be a set, and a morphism may be a function — but they *don't need to be!* We'll see examples where they're not.

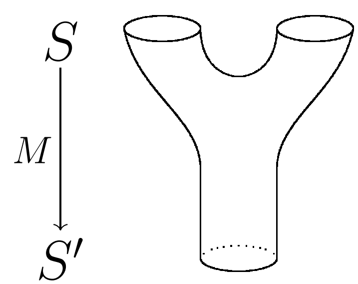
So: ask not what an object is 'made of'. Instead, ask how it's related to other objects... via morphisms!

Categories are already revolutionizing mathematics. What about physics?

I claim:

Quantum theory makes more sense when seen as part of a theory of spacetime — but this can only be clearly understood using categories.

Why? The 'weird' features of quantum theory come from the ways that Hilb is less like Set than nCob — the category where objects are choices of 'space' and morphisms are choices of 'spacetime':

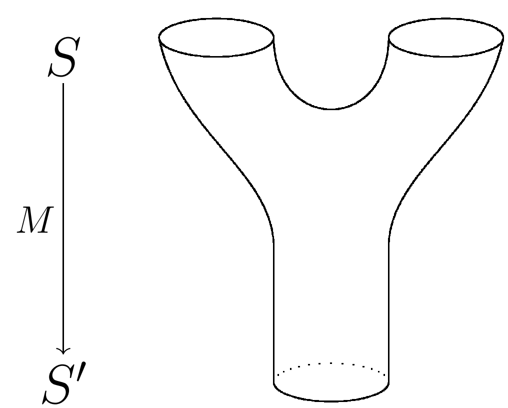


	object $\bullet$	morphism $\bullet \longrightarrow \bullet$
SET THEORY	**set**	**function between sets**
QUANTUM THEORY	**Hilbert space** **(state)**	**operator between Hilbert spaces** **(process)**
GENERAL RELATIVITY	**manifold** **(space)**	**cobordism between manifolds** **(spacetime)**

Objects and Morphisms

Every category has *objects* and *morphisms*:

- **In Set an object is a set, and a morphism is a function.**
- **In Hilb an object is a Hilbert space, and a morphism is a linear operator.**
- **In nCob an object is an $(n-1)$-dim manifold, and a morphism is a cobordism between such manifolds:**



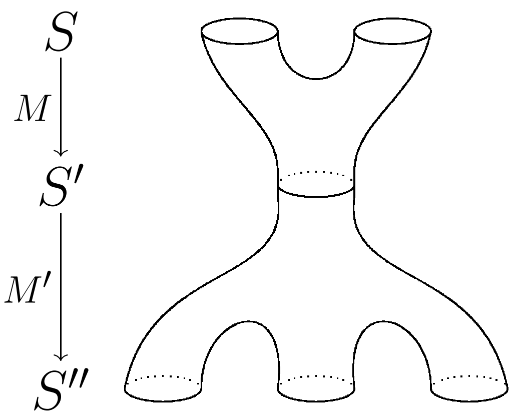
Composition

Every category lets us *compose* morphisms in an associative way:

- **In Set, we compose functions as usual.**
- **In Hilb, we compose operators as usual:**

$$(T'T)\psi = T'(T\psi).$$

- **In nCob, we compose cobordisms like this:**



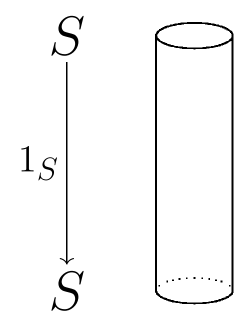
Identity Morphisms

Every category has an *identity morphism* $1_x \colon x \to x$ for each object x:

- **In Set, $1_S \colon S \to S$ is the identity function.**
- **In Hilb, $1_H \colon H \to H$ is the identity operator:**

$$1_H \psi = \psi.$$

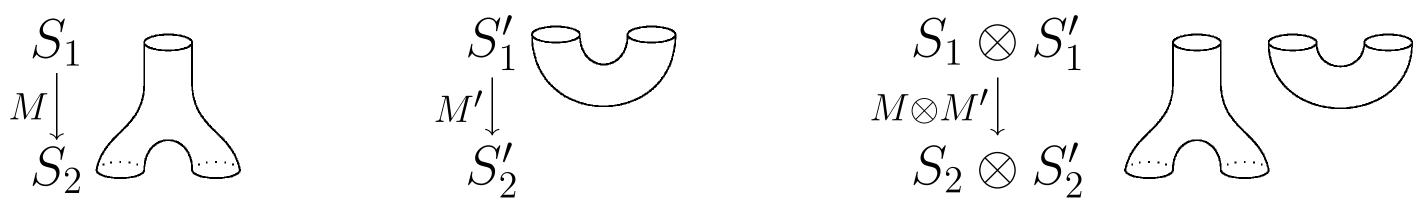
- **In nCob, identity morphisms look like this:**



Monoidal Categories

In fact, all our examples are *monoidal* categories — they have a *tensor product* and *unit object*:

- **In Set, the tensor product is $\times$, and the unit object is the 1-element set.**
- **In Hilb, the tensor product is $\otimes$, and the unit object is $\mathbb{C}$.**
- **In nCob, the tensor product looks like this:**



and the unit object is the empty manifold.

(A bunch of axioms must hold, and they do....)

Now for the first big difference: **the tensor product in Set is 'cartesian', while those in nCob and Hilb are not!**

A monoidal category is *cartesian* when you can duplicate data:

$$\Delta_x \colon x \to x \otimes x$$

and delete it:

$$e_x \colon x \to 1$$

so that these diagrams commute:

$$\begin{array}{ccc} x & \xrightarrow{\Delta_x} & x \otimes x \\ {\scriptstyle 1_x}\downarrow & & \downarrow{\scriptstyle e_x \otimes 1_x} \\ x & \xleftarrow{\sim} & 1 \otimes x \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\Delta_x} & x \otimes x \\ {\scriptstyle 1_x}\downarrow & & \downarrow{\scriptstyle 1_x \otimes e_x} \\ x & \xleftarrow{\sim} & x \otimes 1 \end{array}$$

In Set, you can do this. In Hilb you can't: you can neither clone a quantum, nor cleanly delete quantum information. ***Nor can you do this in*** **nCob!**

Duality for Objects

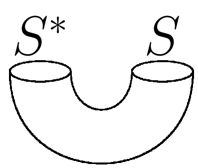
Another big difference: Both nCob and Hilb have 'duals for objects', but Set does not. This is why quantum teleportation seems odd.

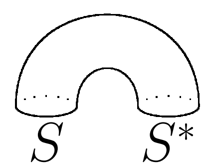
A monoidal category has ***duals for objects*** if every object x has an object x^* with morphisms

$$\epsilon_x \colon x^* \otimes x \to 1, \qquad \iota_x \colon 1 \to x \otimes x^*$$

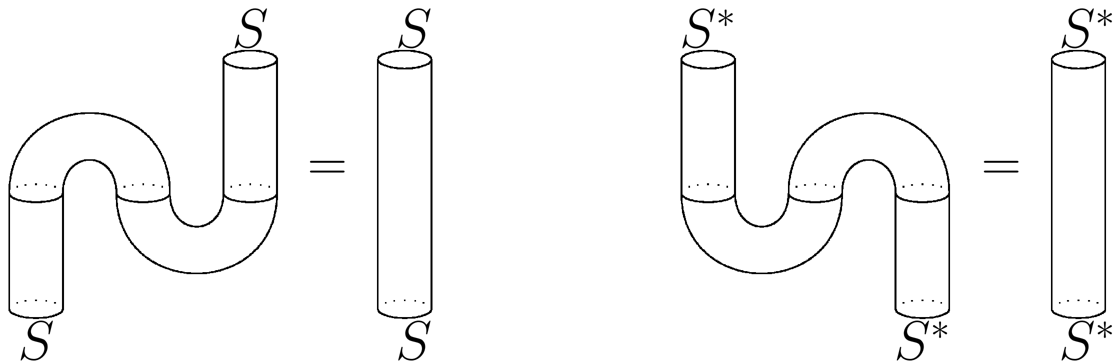
satisfying the ***zig-zag identities***.

In nCob, S^* is S with its orientation reversed. We have

$\epsilon_S =$ 

$\iota_S =$ 

and the zig-zag identities look like this:



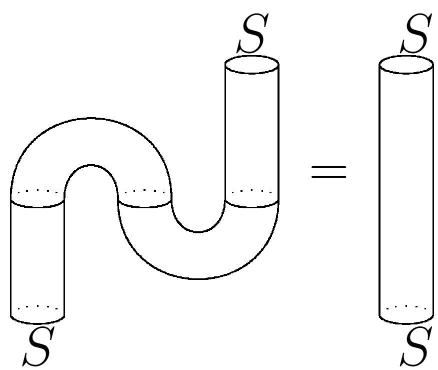
In Hilb, H^* is the dual Hilbert space. We have

$$\begin{array}{rrcl} \epsilon_H \colon & H^* \otimes H & \to & \mathbb{C} \\ & \ell \otimes \psi & \mapsto & \ell(\psi) \end{array} \qquad \begin{array}{rrcl} \iota_H \colon & \mathbb{C} & \to & H \otimes H^* \\ & c & \mapsto & c\,1_H \end{array}$$

and the zig-zag identities say familiar things about linear algebra.

But... ***there is no 'dual' of a set!***

Abramsky and Coecke have shown that quantum teleportation relies on the zig-zag axiom:



A particle interacts with one of a pair of particles prepared in the Bell state. Its quantum state gets transferred to the other member of the pair!

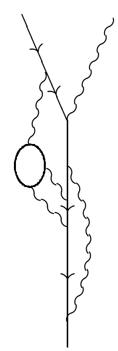
Read their paper *A Categorical Semantics of Quantum Protocols* for details.

In summary:

> **Quantum theory seems counterintuitive if we expect Hilb to act like Set, since it acts more like nCob. Superficially, Hilbert spaces and operators seem like sets and functions. But, they're really more like *spaces* and *spacetimes!***

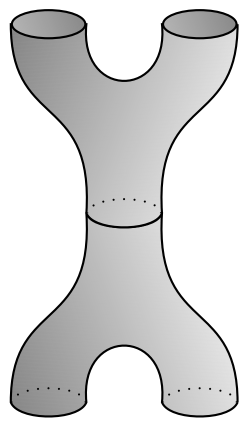
This is a clue.

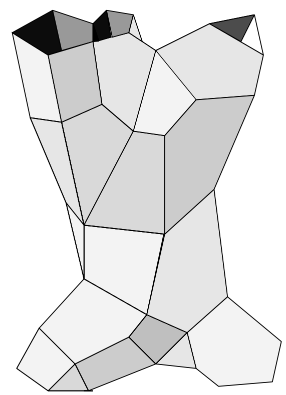
Perhaps Feynman was the first to get it...



...or maybe Penrose, with his spin networks.

Both string theory and spin foam models are trying to exploit such clues. They are groping towards a language for quantum spacetime that will usefully blur the distinction between *pieces of spacetime geometry*:



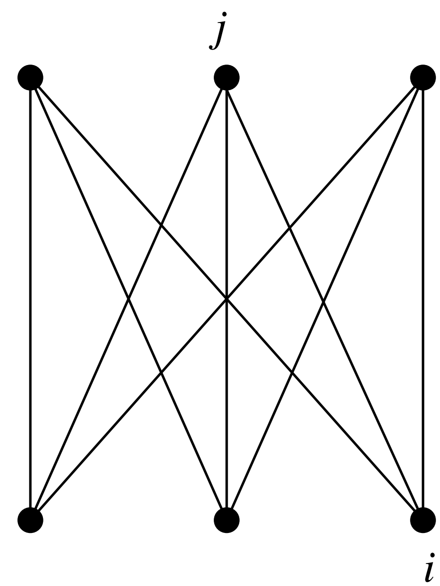


and *quantum processes*.

At this point we should think of them, not as predictive theories, but as explorations of the mathematical possibilities!

But what's really going on here?

To dig deeper, it's worth pondering Heisenberg's 'matrix mechanics'. Here a matrix U_{ij} describes the amplitude for a system to go from its jth state to its ith state:



A matrix of natural numbers is secretly a *span* of finite sets:

$$\begin{array}{ccccc} & & M & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ S_1 & & & & S_2 \end{array}$$

Spans of measure spaces give more general operators between Hilbert spaces. We see this in path integrals.

On the other hand, a cobordism is a *cospan*:

$$\begin{array}{ccccc} & & M & & \\ & \nearrow^{i_1} & & \nwarrow^{i_2} & \\ S_1 & & & & S_2 \end{array}$$

The similarities between nCob and Hilb are shared by many categories where the morphisms are (co)spans. Composing spans generalizes matrix multiplication.

So: *try to understand physics from a 'span-ish' perspective!*