Volume 155, number 2,3 PHYSICS LETTERS A 6 May 1991

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Topological lower bound on the energy of a twisted rod

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Received 8 June 1990; revised manuscript received 19 February 1991; accepted for publication 28 February 1991
Communicated by A.R. Bishop

If one end of an elastic rod is rotated by an angle of 2π relative to the other, the "body frame" along the rod traces out a noncontractible loop in SO(3). This is not the case for a rotation by 4π. A lower bound is derived for the energy of a thin elastic rod whose body frame traces out a noncontractible loop in SO(3).

If one takes an elastic rod, holds one end fixed, and twists the other through an angle of 2π, the twist cannot be undone by moving either end as long as the orientations of the ends are fixed. However, if one twists by an angle of 4π, the twist *can* be undone by moving the ends of the rods holding their orientations fixed. This is because the rotation group in three dimensions, SO(3), is doubly connected. Here we use this fact to derive lower bounds on the energy of a thin elastic rod with one end twisted by an angle of 2π. While there have been a number of applications of topology to continuum mechanics [1,2], this rather simple result seems not to have been noted before.

The state of a thin elastic rod may be described by a function F from the interval $[0, L]$, where L is the length of the rod, to SO(3). For each point $s\in[0, L]$, $F(s)$ describes the "body frame" of the rod as rotated from the standard frame (e_1, e_2, e_3). We may identify a tangent vector $\boldsymbol{\omega}$ at any point $x\in\mathrm{SO}(3)$ with a vector $(\omega_1, \omega_2, \omega_3)$ in the Lie algebra $\mathrm{so}(3)\cong\mathbb{R}^3$ by left translation of the tangent space at x to the identity in SO(3). The elastic energy of the rod is then given by

$$E=\frac{1}{2}\int_0^L \sum_{i=1}^3 I_i\omega_i^2\,\mathrm{d}s \tag{1}$$

(under the approximations made in ref. [3]), where $\boldsymbol{\omega}$ is the tangent vector $\mathrm{d}F/\mathrm{d}s$, and I_i are the principal moments of inertia: I_1 and I_2 for the cross-section of the rod and I_3 for the torsional rigidity of the rod. In particular, $I_1=I_2$ for a homogeneous rod with a circular cross-section.

As a digression, note that if one interprets the parameter s in eq. (1) as time, then E equals the *action* for the time evolution of a rigid body with moments of inertia I_i and angular velocity $\boldsymbol{\omega}$. Thus the problem of the thin elastic rod may be mapped onto the time evolution of a rotating rigid body. This was apparently first noted by Kirchhoff [4].

Give SO(3) the Riemannian metric g such that

$$\|\omega\|^2=\sum_{i=1}^3 I_i\omega_i^2\,,$$

cf. ref. [5]. Let g_0 denote this metric in the special case where $I_i=1$ for all i. Note that $g\geqslant I_{\min}g_0$, where $I_{\min}$ denotes the minimum of the I_i. Using this and the Cauchy–Schwarz inequality we have

$$E=\int_0^L g(\omega,\omega)\,ds$$

$$\geqslant I_{\min}\int_0^L g_0(\omega,\omega)\,ds$$

$$\geqslant \frac{I_{\min}}{L}\left(\int_0^L \|\omega\|\,ds\right)^2, \tag{2}$$

where $\|\omega\|$ denotes the length of ω with respect to the metric g_0.

There are two homotopy classes of loops in SO(3), the contractible loops (such as a rotation through 4π about any axis) and the noncontractible ones (such as a rotation through 2π). We can find a lower bound on the energy of a rod whose body frame F traces out a noncontractible loop in SO(3) using (2). This inequality implies that the energy is greater than or equal to $I_{\min}/L$ times the square of the length of the shortest noncontractible loop in SO(3) relative to the metric g_0.

Let α:SU(2)→SO(3) be the standard two-fold cover. (For a treatment of the relation between SO(3), SU(2) and S^3 see ref. [6]; for basic facts about covering spaces and homotopy of paths see ref. [7].) Recall that any contractible loop in SO(3) lifts to a loop in SU(2), while a noncontractible loop lifts to a path joining antipodal points $\pm x$ in SU(2). Let $\tilde{g}_0$ denote the lift of the metric g_0 on SO(3) to SU(2). Using the standard identification of SU(2) with S^3, the invariance of the metric $\tilde{g}_0$ on SU(2) implies that it is a constant multiple of the standard metric on S^3. Thus the shortest path between antipodal points follows a great circle on S^3. The loop in SO(3) traced out by rotating through the angle 2π about any axis $\boldsymbol{n}$ ($\|\boldsymbol{n}\|=1$) lifts to a great circle between antipodal points in SU(2), given by

$$\phi\mapsto\cos\tfrac{1}{2}\phi+\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\tfrac{1}{2}\phi\,,$$

as ϕ goes from 0 to 2π. This path has length

$$\int_0^{2\pi}\|\boldsymbol{n}\|\,d\phi=2\pi\,,$$

relative to the metric $\tilde{g}_0$. Thus we have the following lower bound on the energy E of a rod whose body frame traces out a noncontractible loop in SO(3):

$$E\geqslant\frac{4\pi^2 I_{\min}}{L}\,. \tag{3}$$

It also follows from the argument above that this lower bound is attained by a loop in SO(3) corresponding to rotation with constant angular velocity about an axis e_i such that $I_i=I_{\min}$. If $I_3>I_2\geqslant I_1$, this minimum corresponds to pure bending ($\omega_3=0$), while if I_1, $I_2>I_3$ the minimum corresponds to pure twisting ($\omega_1=\omega_2=0$).

Note also that between any two points in SO(3) there are two homotopy classes of paths, and each class will have a lower bound on its length. Thus for any fixed orientations of the ends of a rod there will be two lower bounds on the rod's energy, one for each homotopy class.

The lower bound (3) also holds for any rod that is bent into a loop. Here we replace the condition that the frame $F(s)$ traces out a noncontractible loop in SO(3) by the condition that both ends of the rod are at the same point in space. Let $\boldsymbol{x}(s)$ denote the space curve in $\mathbb{R}^3$ that the rod describes, and let $\kappa(s)$ denote the curvature of this curve. Assuming that $\boldsymbol{x}(L)=\boldsymbol{x}(0)$, it is known [8] that

$$\int_0^L \kappa\,ds\geqslant 2\pi\,.$$

Moreover, it is easily seen that

$$E\geqslant I_{\min}\int_0^L \kappa^2\,ds\,.$$

Using the Cauchy–Schwarz inequality we have

$$\left(\int_0^L \kappa\,ds\right)^2\leqslant L\int_0^L \kappa^2\,ds\,,$$

so that $E\geqslant 4\pi^2 I_{\min}/L$.

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