# Quantum Quandaries: A Category–Theoretic Perspective

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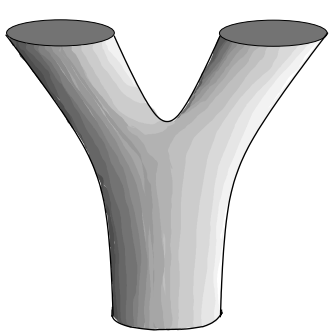
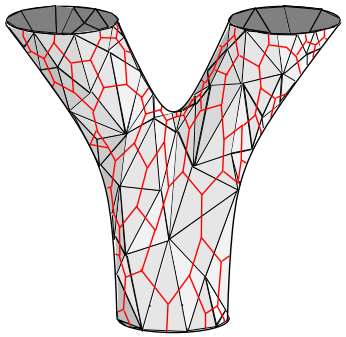


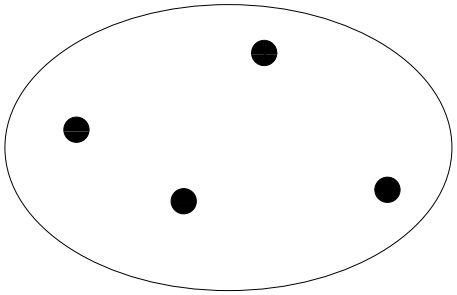


figures by Aaron Lauda

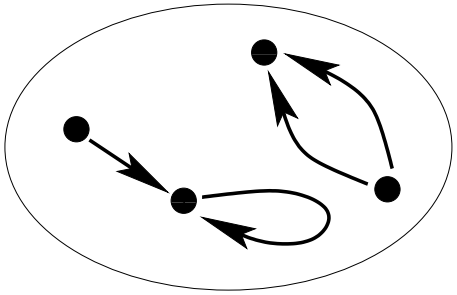
**for more, see**
**http://math.ucr.edu/home/baez/quantum/**

# The Big Idea

**Once upon a time, mathematics was all about *sets*:**



**In 1945, Eilenberg and Mac Lane introduced *categories*:**



**Category theory puts *processes* (morphisms):**

$$\bullet \longrightarrow \bullet$$

**on an equal footing with *things* (objects):**

$$\bullet$$

**An object of a category may be a set, and a morphism may be a function — but they *don't need to be!* We'll see examples where they're not.**

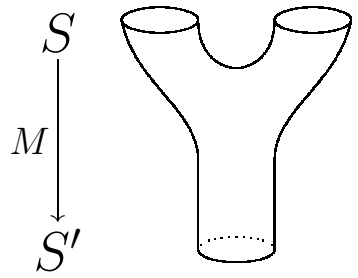
**So: ask not what an object is 'made of'. Instead, ask how it's related to other objects... via morphisms!**

**Categories are already revolutionizing mathematics. What about physics?**

**I claim:**

***Quantum theory makes more sense when seen as part of a theory of spacetime — but this can only be clearly understood using categories.***

**Why? The 'weird' features of quantum theory come from the ways that Hilb is less like Set than $n$Cob — the category where objects are choices of 'space' and morphisms are choices of 'spacetime':**

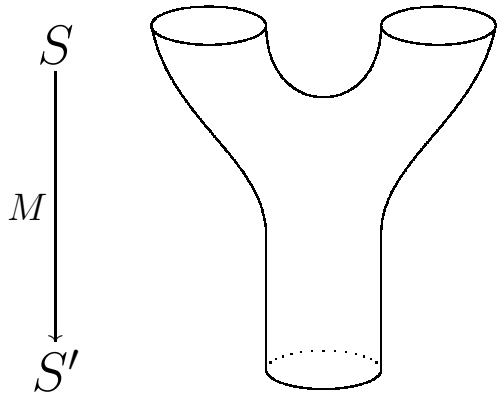


| | **object** $\bullet$ | **morphism** $\bullet \to \bullet$ |
|---|---|---|
| **SET THEORY** | **set** | **function between sets** |
| **QUANTUM THEORY** | **Hilbert space** **(state)** | **operator between Hilbert spaces (process)** |
| **GENERAL RELATIVITY** | **manifold** **(space)** | **cobordism between manifolds (spacetime)** |

## Objects and Morphisms

**Every category has *objects* and *morphisms*:**

- **In Set an object is a set, and a morphism is a function.**
- **In Hilb an object is a Hilbert space, and a morphism is a linear operator.**
- **In $n$Cob an object is an $(n-1)$-dim manifold, and a morphism is a cobordism between such manifolds:**



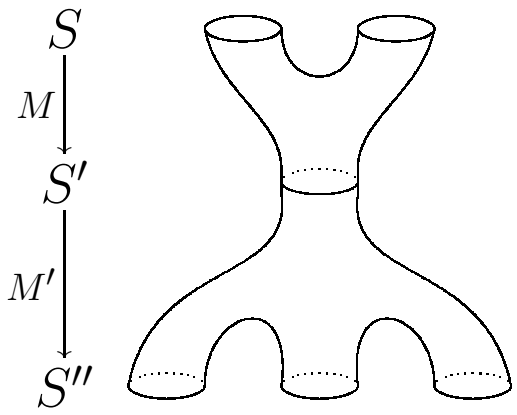
## Composition

**Every category lets us *compose* morphisms in an associative way:**

- **In Set, we compose functions as usual.**
- **In Hilb, we compose operators as usual:**

$$(T'T)\psi = T'(T\psi).$$

- **In $n$Cob, we compose cobordisms like this:**



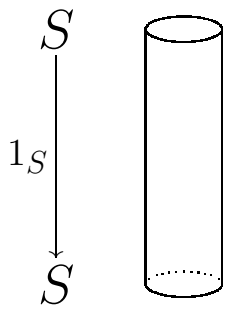
## Identity Morphisms

**Every category has an *identity morphism* $1_x \colon x \to x$ for each object $x$:**

- **In Set, $1_S \colon S \to S$ is the identity function.**
- **In Hilb, $1_H \colon H \to H$ is the identity operator:**

$$1_H \psi = \psi.$$

- **In $n$Cob, identity morphisms look like this:**



# Monoidal Categories

**In fact, all our examples are *monoidal* categories — they have a *tensor product* and *unit object*:**

- **In Set, the tensor product is $\times$, and the unit object is the 1-element set.**
- **In Hilb, the tensor product is $\otimes$, and the unit object is $\mathbb{C}$.**
- **In $n$Cob, the tensor product looks like this:**

**and the unit object is the empty manifold.**

**(A bunch of axioms must hold, and they do....)**

***Now for the first big difference:*** **the tensor product in Set is 'cartesian', while those in $n$Cob and Hilb are not!**

**A monoidal category is *cartesian* when you can duplicate data:**

$$\Delta_x \colon x \to x \otimes x$$

**and delete it:**

$$e_x \colon x \to 1$$

**so that these diagrams commute:**

$$\begin{array}{ccc} x & \xrightarrow{\Delta_x} & x \otimes x \\ {\scriptstyle 1_x}\downarrow & & \downarrow{\scriptstyle e_x \otimes 1_x} \\ x & \xleftarrow{\sim} & 1 \otimes x \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\Delta_x} & x \otimes x \\ {\scriptstyle 1_x}\downarrow & & \downarrow{\scriptstyle 1_x \otimes e_x} \\ x & \xleftarrow{\sim} & x \otimes 1 \end{array}$$

**In Set, you can do this. In Hilb you can't: you can neither clone a quantum, nor cleanly delete quantum information. *Nor can you do this in* $n$Cob!**

## Duality for Objects

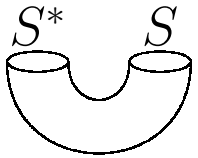
***Another big difference:*** **Both $n$Cob and Hilb have 'duals for objects', but Set does not. This is why quantum teleportation seems odd.**

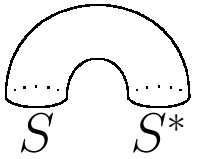
**A monoidal category has *duals for objects* if every object $x$ has an object $x^*$ with morphisms**

$$\epsilon_x \colon x^* \otimes x \to 1, \quad \iota_x \colon 1 \to x \otimes x^*$$

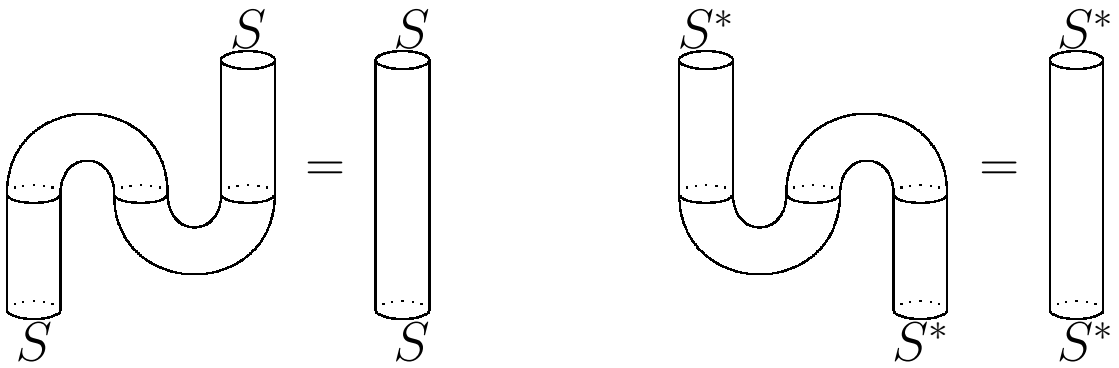
**satisfying the *zig-zag identities*.**

**In $n$Cob, $S^*$ is $S$ with its orientation reversed. We have**

$\epsilon_S \;=$ 


$\iota_S \;=$ 


**and the zig-zag identities look like this:**



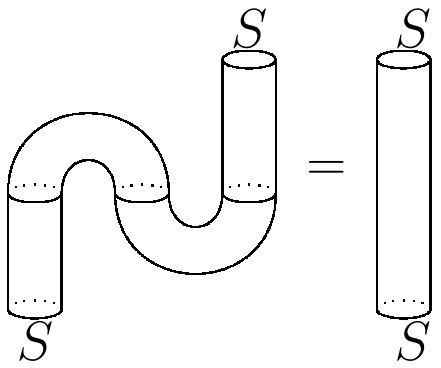

**In Hilb, $H^*$ is the dual Hilbert space. We have**

$$\epsilon_H\colon \begin{array}{ccc} H^* \otimes H & \to & \mathbb{C} \\ \ell \otimes \psi & \mapsto & \ell(\psi) \end{array} \qquad \iota_H\colon \begin{array}{ccc} \mathbb{C} & \to & H \otimes H^* \\ c & \mapsto & c\,1_H \end{array}$$

**and the zig-zag identities say familiar things about linear algebra.**

**But... *there is no 'dual' of a set!***

**Abramsky and Coecke have shown that quantum teleportation relies on the zig-zag axiom:**




**A particle interacts with one of a pair of particles prepared in the Bell state. Its quantum state gets transferred to the other member of the pair!**

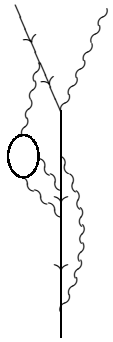
**Read their paper *A Categorical Semantics of Quantum Protocols* for details.**

***In summary:***

> **Quantum theory seems counterintuitive if we expect Hilb to act like Set, since it acts more like $n$Cob. Superficially, Hilbert spaces and operators seem like sets and functions. But, they're really more like *spaces* and *spacetimes!***

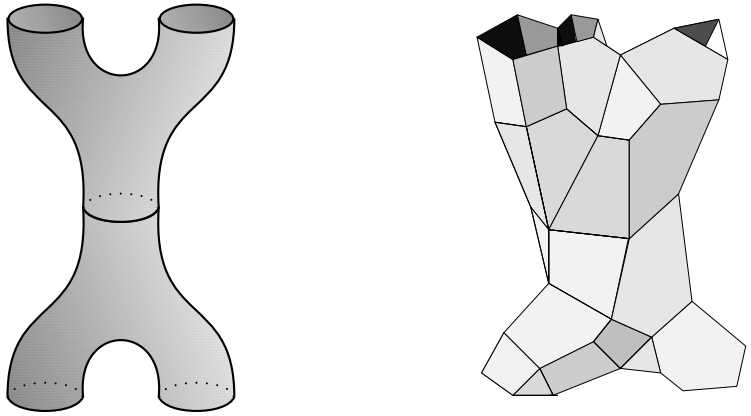
**This is a clue.**

**Perhaps Feynman was the first to get it...**



**...or maybe Penrose, with his spin networks.**

**Both string theory and spin foam models are trying to exploit such clues. They are groping towards a language for quantum spacetime that will usefully blur the distinction between *pieces of spacetime geometry*:**

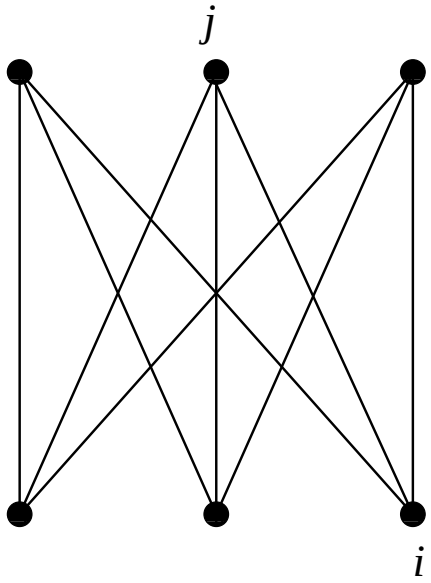


**and *quantum processes*.**

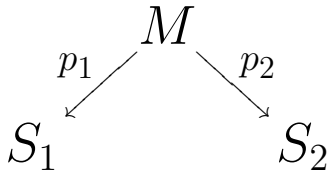
**At this point we should think of them, not as predictive theories, but as explorations of the mathematical possibilities!**

***But what's really going on here?***

**To dig deeper, it's worth pondering Heisenberg's 'matrix mechanics'. Here a matrix $U_{ij}$ describes the amplitude for a system to go from its $j$th state to its $i$th state:**

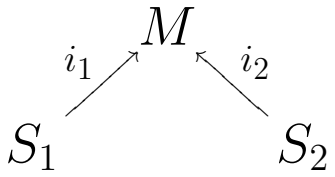


**A matrix of natural numbers is secretly a *span* of finite sets:**




**Spans of measure spaces give more general operators between Hilbert spaces. We see this in path integrals.**

**On the other hand, a cobordism is a *cospan*:**




**The similarities between $n$Cob and Hilb are shared by many categories where the morphisms are (co)spans. Composing spans generalizes matrix multiplication.**

**So: *try to understand physics from a 'span-ish' perspective!***