Notes on the Lie Operad

Todd H. Trimble
Department of Mathematics
University of Chicago

These notes have been reLaTeXed by Tim Hosgood, who takes responsibility for any errors.

The purpose of these notes is to compare two distinct approaches to the Lie operad. The first approach closely follows work of Joyal [9], which gives a species-theoretic account of the relationship between the Lie operad and homology of partition lattices. The second approach is rooted in a paper of Ginzburg-Kapranov [7], which generalizes within the framework of operads a fundamental duality between commutative algebras and Lie algebras. Both approaches involve a bar resolution for operads, but in different ways: we explain the sense in which Joyal's approach involves a "right" resolution and the Ginzburg-Kapranov approach a "left" resolution. This observation is exploited to yield a precise comparison in the form of a chain map which induces an isomorphism in homology, between the two approaches.

1 Categorical Generalities

Let $\mathcal{FB}$ denote the category of finite sets and bijections. $\mathcal{FB}$ is equivalent to the permutation category $\mathbf{P}$, whose objects are natural numbers and whose set of morphisms is the disjoint union $\sum_{n\geqslant 0} S_n$ of all finite symmetric groups. $\mathbf{P}$ (and therefore also $\mathcal{FB}$) satisfies the following universal property: given a symmetric monoidal category $\mathcal{C}$ and an object A of $\mathcal{C}$, there exists a symmetric monoidal functor $\mathbf{P} \to \mathcal{C}$ which sends the object $\mathbf{1}$ of $\mathbf{P}$ to A and this functor is unique up to a unique monoidal isomorphism. (Cf. the corresponding property for the braid category in [11].)

Let V be a symmetric monoidal closed category, with monoidal product $\oplus$ and unit 1. For the time being we assume V is complete and cocomplete: later we will need to relax this condition.

Definition 1. A V-species is a functor $\mathcal{FB} \to V$. The category of V-species and their natural transformations is denoted $V^{\mathcal{FB}}$.

The category $V^{\mathcal{FB}}$ carries several monoidal structures. One is the Day convolution ([3]) induced by the monoidal product $\oplus$ on $\mathcal{FB}$. To set this up. we work in the context of V-enriched category theory (see [12]), and recall that any locally small category $\mathcal{C}$ (such as $\mathcal{FB}$) can be regarded as a V-category, by composing hom: $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}et$ with the symmetric monoidal functor $\mathcal{S}et \to V$ sending a set U to a U-indexed coproduct of copies of 1. Then Day convolution is given abstractly by the formula

$$(F \otimes G)[S] = \int^{W,X \in \mathcal{FB}} F[W] \otimes G[X] \otimes \mathcal{FB}(S, W \oplus X).$$

Since each finite bijection $\phi\colon S \to W \oplus X$ induces a decomposition of S as a disjoint union $T + U$ of subsets (setting $T = \phi^{-1}(W)$ and $U = \phi^{-1}(X)$), this coend formula may be simplified:

$$(F \otimes G)[S] = \sum_{S=T+U} F[T] \otimes G[U].$$

(For the remainder of this paragraph, categorical terms such as "category", "functor", "colimit", etc. refer to their V-enriched analogues.) The product $F \otimes G$ preserves colimits in the separate arguments F and G (i.e., $- \otimes G$ and $F \otimes -$ are cocontinuous for all F and G). Since F and G may be canonically presented as colimits of representables, one may define a symmetric monoidal structure on this product, uniquely up to monoidal isomorphism, so that the Yoneda embedding $y : \mathcal{FB}^{\mathrm{op}} \to V^{\mathcal{FB}}$ is a symmetric monoidal functor, i.e., so that there is a coherent isomorphism $\hom(W \oplus X, -) \cong \hom(W, -) \otimes \hom(X, -)$. In conjunction with the universal property of $\mathcal{FB}$, we may state a universal property of $V^{\mathcal{FB}}$: let C be a (V-)category which is symmetric monoidally cocomplete (meaning its monoidal product is separately cocontinuous), and let A be an object of C. Then there exists a cocontinuous symmetric monoidal functor $V^{\mathcal{FB}} \to C$ sending $\hom(\mathbf{1}, -)$ to A, and this functor is unique up to monoidal isomorphism.

This universal property may be exploited to yield a second monoidal structure on $V^{\mathcal{FB}}$. Let $[V^{\mathcal{FB}}, C]$ denote the category of cocontinuous symmetric monoidal functors $V^{\mathcal{FB}} \to C$; then the universal property may be better expressed as saying that the functor $[V^{\mathcal{FB}}, C] \to C$ which evaluates a species $F : \mathcal{FB} \to V$ at $\mathbf{1} \in \mathcal{FB}$ is an equivalence. In the case $C = V^{\mathcal{FB}}$, the left-hand side of the equivalence carries a monoidal structure given by endofunctor composition. This monoidal structure transports across the equivalence to yield a monoidal product on $V^{\mathcal{FB}}$, denoted by $\circ$.

An explicit formula for $\circ$ is given as follows. Under one set of conventions, a V-species F may be regarded as a right module over the permutation category $\mathbf{P}$, so that the component $F[n]$ carries an action $F[n] \otimes S_n \to F[n]$. The n-fold Day convolution $G^{\otimes n}$ carries, under the same conventions, a left S_n-action $S_n \otimes G^{\otimes n} \to G^{\otimes n}$. Thn the coend formula for $F \circ G$ may be written as

$$(F \circ G)[S] = \sum_{n \geq 0} F[n] \otimes_{S_n} G^{\otimes n}[S].$$

A special case of this "substitution product" $\circ$ is the analytic functor construction. For each object X in V there is a V-species $\widehat{X}$ such that $\widehat{X}[0] = X$ and $\widehat{X}[n] = 0$ otherwise. Letting X^n denote the n-fold tensor product in V, we have

$$(F \circ \widehat{X})[0] = \sum_{n \geq 0} F[n] \otimes_{S_n} X^n$$

and $(F \circ \widehat{X})[n] = 0$ otherwise; we write the right-hand side of the above equation as $F(X)$. This defines a functor $F(-) : V \to V$, which we call the **analytic** functor attached to the species F. We often commit an abuse of language and write $F(X)$ for the analytic functor, thinking of the X as a variable or placeholder for an argument, much as one abuses language by referring to a function $f(x) = \sin(x)$.

The analytic functor $F(-) : V \to V$ determines, up to isomorphism, its generating species F; we describe this determination for the category $V = Vect_k$ of vector spaces over a ground field k. Let $F_n(X)$ denote the n^{th}-degree component $F[n] \otimes_{S_n} X^n$, and let X be a vector space freely generated from a set $\{x_1, \ldots, x_n\}$ whose cardinality equals that degree. Then the species value $F[n]$ can be recovered as the subspace of $F_n(X)$ spanned by equivalence classes of those expressions $\tau \otimes x_{i(1)} \otimes \ldots \otimes x_{i(n)}$ in which each x_i occurs exactly once. We use the notations $F[n]$ and $F(X)[n]$ interchangeably for these species values.

If $G[0] = 0$, we have $G^{\otimes n}[S] = 0$ whenever n exceeds the cardinality $|S|$, in which case $F \circ G$ makes sense for V *finitely* cocomplete. For general n we have in that case

$$G^{\otimes n}[S] = \sum_{S = T_1 + \ldots + T_n} G[T_1] \otimes \ldots \otimes G[T_n]$$

where the sum is indexed over ordered partitions of S into n nonempty subsets T_i. The group S_n permutes such ordered partitions in such a way that the orbits correspond to unordered partitions, which are tantamount to equivalence relations on S. Let $Eq(S)$ denote the set of such equivalence relations, and let $\pi : S \to S/R$ denote the canonical projection of S onto the set of R-equivalence classes. Then the substitution product may be rewritten as

$$(F \circ G)[S] = \sum_{R \in Eq(S)} F[S/R] \otimes \bigotimes_{x \in S/R} G[\pi^{-1}(x)]$$

whenever $G[0] = 0$.

2 Operads

Definition 2. An **operad** in V is a monoid in the monoidal category $(V^{\mathcal{FB}}, \circ)$.

The unit for the monoidal product $\circ$ will be denoted X; it is defined by $X[n] = 0$ if $n \geq 1$, and $X[1] = I$ where I is the monoidal unit of V. (We repeat that we also use X as abusive notation for a placeholder or variable with values ranging over objects of V.)

Clearly an operad M induces a monad $M \circ - : V^{\mathcal{FB}} \to V^{\mathcal{FB}}$, which in turn restricts to the analytic monad $M(-) : V \to V$ along the embedding $\widehat{(-)} : V \to V^{\mathcal{FB}}$ if V is cocomplete. Many algebraic structures arising in practice are algebras of analytic monads. For $V = Vect_k$, we have, e.g.,

1. The tensor algebra $T(X) = 1 + X + X^{\otimes 2} + \ldots$, denoted $\frac{1}{1-X}$. The algebras of $T(-)$ are associative algebras. The species value $T[n]$ is the space

freely generated from the set of linear orders on an n-element set, with the evident S_n-action.

2. The symmetric algebra $S(X) = 1 + X + X^{\otimes 2}/S_2 + \ldots$, denoted $\exp(X)$. Algebras of $S(-)$ are commutative associative algebras. The species value $\exp[n]$ is the trivial 1-dimensional representation of S_n.

3. The Lie operad $L[-]$ may be presented as an operad generated by a binary operation $[-,-] \in L[2]$, subject to the Jacobi relation $[-,[-,-]] + [-,[-,-]]\sigma + [-,[-,-]]\sigma^2 = 0$ and the alternating relation $[-,-] + [-,-]\tau = 0$, where σ is a 3-cycle and τ is a 2-cycle. The algebras of the analytic monad $L(-)$ are Lie algebras.

By way of contrast, Boolean algebras are not algebras of an analytic monad, since the equation $x \wedge x = x$ inevitably involves the use of a diagonal map not available in $\mathcal{FB}$.

Although analytic monads are obviously important, we stress that they are simply restrictions of monads $V^{\mathcal{FB}} \to V^{\mathcal{FB}}$, and that it is often more flexible to work in the latter setting. For example, if V is only *finitely* cocomplete, then analytic monads cannot be defined in general; for example, the free commutative monoid construction does not define an analytic monad on finite-dimensional vector spaces. However, if V is finitely cocomplete and M is a V-operad such that $M[0] = 0$, then there *is* a monad $M \circ -$ acting on the orthogonal complement $V^{\perp} \hookrightarrow V^{\mathcal{FB}}$, i.e., the full subcategory of V-species G such that $G[0] = 0$.

This situation occurs often. For example, consider the operad $M(X) = \exp(X) - 1$, whose $Vect_k$-algebras are commutative monoids without unit. This operad induces a monad $M \circ -$ on $V^{\perp}$ where V is the category of *finite-dimensional* vector spaces; the algebras are again commutative ($V^{\perp}$-)monoids without unit. The operad itself can be regarded as the free commutative algebra without unit, $M \circ X$, generated by the monoidal unit X considered as living in the subcategory $V^{\perp}$. For other reasons, Markl ([14]) has also considered algebras over monads $M \circ -$ and $- \circ M$, more general than algebras of analytic monads; he refers to the former as "M-modules".

A look ahead

In the next few sections, we will reprise the beautiful work of Joyal which leads up to a computation of the Lie species L. Our general methodology is to reinterpret Joyal's approach via virtual species by appeal to dg-structures on finite-dimensional super vector spaces; ultimately we feel that a proper approach to virtual species should draw on the standard model category structure on this category.

A second point of our approach is to place Joyal's calculations within the context of a particular bar construction. This will better enable us to compare these calculations with those of Ginzburg-Kapranov, which involve a slightly different bar construction.

3 Poincaré-Birkhoff-Witt

From this section on, we fix a ground field k of characteristic 0, and V henceforth denotes the category of finite-dimensional *vector spaces over k.*

To study the Lie operad $L[-]$, Joyal and others (e.g. [8]) take as starting point the Poincaré-Birkhoff-Witt (PBW) theorem. If L is a Lie algebra, then its universal enveloping algebra $U(L)$ carries a canonical filtration, inherited as a quotient of the tensor algebra $T(L)$ equipped with the degree filtration. Embedded in $T(L)$ as a filtered subspace is the symmetric algebra $S(L)$, whose homogeneous components $S_n(L)$ may be realized as images of symmetrizing operators acting on components $T_n(L) = L^{\otimes n}$:

$$\pi_n : L^{\otimes n} \to L^{\otimes n}$$
$$v_1 \otimes \ldots \otimes v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$

We obtain a composite of maps of filtered spaces

$$S(L) \to T(L) \to U(L)$$

and the PBW theorem concerns the application of the associated graded space functor to this composite (denoted ϕ):

Theorem 1. *The graded map ϕ_{gr} induces an isomorphism between $U_{gr}(L)$ and $S_{gr}(L)$ as graded spaces.*

If $L(X)$ is the free Lie algebra on X, then $U(L(X)) \cong T(X)$ (as algebras even) by an adjoint functor argument. Assembling some prior notation, it follows from PBW that there exists an isomorphism of analytic functors on $Vect_k$:

$$\frac{1}{1-X} \cong (\exp \circ L)(X)$$

which in turn determines a species-isomorphism, which componentwise is an isomorphism of S_n-representations:

$$\frac{1}{1-X}[n] \cong (\exp \circ L)[n].$$

Notice both sides makes sense as V-species. A guiding idea behind the species methodology is that the components of such species are structural analogues of coefficients of formal power series. This analogy can be made precise. Let $V[[x]]$ denote the rig (ring without additive inverses) of isomorphism classes of V-species, with $+$ given by coproduct and $\cdot$ given by $\otimes$. Let $\mathbb{N}[[x]]$ denote the rig of formal power series

$$\sum_{n \geq 0} \frac{a_n x^n}{n!}$$

where the coefficients a_n are natural numbers. This may be equivalently defined to be the rig of sequences a_n of natural numbers where two sequences are multiplied according to the rule

$$(a \cdot b)_n = \sum_p \binom{n}{p} a_p b_{n-p}$$

The rigs $V[[x]]$ and $\mathbb{N}[[x]]$ also have a partially defined composition operation $\circ$, where $f \circ g$ is defined whenever $g(0) = 0$. In the case of $V[[x]]$, it is of course the operation which is descended from the substitution product by passing to isomorphism classes.

Proposition 1. *([10]) The function* $\dim : V[[x]] \to \mathbb{N}[[x]]$*, sending F to the sequence of coefficients $a_n = \dim(F[n])$, is a rig homomorphism which preserves the $\circ$ operation.*

It follows from the proposition and the preceding species isomorphism that $\dim(L[n]) = (n-1)!$, since $\dim(L)(x)$ is the formal power series expansion of $-\log(1-x)$. We are interested in finding an appropriate lift of $-\log(1-x) \in \mathbb{N}[[x]]$ to a species $-\log(1-X)$ in $V[[x]]$, and hence an identification between $-\log(1-X)$ and the Lie species $l[-]$.

4 Virtual species

Before we construct the species $\log(1-X)$, it is convenient to complete the rig $V[[x]]$ to a ring. One proceeds exactly as in K-theory, where one passes from vector bundles to virtual bundles.

Definition 3. $\mathbf{V}$ is the category of $\mathbb{Z}_2$-graded finite-dimensional vector spaces, with monoidal product given by the formula

$$(V \otimes W)_p = \sum_{m+n=p} V_m \otimes W_n$$

and symmetry given by the formula

$$\sigma(x_m \otimes y_n) = (-1)^{mn} y_n \otimes x_m$$

for $x_m \in V_m$ and $y_n \in W_n$.

Definition 4. Let F and G be $\mathbf{V}$-species. Then $F \sim G$ (F and G are **virtually equivalent**) if $F_0 \oplus G_1 \cong F_1 \oplus G_0$ as V-species.

- **Remark:** Exact sequences in $V^{\mathcal{FB}}$ split. One can check this assertion componentwise, where exact sequences of kS_n-modules split since we assumed $char(k) = 0$.

Lemma 1. *The relation $\sim$ is an equivalence relation.*

Proof. Transitivity follows from cancellation: $F \oplus H \cong G \oplus H$ implies $F \cong G$. This in turn follows from the remark. □

It is of course crucial here to work with finite-dimensional vector spaces throughout, in order to avoid the Eilenberg swindle.

Lemma 2. *The relation $\sim$ is respected by $\oplus$, $\otimes$, and $\circ$.*

The proof is left to the reader; see also [9] and [16].

A **virtual species** is a virtual equivalence class of $\mathbf{V}$-species. The ring of equivalence classes is denoted $\mathbf{V}[[x]]$.

Many of our calculations refer to manipulations in the ring $\mathbf{V}[[x]]$ of virtual species, but methodologically it is useful to distinguish the various ways in which virtual equivalences arise. Part of the philosophy behind species is that clarity is promoted and calculations are under good combinatorial control when power series operations $+$, $\cdot$, and $\circ$ can be viewed as arising from categorified functorial operations $\oplus$, $\otimes$, and $\circ$. Put differently, passage from $\mathbf{V}^{\mathcal{FB}}$ to $\mathbf{V}[[x]]$ loses categorical information, and it helps to recognize when a virtual equivalence $F \sim G$ comes from an isomorphism $F \cong G$ in $\mathbf{V}^{\mathcal{FB}}$.

In practice, many virtual equivalences which do *not* come from isomorphisms in $\mathbf{V}^{\mathcal{FB}}$ come about by applying the following remark.

- If an object C of $\mathbf{V}^{\mathcal{FB}}$ carries a differential structure (meaning V-species maps $\partial : C_0 \to C_1$ and $\partial : C_1 : C_0$ satisfying both instances of $\partial^2 = 0$), then C is virtually equivalent to its homology $H(C)$. This observation is tantamount to a structural Euler formula $C_0 - C_1 \sim H_0(C) - H_1(C)$, which obtains because exact sequences in $\mathbf{V}^{\mathcal{FB}}$ split.

This principle can be quite powerful. Its application does not commit one to any particular choice of differential structure on C, so that one is enabled to choose differentials to suit the local occasion. The downside is that because there is no canonical way to split exact sequences, it is sometimes harder to give precise formulas that exhibit such virtual equivalences $C \sim H(C)$.

5 Logarithmic species

One of our goals is to lift the inversion

$$\frac{1}{1-x} = \exp(L(x)) \qquad \textit{implies} \qquad -\log(1-x) = L(x)$$

from the ring $\mathbb{Z}[[x]]$ to the ring $\mathbf{V}[[x]]$. In either ring, a necessary condition for $F(x)$ to have an inverse $F^{-1}(x)$ (with respect to $\circ$) is that the 0^{th} coefficient $F[0]$ be 0. Thus, instead of inverting $\exp(X)$, we invert $\exp(X) - 1$. Suppose then that $\log(1+X)$ is a virtual inverse of $\exp(X) - 1$:

$$\log(1+X) \circ (\exp(X) - 1) \sim X \sim (\exp(X) - 1) \circ \log(1+X).$$

Proposition 2. *If F and G are $\mathbf{V}$-species such that $F[0] = 0 = G[0]$, then* $\exp(F \oplus G) \cong \exp(F) \otimes \exp(G)$.

Proof. It is immediate that $\exp(F) = \sum_{n \geq 0} F^{\otimes n}/S_n$ is the free commutative monoid in $(\mathbf{V}^{\mathcal{FB}}, \otimes)$ generated from F, and the assertion says that the left adjoint exp preserves coproducts. □

Defining $\log((1+F) \otimes (1+G))$ to be

$$\log(1+X) \circ (F + F + F \otimes G),$$

it follows that $\log((1+F) \otimes (1+G)) \sim \log(1+F) + \log(1+G)$. In particular,

$$\log(\frac{1}{1-X}) \sim -\log(1-X)$$

where of course $-(F_0, F_1)$ is defined to be (F_1, F_0). The species $\log(1-X)$ is easily obtained from $\log(1+X)$ by the following result.

Proposition 3. $F(-X)[S] \cong (-1)^{|S|} F[S] \otimes \Lambda[S]$, *where $\Lambda[S]$ denotes the top exterior power $\Lambda^{|S|}(kS)$ of the vector space kS freely generated from S.*

Proof. The $\mathbf{V}$-species X, which by definition is the unit with respect to $\circ$, is given by $(X[1]_0, X[1]_1) = (k, 0)$ and $X[n] = 0$ otherwise. Thus $(-X[1]_0, -X[1]_1) = (0, k)$ and $-X[n] = 0$ otherwise. Hence

$$\begin{aligned} (F \circ (-X))[S] &= \sum_{R \in Eq(S)} F[S/R] \otimes \bigotimes_{x \in S/R} (-X)[\pi^{-1}(x)] \\ &\cong F[S] \otimes (-X)[1]^{\otimes |S|} \end{aligned}$$

Now $(-X)[1]^{\otimes n}$ is 1-dimensional and is concentrated in degree $n \pmod 2$ (whence the sign $(-1)^{|S|}$). A transposition in S_n induces a sign change in $(-X[1]^{\otimes n})_{n \pmod 2}$, by definition of symmetry: $\sigma(x_1 \otimes y_1) = -y_1 \otimes x_1$. This proves the claim. □

We proceed to compute the inverse $\log(1+X)$ to $\exp(X) - 1$. Recalling an earlier remark, $F(X) = \exp(X) - 1$ is the operad such that algebras of the monad $F \circ -$ (acting on $\mathbf{V}$-species G such that $G[0] = 0$) are commutative algebras without unit.

The underlying species F satisfies $F[0] = 0$, $F[1] = 1$ (i.e., $= (k, 0)$). Joyal gives a general method due to G. Labelle for inverting such species. Introduce an operator

$$\begin{aligned} \delta_F \colon \mathbf{V}[[x]] &\to \mathbf{V}[[x]] \\ H &\mapsto H \circ F - H \end{aligned}$$

so that $(1 + \delta_F)(H) = H \circ F$; here 1 denotes the identity functor. Observe that δ_F preserves sums, because $- \circ F : \mathbf{V}^{\mathcal{FB}} \to \mathbf{V}^{\mathcal{FB}}$ is the restriction of a cocontinuous monoidal functor which thus preserves coproducts.

Let $\widehat{0}$ denote the bottom element of the lattice $eq(S)$ ordered by inclusion of equivalence relations. This $\widehat{0}$ is the discrete equivalence relation on S, so that $S/\widehat{0} \cong S$. We have

$$\delta_F(H)[S] = (\sum_{R \in Eq(S)} H[S/R] \otimes \bigotimes_{x \in S/R} F[\pi^{-1}(x)]) - H[S]$$

and since $H[S] \cong H[S/\widehat{0}] \otimes \bigotimes_{x \in S/\widehat{0}} F[1]$ by our assumptions on F, we may rewrite the right-hand side (up to virtual equivalence) as

$$\sum_{\widehat{0} < R} H[S/R] \otimes \bigotimes_{x \in S/R} F[\pi^{-1}(x)].$$

Define the **V**-*species* $\delta_F(H)$ by the above expression, so that δ_F will be used to denote a functor on **V**-species H, in addition to an operator on $\mathbf{V}[[x]]$. In particular, when $F(X) = \exp(X) - 1$, we have

$$\delta_F(H) = \sum_{\widehat{0} < R} H[S/R].$$

In general, the n^{th} iterate $\delta_F^n(H)[S]$ is a sum of the form

$$\sum_{\widehat{0} < R_1 < \ldots < R_n} terms$$

where terms are indexed by strictly increasing chains of equivalence relations on S. As soon as $n \geq |S|$, there are no chains of that length, so this sum will be empty. In this way, for each finite S, $\delta_F^n(H)[S] = 0$ for all sufficiently large n, and so the expression

$$(1 + \delta_F)^{-1}(H) := \sum_{n \geq 0} (-1)^n \delta_F^n(H)$$

makes sense as a functor on **V**-species H.

We may now construct the inverse species $F^{-1}(X)$:

$$F^{-1}(X) = (1 + \delta_F)^{-1}(X) := \sum_{n \geq 0} (-1)^n \delta_F^n(X)$$

Proposition 4. *$(F^{-1} \circ F)(X) \sim X$.*

Proof. We have

$$(F^{-1} \circ F)(X) \sim (1 + \delta_F)(F^{-1})(X)) \qquad = (1 + \delta_F)(\sum_{n \geq 0} (-1)^n \delta_F^n(X))$$

$$\sim \sum_{n \geq 0} (-1)^n \delta_F^n(X) + \sum_{n \geq 0} (-1)^n \delta_F^{n+1}(X)$$

which telescopes down to $\delta_F^0(X) = X$. □

- **Remark:** Under our hypotheses on F, one may prove by induction that $F \circ G \sim F \circ H$ implies $G \sim H$. From this and the proposition, it easily follows that $(F \circ F^{-1})(X) \sim X$.

6 A bar construction

When F carries an operad structure, this construction of F^{-1} admits a more categorical interpretation. Observe that there is an embedding

$$\delta_F^n(X) \hookrightarrow F \circ \ldots \circ F = F^{\circ n}.$$

Let us regard the operad F as a monoid with multiplication $m : F \circ F \to F$ and unit $u : X \to F$. There is a (necessarily unique) operad map $\varepsilon : F \to X$, called an *augmentation*, and this may be used to turn X into a left F-module and also into a right F-module, in the usual way. We may thus form a two-sided bar construction $B(X, F, X)$, whose component in dimension n is isomorphic to $F^{\circ n}$.

The bar construction $B(X, F, X)$ is a simplicial object in an additive category, and hence gives rise to a $\mathbb{Z}$-graded chain complex, where each differential is a signed sum of face maps of the form

$$\partial_i : F^{\circ(n+1)} \stackrel{F^{\circ(n-i)} m F^{\circ(i-1)}}{\to} F^{\circ n}.$$

By reduction of the grading, we may regard $B(X, F, X)$ as a $\mathbb{Z}_2$-graded chain complex, provided that the two components are taken as species valued in the category of (possibly infinite-dimensional) vector spaces.

However, since we are dealing with virtual species, we want to cut back to $\mathbf{V}$-valued species, where the components are finite-dimensional. To this end, notice that each of the maps ∂_i restricts to a map

$$\delta_F^{n+1}(X) \to \delta_F^n(X)$$

and the $\delta_F^n(X)$ form a chain subcomplex. We regard this chain complex as our preferred **bar construction** for F, or more precisely a *right* bar construction $B_r(F)$, as we now explain in more detail.

There is an exact sequence

$$0 \to \delta_F \to (-) \circ F \to (-) \circ X \to 0$$

making the functor δ_F, for an operad F, analogous to tensoring on the *right* with an augmentation ideal IG of a group ring $\mathbb{Z}G$, sitting in an exact sequence

$$0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Here δ_F (resp. $- \otimes IG$) is regarded as a formal or virtual difference between $- \circ F$ and $- \circ X$ (resp. $- \otimes \mathbb{Z}G$ and $- \otimes \mathbb{Z}$). In forming $F^{-1}(X)$ as

$$X + (-\delta_F)(X) + (-\delta_F)^2(X) + \ldots,$$

we interpret the additive inverse $-\delta_F$ as the result of composing δ_F with the degree 1 shift operator Σ acting on $\mathbb{Z}_2$-graded species, which plays the structural role of additive inverse. Hence the construction for $F^{-1}(X)$ above is analogous to the normalized bar resolution

$$\mathbb{Z} + \Sigma(IG) + (\Sigma(IG))^{\otimes 2} + \dots$$

originally introduced by Eilenberg-Mac Lane [4].

- **Remark:** The analogy between the complex $B_r(F)$ and the classical normalized bar resolution is imperfect, however, because the monoidal product $- \circ -$ behaves rather differently from the tensor product $- \otimes -$. In particular, $- \circ -$ does not preserve colimits on each side; only $- \circ F$ preserves colimits. This explains why, in our geometric series construction for $F^{-1}(X)$, we must consistently apply the substitution product $- \circ F$ with F appearing on the right. Also, whereas the classical bar resolution produces a free algebra (or cofree coalgebra) construction on the shifted augmentation ideal via a geometric series, the same geometric series construction in the operadic context does not yield a cofree co-operad, again because of failure of the monoidal product $- \circ -$ to preserve coproducts in each of its separate arguments. We return to this point later.

To give further weight to the sense in which $B_r(F)$ is a bar construction, we assemble the components $\delta_F^n(X) \circ F$ of $F^{-1} \circ F$ into an acyclic chain complex $E_r(F)$ of $\mathbf{V}$-species that will be a (right) F-free resolution of the unit operad X. Indeed, there is an embedding

$$\delta_F^n(X) \circ F \hookrightarrow F^{\circ(n+1)}$$

where now $F^{\circ(n+1)}$ is the degree n component of the two-sided bar resolution $B(X, F, F)$ (as simplicial object). Again, the face maps

$$\partial_i : F^{\circ(n+1)} \circ F \to F^{\circ n} \circ F$$

restrict to maps

$$\delta_F^{n+1}(X) \circ F \to \delta_F^n(X) \circ F$$

so that the $\delta_F^n(X) \circ F$ are components of a subcomplex $E_r(F)$. This endows the $\mathbb{Z}_2$-graded species $F^{-1} \circ F$ with a dg-structure.

Then, we may regard the virtual equivalence

$$(F^{-1} \circ F)(X) \sim X$$

as arising from a homotopy equivalence between dg-species:

$$\begin{array}{ccccccccc}
\dots \longrightarrow & \delta_F^{n+1}(X) & \longrightarrow & \delta_F^n(X) & \longrightarrow \dots \longrightarrow & \delta_F^0(X) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \mathrm{id} & \\
& 0 & & 0 & & X &
\end{array}$$

Indeed, we have an augmentation map $B(X, F, F) \to X$ between simplicial objects (regarding X on the right as a constant simplicial object). This restricts to an augmentation map $E_r(F) \to X$ which, by the Dold-Kan correspondence, corresponds to a map between chain complexes as displayed above. To check that the augmentation $E_r(F) \to X$ is a homotopy equivalence, one checks that the standard contracting homotopy on $B(X, F, F)$, with components

$$F^{\circ n} \circ u : F^{\circ n} \circ X \to F^{\circ n} \circ F,$$

restricts to a contracting homotopy on $E_r(F)$. In more detail, each component $\delta_F^n(X) \circ F$ breaks up as a coproduct

7 The Lie species

We now return to our example $F(X) = \exp(X) - 1$ and the construction of the inverse $F^{-1}(X) = \log(1 + X)$. By iterating the functor

$$\delta_F : H \mapsto (S \mapsto \sum_{\widehat{0}<R} H[S/R]),$$

we derive an expression of $\delta_F^n(X)[S]$ as the space whose basis elements are n-fold chains of strict inclusions of equivalence relations

$$\widehat{0} < R_1 < \ldots R_n = \widehat{1}$$

where $\widehat{1}$ denotes the top element of the lattice $Eq(S)$, namely the indiscrete equivalence relation with one equivalence class. Each such chain may be identified with an $(n-1)$-fold chain in the poset $Eq(S) - \{\widehat{0}, \widehat{1}\}$, in other words as cells of dimension $n - 2$ in the simplicial complex underlying the nerve of this poset. Let $C_i[S]$ denote the set of cells of dimension i (or rather the vector space it generates). Then for $|S| > 2$, we may identify the complex $B_r(F)[S]$,

$$0 \to \delta_F^{|S|-1}(X)[S] \to \ldots \to \delta_F^2(X)[S] \to \delta_F^1(X)[S] \to X[S] \to 0,$$

with

$$0 \to C_{|S|-3}[S] \to \ldots \to C_0[S] \to k \to 0 \to 0$$

It follows that $\log(1 + X)[S]$ is virtually equivalent to *reduced* homology of the simplicial complex C associated with $Eq(S)$.

Definition 5. A finite lattice is geometric if every element x is a joint of atoms, if every maximal chain beginning at 0 and ending at x has the same length $\rho(x)$, and if $\rho(X \vee y) + \rho(x \wedge y) \leq \rho(x) + \rho(y)$. (A finite lattice is modular iff this last inequality is an equality.)

Now the lattice of equivalence relations is a *geometric lattice*, and a result of Folkman is that the reduced homology of such a lattice is trivial except in top degree; more exactly, Folkman's result (the theorem below) implies that the nerve of the poset $Eq(S) - \{\widehat{0}, \widehat{1}\}$ is a bouquet of spheres of dimension $|S| - 3$, and the number of spheres is given by the value $\mu(\widehat{0}, \widehat{1})$ of Rota's Möbius function on the lattice $Eq(S)$.

Theorem 2. *([5]) If L is a finite geometric lattice, then the reduced homology of $L - \{0, 1\}$ is trivial except in top dimension, where the Betti number is the value $\mu(0, 1)$ of Rota's Möbius function on the lattice.*

Corollary 1. $\log(1 + X)[S] \sim (-1)^{|S|-1} H_{|S|-1}(B_r(F)[S])$.

Proof. By the structural Euler formula, $F^{-1}(X)[S] = \log(1 + X)$ as given by the $\mathbb{Z}_2$-graded chain complex $B_r(F)[S]$ is virtually equivalent to its homology, which in turn is equivalent to the reduced homology of $C[S]$. By Folkman's theorem, the reduced homology of $C[S]$ is concentrated in top degree $|S| - 3$, corresponding to the homology of $B_r(F)[S]$ sitting in degree $|S| - 1$. □

Corollary 2. *The Lie species $L[S]$ is isomorphic to $H_{|S|-1}(B_r(F)[S]) \otimes \Lambda[S]$.*

Proof. Starting from the PBW theorem

$$\frac{1}{1 - X} \cong \exp(L(X)),$$

we were led to the virtual equivalence

$$-\log(1 - X) \sim L(X).$$

Applying Joyal's rule of signs to the previous theorem, $-\log(1 - X)[S]$ is virtually equivalent to

$$-(-1)^{|S|}(-1)^{|S|-1} H_{|S|-1}(B_r(F)[S]) \otimes \Lambda[S]$$

which boils down to $H_{|S|-1}(B_r(F)[S]) \otimes \Lambda[S]$. As virtual species, both this and the Lie species value $L[S]$ are concentrated in degree 0 (mod 2). But if two **V**-species concentrated in degree 0 are virtually equivalent, they are isomorphic. □

This concludes our rendition of Joyal's calculation of $L[S]$. This calculation does not specify the operad structure of $L[-]$, but in view of the formula just given for $L[n]$, we make the following remark: given an operad in chain complexes with components $C[n]$, the homology operad $H_*(C)$ contains a suboperad with components $H_{n-1}(C[n])$. Unfortunately, this remark does not apply to $C[n] = B_r(F)[n] \otimes \Lambda[n]$, since the bar construction $B_r(F)[n]$ carries no obvious operad structure. A different bar construction $B_l(F)$, or rather its dual cobar construction, is better suited for applying the present remark.

8 Free operads

Let F be a V—operad such that $F[O] = 0$ and $F[1] = 1$. Our earlier construction for $F^{-1}(X)$, namely $\sum_{n \geq 0}(-\delta_F)^n(X)$ is not a free operad on the suspension $-\delta_F(X)$ of the augmentation ideal $\delta_F(X)$, due to the failure of $- \circ -$ to preserve separate colimits.

To construct free operads, it is convenient to use the language of trees [2].

Definition 6. A rooted tree is an (undirected) acyclic connected finite graph together with a distinguished node called the root.

Let N be the set of nodes of a rooted tree T with root r, and E the set of edges. There is a bijection $\phi_-: E \xrightarrow{\sim} N - \{r\}$, uniquely determined by the requirement that the node $\phi_-(e)$ lie on the edge e. Let $\phi_+(e)$ denote the other node lying on e. One may give a rooted tree a canonical directed graph structure, with source—target map $\langle \phi_-, \phi_+ \rangle : E \to N \times N$. The free category on a directed graph has a terminal object r iff the directed graph comes from a tree rooted at r. The free category on a rooted tree T defines a partial order on its set of nodes: an atomic element in this poset is called a *leaf* of T. A directed subgraph of T whose free category possesses a terminal object is called a *subtree*. In particular, the slice over an object x comes from a subtree T_x called the *tree over* x.

A rooted tree T may be characterized up to isomorphism by a function $f_T : N - \{r\} \to N$, sending a node to its unique successor in the induced partial order. Identifying functions of this form with based endofunctions $(N, r) \to (N, r)$, the endofunctions which arise in this way are precisely those whose iterates converge to the constant function at r. If x is a node in T, the set $\sigma_x = f_T^{-1}(x)$ is called the *sprout over* x.

Up to isomorphism, a rooted tree whose set of leaves is S is characterized in terms of S by the following data: an element C of the free commutative monoid $\exp(\mathcal{P}S)$ on the power set of S (i.e. a multiset of subsets of S), together with a linear order on each set of repetitions in C of a given subset. To obtain such a structure C_T from a rooted tree T, assign to each node x the set $\lambda(x)$ of leaves $\leq x$ and let C_T be the multiset $\{\lambda(x) : x \text{ is a node of } T\}$. There is a repetition $\lambda(y) = \lambda(x)$ whenever $\sigma_x = \{y\}$; in that case impose the order $\lambda(y) < \lambda(x)$. if on the other hand $\{y\}$ is a proper subset of σ_x, there is a proper inclusion $\lambda y \subset \lambda(x)$. The set C_T contains S as $\lambda(r)$, and contains each singleton $\{s\}$ in S as $\lambda(s)$. The absence of cycles in the tree T implies a *trichotomy law*: for all $U, V \in C_T$, either $U \subseteq V$ or $V \subseteq U$ or $U \cap V = \varnothing$. In view of the essential equivalence between trees and such multisets, we make the following definition.

Definition 7. A tree on a finite set S is a finite multiset C of subsets of S containing S and each singleton of S, and satisfying the trichotomy law, together with a linear ordering on each set of repetitions. The set of trees on S is denoted by $\mathcal{T}[S]$. For $T \in \mathcal{T}[S]$, the set of singletons $\{s\} \in T$ is identified with S.

$\mathcal{T}[-]$ carries a structure of $\mathcal{S}et$-species: to each bijection $\phi: S \to T$ we form a map $\mathcal{T}[S] \to \mathcal{T}[T]$ by relabelling along it.

Next, we introduce a grafting operation on trees. Suppose given a tree $T \in \mathcal{T}[X]$ and for each $x \in X$ a tree T_x rooted at x. Then the discrete graph on X (having no edges and denoted again by X) is obviously embedded in T and in the disjoint union of the T_x; the pushout of these two embeddings in the category of undirected graphs gives an acyclic connected graph. Defining the root of the pushout as the image of the root of T, the result is a rooted tree called a *grafting*, denoted by $m(T; T_x : x \in X)$. Let R be an equivalence relation on S; put $X = S/R$. and let $\pi\colon S \to X$ be the canonical quotient. Then the grafting operation just described induces a map

$$m[S]\colon \sum_{R \in Eq(s)} \mathcal{T}[S/R] \otimes \bigotimes_{x \in S/R} \mathcal{T}[\pi^{-1}(x)] \longrightarrow \mathcal{T}[S]$$

There is a map $u\colon X \to T$ which sends the element r of $X[1]$ to the rooted tree consisting only of r. The triple $(\mathcal{T}, m, u)$ defines a $\mathcal{S}et$-operad: satisfaction of the operad axioms follows easily from universal properties of pushouts. In fact, it is not difficult to show that $\mathcal{T}$ is the free operad generated by the $\mathcal{S}et$-species $\exp(X) - 1$. The unit map $\eta\colon (\exp(X) - 1) \to \mathcal{T}$ which generates it is given componentwise by maps $(\exp(X) - 1)[S] \to \mathcal{T}[S]$ each of which, for $|S| > 0$, sends the unique element in the domain to the "sprout" on S: the multiset $S + \sum_{s \in S}\{s\}$. It is clear that every structure of species $\mathcal{T}$ is obtained by grafting together a collection of such sprouts and instances of $u[1]$.

More generally, letting $\mathcal{C}$ be a complete cocomplete symmetric monoidal closed category, we describe the free operad $\mathcal{O}(G)$ on a $\mathcal{C}$-species G such that $G[0] = 0$. Let $N(T)$ denote the set of nodes in a tree $\mathcal{T} \in \mathcal{T}[S]$. As a species,

$$\mathcal{O}(G)[S] = \sum_{T \in \mathcal{T}[S]} \bigotimes_{x \in N(T) - S} G[\sigma_x]$$

for $|S| > 0$ and $\mathcal{O}(G)[0] = 0$. If in addition $G[1] = 0$, a summand corresponding to a tree T is 0 whenever there are nodes $x \in N(T) - S$ such that σ_x is a singleton: i.e., when there are repetitions of subsets. So if $G[1] = 0$, the summation can be restricted to multisets without repetitions, i.e., to rooted trees where each node is the join of the leaves below it. These shall be called *proper trees*.

- **Remark:** The description we have given of $\mathcal{O}(G)$ can be formulated as a "wreath product" $T \int G$, as in unpublished notes of Kelly, similar to the wreath product constructions pertaining to Kelly's "clubs" [13]. The essential point is that the multiplication on the operad $\mathcal{O}(G)$ is induced from the grafting multiplication on . The unit map $u\colon X \to \mathcal{O}(G)$ is also induced from that of $\mathcal{T}$; the map $u[1]$ factors through the summand where the iterated tensor product is indexed over an empty set (empty products are interpreted as the unit I for the tensor).

There is a familiar free—forgetful adjunction arising from the free operad construction: it will be useful to describe the counit of this adjunction using a kind of term-rewrite system. To begin, let $T \in \mathcal{T}[S]$ be a tree described as a multiset, and U a subset of S occurring in T neither as $\lambda(r)$ nor as $\lambda(s)$ for $s \in S$. Then we may delete U from T to obtain another tree $T/U \in \mathcal{T}[S]$, called the contraction of T along U (deletion corresponds to contraction of an internal edge in the tree). Contractions may be continued until one reaches a sprout, i.e., the element in the image of the unit $\eta\colon (\exp(X) - 1)[S] \to \mathcal{T}[S]$. This iterated contraction reproduces the unique $\mathcal{S}et$-species map $\mathcal{T} \to \exp(X) - 1$, necessarily a morphism of operads.

If a tree is regarded as the result of grafting together a collection of sprouts. then each contraction may be regarded as a tree surgery, where a subtree T' obtained by grafting together two sprouts is replaced by a sprout having the same leaves as the subtree. More formally. suppose given two subsprouts with leaf—sets of the form σ_x and σ_y, where $y \in \sigma_x$ and $\lambda(y) = U$, the deleted set, T', and the sprout which replaces it in T/U, both have x as root: we use the notation x/y to denote the root. in the latter case. Both have the same set of leaves. which we may write as $\sigma_{x/y}$. Observe there is an inclusion $\sigma_y \hookrightarrow \sigma_{x/y}$ and a surjection $\pi_y \colon \sigma_{x/y} \to \sigma_x$: indeed. defining an equivalence relation R on the set $\sigma_{x/y}$ by zRw if $z = w$ or $z, w \in \sigma_y$, the set σ_x is canonically identified with the set of R—equivalence classes on $\sigma_{x/y}$.

Now suppose G carries an operad structure. In $(G \circ G)[\sigma_{x/y}]$ there is a summand of the form

$$G[\sigma_x] \otimes \bigotimes_{z \in \sigma_x} G[\pi_y^{-1}(z)]$$

where $|\pi_y^{-1}(z)| = 1$ if $z \neq y$. There is a unit map $I \cong X[\pi_y^{-1}(z)] \to G[\pi_y^{-1}(z)]$ for $z \neq y$, and of course $\pi_y^{-1}(y) = \sigma_y$: therefore there is an induced map

$$G[\sigma_x] \otimes G[\sigma_y] \longrightarrow (G \circ G)[\sigma_{x/y}]$$

whence, after composing with the multiplication on G, a map

$$G[\sigma_x] \otimes G[\sigma_y] \longrightarrow G[\sigma_{x/y}]$$

Expand this by tensoring with identity maps to obtain a "G—contraction" map

$$m_{T/U} \colon \bigotimes_{x \in N(T)-S} G[\sigma_x] \longrightarrow \bigotimes_{x \in N(T/U)-S} G[\sigma_x]$$

on the T-summand of $\mathcal{O}(G)[S]$. Such G—contractions may be iterated until a sprout is reached, leading to a map

$$\bigotimes_{x \in N(T)-S} G[\sigma_x] \longrightarrow G[S]$$

and by the operad axioms, this map is independent of the order in which contractions are performed. Finally, all of these maps assemble to give a map

$$\epsilon_G[S] \colon \mathcal{O}(G)[S] \longrightarrow G[S]$$

which is the component at S of the counit ϵ_G of the adjuiiction.

We remark that if F is an operad and $F[1] = 1$, then ϵ_F restricts to a map $\mathcal{O}(\delta_F(X)) - X \to \delta_F(X)$. In particular there are contraction maps

$$\mu_{T/U} \colon \bigotimes_{x \in N(T)-S} \delta_F(X)[\sigma_x] \longrightarrow \bigotimes_{x \in N(T/U)-S} \delta_F(X)[\sigma_x].$$

9 The left bar construction

From this section on, F denotes a $\mathcal{V}$—operad with $F[0] = 0$ and $F[1] = 1$. We recall (see [6, 7]) a chain complex structure $B_l(F)$ on the free operad generated from the suspension $G = -\delta_F(X)$ (In this case $\mathcal{O}(G)[S]$ is a coproduct over *proper* trees on S, so that both its $\mathbb{Z}_2$—graded components are finite-dimensional.)

The grading on $B_l(F)[S]$ is determined from the construction of $\mathcal{O}(-\delta_F(X))$ as a $\mathcal{V}$—species: for example. if F is the V—operad $\exp(X) - 1$, and T is a proper tree then the summand

$$\bigotimes_{x \in N(T)-S} (-\delta_F(X))[\sigma_x]$$

is concentrated in degree given by the number n of sprouts σ_x (taken modulo 2). Taking $F(X) = \exp(X) - 1$ as basic, we define differentials for $B_l(F)$. Let $\mathcal{T}_j[S]$ denote those proper trees on S consisting of exactly j subsets of S aside from S and $\{s\}$ for $s \in S$. Arguing as we did in the proof of proposition 3, $\mathcal{O}(-\delta_F(X))$ lifts to a $\mathbb{Z}$-graded species $\mathcal{O}(\Sigma\delta_F(X))$ whose $(j+1)$-st component is given by

$$\sum_{T \in \mathcal{T}_j[S]} \Lambda[T].$$

If $\mathcal{P}S$ denotes the set $\mathcal{P}S - \{S\} - \cup_{s \in S}\{\{s\}\}$, and $\Lambda^j[S]$ the j-th exterior power of a space freely generated from S, there is a monomorphism

$$\sum_{T \in \mathcal{T}_j[S]} \Lambda[T] \hookrightarrow \Lambda^j[\mathcal{P}S].$$

To see this, first observe that every tree on S contains S and all singletons $\{s\}$, so no essential information is lost if we ignore these: thus there is a species isomorphism

$$\sum_{T \in \mathcal{T}_j[S]} \Lambda[T] \cong \sum_{T \in \mathcal{T}_j[S]} \Lambda[T - \{S\} - \cup_{s \in S}\{\{s\}\}].$$

MISSING PAGE 16 OF THE ORIGINAL

to work with species valued in the category of $\mathbb{Z}_2$-graded spaces (rather than just the finite—dimensional ones). We have inclusions

$$-F \circ \mathcal{O}(-\delta_F(X)) \xhookrightarrow{\eta \circ i} \mathcal{O}(-F) \circ \mathcal{O}(-F) \xhookrightarrow{m} \mathcal{O}(-F)$$

and a differential structure on $\mathcal{O}(-F)$ from the preceding discussion: by restriction we get a differential structure on $-F \circ \mathcal{O}(-\delta_F(X))$, and hence a complex $E_l(F)$. To show $E_l(F) \to X$ is a homotopy equivalence. it suffices to exhibit a contracting homotopy for $E_l(F)[S]$ when $|S| > 1$. We have

$$-F - \circ\mathcal{O}(-\delta_F(X)) \cong (-\delta_F(X) \circ \mathcal{O}(-\delta_F(X))) \oplus (-X \circ \mathcal{O}(-\delta_F(X)))$$

and now we write down the restriction of the contracting homotopy h to each summand. For the first summand we have

$$-\delta_F(X) \circ \mathcal{O}(-\delta_F(X)) \xrightarrow{\eta \circ 1} \mathcal{O}(-\delta_F(X)) \circ \mathcal{O}(-\delta_F(X)) \xrightarrow{m} \mathcal{O}(-\delta_F(X))$$

which we further compose with

$$\mathcal{O}(-\delta_F(X)) \xrightarrow{i} -X \circ \mathcal{O}(-\delta_F(X)) \xrightarrow{-u \circ 1} -F \circ \mathcal{O}(-\delta_F(X))$$

to obtain the first restriction h_1; here the map i is a degree 1 shift of the canonical isomorphism $\mathcal{O}(-\delta_F(X)) \cong X \circ \mathcal{O}(-\delta_F(X))$. The restriction h_2 of the contracting homotopy to the second summand $-X \circ \mathcal{O}(-\delta_F(X))$ is just a zero map. The proof that $h = h_1 + h_2$ is a contracting homotopy (which only uses simplicial identities) is standard and left to the reader.

The virtual equality $F \circ \mathcal{O}(-\delta_F(X)) \sim X$ follows from the fact that $E_l(F) \to X$ induces an isomorphism in homology. Thus $\mathcal{O}(-\delta_F(X))$ and $\sum_{n \geq 0} (-\delta_F)^n(X)$ both represent the virtual species $F^{-1} \in \mathcal{V}[[x]]$. In the next section we exhibit a chain map between their associated bar constructions $B_r(F)$, $B_l(F)$ which is a quasi-isomorphism.

10 A quasi-isomorphism

Under the hypotheses on an operad F used in the previous section, we describe a chain map $B_l(F) \to B_r(F)$ which induces an isomorphism] in homology.

As usual, it is easiest to begin with the case $F(X) = \exp(X) - 1$. First we exhibit a map

$$\mathcal{O}(-\delta_F(X)) \longrightarrow \sum_{n \geq 0} (-\delta_F)^n(X)$$

which comes from a map of $\mathbb{Z}$-graded species with components

$$\sum_{T \in \mathcal{T}_j[S]} \Lambda[T] \longrightarrow (\delta_F)^{j+1}(X)[S].$$

Let T be a proper tree on S whose elements, aside from S and the singleton sets, are subsets $U_1, \ldots, U_j$ of S. To each non-empty subset $U \subseteq S$ we may associate an equivalence relation R_U on S defined by $x R_U y$ if $x = y$ or $x, y \in U$. If $R, R' \in Eq(S)$, let the sum $R + R'$ denote their join in this lattice. Suppose the subsets U_i are arranged so that $U_i \subseteq U_j$ implies $i \leq j$, and let $R^{(i)}$ denote $\sum_{k \leq i} R_{U_k}$. Then there are strict inclusions $R^{(i)} < R^{(j)}$ when $i < j$, and the trichotomy law guarantees that $0 < R^{(i)} < i$ for all i. We thus obtain a chain in $\delta_F^{j+1}(X)[S]$ of the form

$$0 < R^{(1)} < \ldots < R^{(j)} < R^{(j+1)} := 1.$$

It is sometimes convenient to omit the last member from this chain, which carries no essential information.

The space $\Lambda[T]$ may be regarded as being generated by a j-fold exterior product $U_1 \wedge \ldots \wedge U_j$. Let T denote the poset whose elements are the sets U_i ordered by inclusion, and let $[j]$ denote the poset $\{1 < \ldots < j\}$. Consider the set $\mathrm{Ord}_b(T, [j])$ of order-preserving bijections $T \to [j]$: to each $\phi \in \mathrm{Ord}_b(T, [j])$ we define a sign $\mathrm{sgn}(\phi)$ as the sign of the permutation $[j] \to [j]$ which sends the element i to $\phi(U_i)$, and define equivalence relations

$$R_\phi^{(i)} = \sum_{k \leq i} R_{\phi^{-1}(k)}$$

for $1 \leq i \leq j$, which form a chain

$$R_\phi = [0 < R_\phi^{(1)} < \ldots < R_\phi^{(j)} < 1].$$

Finally. define a map

$$\begin{aligned} \iota_j[S] \colon \sum_{T \in \mathcal{T}_j[S]} \Lambda[T] &\longrightarrow (\delta_F)^{j+1}(X)[S] \\ U_1 \wedge \ldots \wedge U_j &\longmapsto \sum_{\phi \in \mathrm{Ord}_b(T,[j])} \mathrm{sgn}(\phi) \cdot R_\phi. \end{aligned}$$

This is clearly well-defined and natural in $S \in \mathcal{FB}$.

Proposition 5. *The map $\iota \colon B_l(F) \to B_r(F)$ is a monomorphism of chain complexes.*

Proof. First we remark that chains of the form R_ϕ for $\phi \in \mathrm{Ord}_b(T, [j])$ are chains $[0 < R_1 < \ldots < R_j < 1]$ such that, for all i, the equivalence relation R_{i+1}/R_i on S/R_i (given by the kernel pair of the quotient $S/R_i \to S/R_{i+1}$ with $R_0 = 0$ and $R_{j+1} = 1$ for convenience) is an equivalence relation R_{V_i} associated with a subset $V_i \subseteq S/R_i$. Let us call chains which satisfy this property "good". To prove that ι is monic, it is enough to see that we can retrieve T and $\phi \colon T \to [j]$ from the data of a good chain. This is easy: letting $\pi_i \colon S \to S/R_i$ be the canonical quotient, T is the set of subsets $U_i = \pi_i^{-1}(V_i)$, and ϕ is defined by $\phi(U_i) = i$. To show that ι preserves differentials, consider the differential of R_ϕ:

$$\sum_{i=1}^{j} (-1)^i [\ldots < R_\phi^{(i-1)} < R_\phi^{(i+1)} < \ldots]$$

By the trichotomy property for trees, either the i-th summand is a good chain, or $\phi^{-1}(i) \cap \phi^{-1}(i+1)$ is empty. In the latter case, consider $\phi' \in \mathrm{Ord}_b(T, [j])$ obtained by composing ϕ with the transposition $(i i+1)$. Clearly $\mathrm{sgn}(\phi') = -\mathrm{sgn}(\phi)$, and one easily checks that the i-th summands in $\partial(R_\phi)$ and $\partial(R_{\phi'})$ are the same. Thus, in computing $\partial(\iota(U_1 \wedge \ldots \wedge U_j))$, the "bad" chains cancel: what remains is a linear combination of good chains R_{ϕ_i}, where ϕ_i denotes the restriction of ϕ to $T - \phi^{-1}(i)$. Specifically, we have

$$\partial(\iota(U_1 \wedge \ldots \wedge U_j)) = \sum_{\phi \in \mathrm{Ord}_b(T,[j])} \mathrm{sgn}(\phi) \cdot \sum_{i=1}^{j} (-1)^i R_{\phi_i}$$

and some painstaking care with signs shows this equals

$$\sum_{i=1}^{j}(-1)^i \sum_{\phi\in \mathrm{Ord}_b(\widehat{T/U_i},[j-1])} \mathrm{sgn}(\phi)\cdot R_\phi = \iota(\partial(U_1\wedge\ldots\wedge U_j))$$

so that ι preserves differentials. This completes the proof. □

Theorem 3. *For $F(X)=\exp(X)-1$, $B_r(F)$ and $B_l(F)$ are quasi-isomorphic.*

Proof. From theorem 2, $H_*(B_r(F)[n])$ is concentrated in the top degree $n-1$, and from the results of [7], so is $H_*(B_l(F)[n])$. Since this is top degree, both of these homologies are given by the cycle groups in that degree, and by proposition 5, the chain map ι restricts to a monomorphism

$$Z_{n-1}(B_l(F)[n]) \longrightarrow Z_{n-1}(B_r(F)[n])$$

between these cycle groups. On the other hand, both of these homologies are virtually equivalent since both represent the S_n—character $\log(1+X)[n]$. Since both are concentrated in the same degree, this virtual equivalence implies they are isomorphic. Therefore the monomorphism above must itself be an isomor—phism. This completes the proof. □

Now consider more generally an operad F such that $F[0]=0$, $F[1]=1$. To construct a chain map $\iota\colon B_l(F)\to B_r(F)$, we couple the construction for the special case $F(X)=\exp(X)-1$ with a few simple observations. In this special case there are inclusions

$$\delta_F^{j+1}(X) \hookrightarrow F^{\circ(j+1)} \hookrightarrow \mathcal{O}(F)$$

so that each chain $0<R_1<\ldots<R_j<1$ of equivalence relations on S may be regarded as a (non-proper) tree on S. Abbreviating $\delta_{\exp(X)-1}$ to δ, we have for general F

$$\delta_F^{j+1}(X)[S] = \sum_{T\in\delta^{j+1}(X)[S]} \bigotimes_{x\in N(T)-S} F[\sigma_x]$$

Let $T\in\mathcal{T}_j[S]$ be a proper tree and consider a map $\phi\in\mathrm{Ord}_b(T,[j])$; earlier we produced a chain R_ϕ of equivalence relations, which we now regard as a tree. Observe that there is an isomorphism

$$\bigotimes_{x\in N(T)-S}\delta_F(X)[\sigma_x] \cong \bigotimes_{x\in N(R_\phi)-S} F[\sigma_x]$$

since we assumed $F[1]=1$. From this isomorphism, it is clear that there is an induced chain map

$$\sum_{T\in\mathcal{T}_j[S]}\Lambda[T]\otimes \bigotimes_{x\in N(T)-S}\delta_F(X)[\sigma_x] \longrightarrow \sum_{T\in\delta^{j+1}(X)[S]}\bigotimes_{x\in N(T)-S} F[\sigma_x]$$

which gives the desired map $\iota\colon B_l(F)\to B_r(F)$.

It is not difficult to show, using theorem 3, that ι induces an isomorphism in homology. This may suggest new techniques for deciding whether a given quadratic operad satisfies Koszul duality ([7]). For example, in certain cases one can establish Koszul duality through an appeal to theorem 2.

11 The Lie operad

Let $F(X) = \exp(X) - 1$, and let $B_l(F)'$ denote the dual of the complex $B_l(F)$, i.e., the cochain complex with differentials

$$\sum_{T \in \mathcal{T}_{j-1}[S]} \Lambda[T]' \longrightarrow \sum_{T \in \mathcal{T}_j[S]} \Lambda[T]'$$

obtained by transposing those of $B_l(F)[S]$. In [7] it is shown that the operad multiplication and unit on $\mathcal{O}(-\delta_F(X))$, which is the graded species underlying $B_l(F)'$, respect these differentials, so that $B_l(F)'$ carries a structure of an operad valued in the category of $\mathbb{Z}_2$—graded cochain complexes. It is called the *cobar* construction of F, and may be regarded as a lift of the virtual species $F^{-1}(X) = \log(1+X)$ to an operad.

Since the operation of additive inverse on $\mathcal{V}[[x]]$ lifts to the $\mathbb{Z}_2$—graded suspension (as an involution on $\mathcal{V}^{\mathcal{FB}}$ or on species valued in $\mathbb{Z}_2$-graded complexes), we can transport the monad $B_l(F)' \circ (-)$ across the suspension to lift the virtual species $-F^{-1}(-X) = -\log(1-X)$ to an operad. We can describe this operad using proposition 3. First, there is a $\mathcal{V}$—operad with components

$$(-1)^{|S|-1}\Lambda[S],$$

obtained by transporting the monad exp for free commutative monoids across the $\mathbb{Z}_2$-graded suspension. This is described in detail in [7], where it is called the determinantal operad. Next, if F and G are operads, then there is an obvious operad structure on the species with components $F[S] \otimes G[S]$. Putting this together, there is an identification between operads given componentwise as

$$-B_l(F)'(-X)[S] \cong B_l(F)'[S] \otimes (-1)^{|S|-1}\Lambda[S]$$

and since cohomology with coefficients in a field preserves coproducts and tensor products, there is an operad whose components are

$$H^{n-1}(B_l(F)[n]) \otimes \Lambda[n]$$

in degree 0 (compare with corollary 2 and the remarks which follow it).

Theorem 4. *([**tlot**]) The Lie operad L is isomorphic to this operad.*

Proof. (What follows is a sketch of the proof.) The cohomology space above is a cokernel of a space generated by the set of binary trees in $\mathcal{T}[n]$, so that the operad aboVe is an operad generated by a single binary operation (represented by a sprout with two leaves. called a 2-sprout). The kernel of this cokernel is the space of trees in $\mathcal{T}[n]$ obtained by grafting a collection of 2—sprouts with a 3-sprout (which represents a ternary operation), so that the operad $H^*(B_l(F))$ is generated by a binary operation and subject to a single equation in which the tertiary operation is set equal to zero. Upon twisting with the determinantal operad in degree 3, one may verify by hand that this equation is the Jacobi identity. Twisting with the determinantal operad in degree 2 gives the alternating identity, and these give a complete set of identities for the operad. But these are exactly the identities governing the Lie operad. □

- **Remark:** In [1], an explicit description is given for the correspondence between certain elements in free Lie algebras (Lyndon basis elements) and homology classes for partition lattices. Our notes here provide a conceptual framework for this description, which can be extracted from theorems 3 and 4 (compare with corollary 2).

Definition 8. A ($\mathbb{Z}_2$-graded) homotopy Lie algebra is an algebra over the operad

$$(-1)^{n-1}B_l(F)'[n] \otimes \Lambda[n]$$

(which is valued in the category of $\mathbb{Z}_2$-graded complexes).

- **Remark:** If $F(X) = \frac{X}{1-X}$ is the operad governing associative algebras without unit, then there is a virtual equivalence $F(X) \sim -F^{-1}(-X)$. The right-hand side is represented by an operad with components

  $$(-1)^{|S|-1}B_l(F)'[S] \otimes \Lambda[S],$$

  which is easily (and classically) shown to be homotopy-equivalent to the operad $\frac{X}{1-X}$. Algebras over this (right-hand) operad are called homotopy associative algebras: they are equivalent to the A_∞—algebras of [15].

References

[1] Helene Barcelo. “On the action of the Symmetric Group on the Free Lie Algebra and the Partition Lattice”. PhD thesis. U.C. San Diego, 1988.

[2] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Vol. 347. Lecture Notes in Mathematics. Springer-Verlag, 1973.

[3] B. J. Day. “On closed categories of functors”. In: *Reports of the Midwest Category Seminar IV*. Vol. 137. Lecture Notes in Mathematics. Springer-Verlag, 1970, pp. 1–38.

[4] S. Eilenberd and S. MacLane. “On the groups $H(\pi, n)$, I”. In: *Ann. Math.* 58.1 (1953), pp. 55–106.

[5] J. Folkman. “The homology groups of a lattice”. In: *J. Math. Mech.* 13 (1966), pp. 631–636.

[6] E. Getzler and J. D. S. Jones. "Operads, homotopy algebra, and iterated integrals for double loop spaces". In: (1994). arXiv: `9403055 [hep-th]`.

[7] V. A. Ginzburg and M. M. Kapranov. "Koszul duality for operads". In: *Duke Math. J.* 76 (1994), pp. 203–272.

[8] V. Hinich and V. Schechtman. "Homotopy Lie algebras". In: *Adv. Soviet Math.* 16(2) (1993), pp. 1–28.

[9] A. Joyal. "Foncteurs analytiques et espèces de structures". In: *Combinatoire énumérative*. Vol. 1234. Lecture Notes in Mathematics. Springer-Verlag, 1985, pp. 126–159.

[10] A. Joyal. "Une théorie combinatoire des séries formelles". In: *Advances in Mathematics* 42(1) (1981), pp. 1–82.

[11] A. Joyal and R. Street. "Braided tensor categories". In: *Advances in Mathematics* 102 (1993), pp. 20–78.

[12] G. M. Kelly. *Basic Concepts of Enriched Category Theory*. Vol. 64. London Math. Soc. Lecture Notes Series. Cambridge University Press, 1982.

[13] G. M. Kelly. "On clubs and doctrines". In: *Category Seminar*. Vol. 420. Lecture Notes in Mathematics. Springer-Verlag, 1974, pp. 181–256.

[14] M. Markl. "A compactification of the real configuration space: resolution of diagonals is the operadic completion". Preprint. 1996.

[15] J. D. Stasheff. "Homotopy associativity of H-spaces. II". In: *Trans. Amer. Math. Soc.* 108 (1963), pp. 293–312.

[16] Y-N. Yeh. "The calculus of virtual species and K-species". In: *Combinatoire énumérative*. Vol. 1234. Lecture Notes in Mathematics. Springer-Verlag, 1985, pp. 351–369.