**The Vacuum and Lightcone Quantization of Interaction Hamiltonians**

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**Abstract**

Let $\phi$ denote the conformally invariant neutral free scalar field on $\mathbb{R} \times S^n$. The 'naive' lightcone Hamiltonian for a $\phi^p$ interaction is given by $\int_C {:}\phi^p{:}$ , where $C$ denotes a lightcone in $\mathbb{R} \times S^n$, and the Wick power is relative to the free vacuum. We show that this sesquilinear form annihilates the free vacuum if $n \geq 3$ is odd, $p > 2$, and $p(n-1) \equiv 0 \bmod 4$.

# 1 Introduction

Let $\phi$ denote the neutral free scalar field on $\mathbb{R} \times S^n$. The Wick power ${:}\phi^p(t,u){:}$ at the point $(t,u) \in \mathbb{R} \times S^n$ is a densely defined sesquilinear form on the free field Hilbert space. This permits the construction of $\int_{t=0} {:}\phi^p(t,u){:}\, d^n u$, the 'naive' interaction Hamiltonian for the Cauchy problem, as a densely defined sesquilinear form. For $p > 1$ this form corresponds to a densely defined operator only if the space dimension $n$ equals 1, while for higher dimensions its domain as an operator consists only of the zero vector [1]. This, of course, is why the Hamiltonian is thought to require further 'renormalization'.

Integrating ${:}\phi^p{:}$ along timelike directions has a stronger smoothing effect. For example, for all $p$ and $n$ the sesquilinear form $\int f(t)\, {:}\phi^p(t,u){:}\, dt$ corresponds to a densely defined operator for $f \in C_0^\infty(\mathbb{R})$, as can be shown using the technique of Ref. 2. Integrating over both space and time has a similar smoothing effect, and is useful in the perturbative construction of scattering as an operator-valued formal power series [3].

A less studied case is the integral of $:\phi^p:$ over a characteristic surface. Such integrals appear in the 'Goursat problem', the initial-value problem in which a field is determined by its restriction to a lightcone. The Goursat problem has long been of interest in (mostly nonrigorous) quantum field theory under the name of 'light-cone quantization' [4], and has recently been investigated for classical nonlinear wave equations [5]. The Goursat problem is more conveniently posed on $\mathbb{R} \times S^n$ than on Minkowski space, in part because of the compactness of the cone in the former case. Moreover, for conformally invariant nonlinear wave equations there is a close connection between the Goursat problem on $\mathbb{R}\times S^n$ and scattering in $\mathbb{R}^{n+1}$, since $\mathbb{R}\times S^n$ may be identified with the universal cover of conformally compactified Minkowski space, and scattering may be treated in terms of 'in' and 'out' fields on the lightcones at past and future infinity which form the boundary of Minkowski space as embedded in $\mathbb{R} \times S^n$. (In addition to the above references, see Ref. 6.)

The conformal wave equation on $\mathbb{R} \times S^n$ may be written as $(\Box + c_n)\varphi = 0$, where $c_n = \left(\frac{n-1}{2}\right)^2$ and $\varphi$ denotes a classical field, i.e., a real-valued function. The Hamiltonian of the classical nonlinear wave equation

$$(\Box + c_n)\varphi + p\varphi^{p-1} = 0$$

is equal to

$$\int_C \frac{1}{2}\{(\nabla\varphi)^2 + c_n\varphi^2\} + \varphi^p \, ,$$

where $C$ denotes a lightcone in $\mathbb{R} \times S^n$ [5]. This leads us to consider the sesquilinear form $\int_C :\phi^p:$ as the 'naive lightcone interaction Hamiltonian' for the quantized $\phi^p$ theory. Here we prove that when $n \geq 3$ is odd, $p > 2$, and $p(n-1)$ is divisible by 4, the form $\int_C :\phi^p:$ annihilates the free vacuum. (The case $n = 3$, $p = 4$ which appears in the massless $\phi^4$ theory satisfies these conditions.) In particular, the form's domain as an operator is nontrivial, in contrast to the naive interaction Hamiltonian for the Cauchy problem.

This theorem recalls the notion that the free field and interacting field vacua coincide at past infinity, as implicitly assumed in naive perturbative quantum field theory. In view of the evidence against this idea, however, one should be cautious about making this interpretation.

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## 2 Results

We begin with some notation concerning conformal geometry; for more details see [5, 7]. Let $\mathbf{M}_0$ denote $(n+1)$-dimensional Minkowski space, which we identify with $\mathbb{R}^{n+1}$ given the Lorentzian metric $dx_0^2 - dx_1^2 - \cdots - dx_n^2$. Let $\widetilde{\mathbf{M}}$ denote the universal cover of the conformal compactification of $\mathbf{M}_0$, which we identify with the 'Einstein universe' $\mathbb{R} \times S^n$ and give the coordinates $(\tau, u)$. We specify points of $S^n$ in terms of pairs $(\rho, \omega) \in \mathbb{R} \times S^{n-1}$ as follows. Embedding $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$, so that $u = (u_1, \ldots, u_{n+1})$, and similarly embedding $S^{n-1}$ in $\mathbb{R}^n$, we then map $(\rho, \omega) \in \mathbb{R} \times S^{n-1}$ to $(\cos\rho, \sin\rho\, \omega) \in S^n$.

Let $C \subset \widetilde{\mathbf{M}}$ denote the lightcone given by $\{\rho = \tau - \pi, \rho \in [0, \pi]\}$. We identify $C$ with $S^n$ by means of the map $(\tau, u) \mapsto u$, and use this identification to endow $C$ with a Riemann metric, and thus the volume form $\nu = \sin^{n-1}\rho\, d\rho \wedge d\omega$, where by an abuse of notation $d\omega$ denotes the volume form on $S^{n-1}$.

The real scalar bundle of conformal weight $\frac{n-1}{2}$ over $\widetilde{\mathbf{M}}$ admits a conformally invariant subspace whose sections satisfy the 'conformal wave equation'. In an appropriate parallelization the sections may be represented by real functions on $\widetilde{\mathbf{M}}$, in such a way that the equation is given by $(\Box + c_n)\varphi = 0$, where $\Box$ denotes the D'Alembertian on $\widetilde{\mathbf{M}}$ relative to the Einstein metric.

Let $\mathbf{H}$ denote the Hilbert space of solutions of the conformal wave equation with the conformally invariant complex structure and inner product given in Ref. 5. Elements of $\mathbf{H}$ may be specified in terms of their Cauchy data at $t = 0$, giving an isomorphism $\mathbf{H} \cong H^{1/2}(S^n) \oplus H^{-1/2}(S^n)$, where the $H^k(S^n)$ denote $L^2$-Sobolev spaces. The group $\widetilde{\mathbf{G}} = \widetilde{SO}(2, n+1)$ acts as conformal transformations of $\widetilde{\mathbf{M}}$, and the action of $\widetilde{\mathbf{G}}$ on sections of the weight $\frac{n-1}{2}$ scalar bundle over $\widetilde{\mathbf{M}}$ gives rise to a strongly continuous unitary representation $U$ of $\widetilde{\mathbf{G}}$ on $\mathbf{H}$.

The 'conformally invariant free scalar field' on $\widetilde{\mathbf{M}}$ is the system $(\mathbf{K}, W, \Gamma, v)$, uniquely determined up to unitary equivalence by the following properties: 1) $\mathbf{K}$ is a complex Hilbert space, 2) the 'Weyl system' $W$ is a strongly continuous map from $\mathbf{H}$ to unitaries on $\mathbf{K}$ such that

$$W(f)W(f') = e^{-i\,\mathrm{Im}\langle f,f'\rangle/2}W(f+f')\,,$$

3) $\Gamma$ is a strongly continuous positive-energy unitary representation of $\widetilde{\mathbf{G}}$ on $\mathbf{K}$ such that

$$W(U(g)f) = \Gamma(g)W(f)\Gamma(g)^{-1}\,,$$

and 4) the vector $v \in \mathbf{K}$, the 'free vacuum', is cyclic for $W$ and invariant under $\Gamma$. Let $\mathbf{K}_\infty$ denote the space of $C^\infty$ vectors for $\Gamma$, with its natural topology. Define $\Phi(f)$ to be the self-adjoint generator of $W(tf)$ for $f \in \mathbf{H}$. Then for $f \in \mathbf{H}$, the operator given by

$$a(f) = \frac{1}{\sqrt{2}}\{\Phi(f) + i\Phi(if)\}$$

is a continuous from $\mathbf{K}_\infty$ to itself.

The real Hilbert space $L^2(S^n)$ is a direct sum $\bigoplus_{j=0}^{\infty}\mathbf{E}_j$ of eigenspaces of the Laplacian, where each $\mathbf{E}_j$ has a finite orthonormal basis $\{\psi_{jk}\}$ such that $\Delta\psi_{jk} = -j(n-1+j)\psi_{jk}$ [8]. Defining

$$a_{jk} = E_j^{-1/2}\, a(\psi_{jk}, 0)\,,$$

where $E_j = j + \frac{n-1}{2}$, it follows that

$$[a_{jk}, a^*_{j'k'}]\, u = \delta_{jj'}\, \delta_{kk'}\, u\,.$$

for any $u \in \mathbf{K}_\infty$.

Given the point $x \in \widetilde{\mathbf{M}}$ with coordinates $(t, \rho, \omega)$, the operator $a(x)$ from $\mathbf{K}_\infty$ to $\mathbf{K}_\infty$ is given by the strongly convergent sum

$$a(x) = \sum_{j,k} E_j^{-1/2}\, \psi_{jk}(\rho, \omega)\, e^{-iE_j t}\, a_{jk}\ . \tag{1}$$

As described in [2], given $p \geq 1$ and $x \in \widetilde{\mathbf{M}}$, the 'pth Wick power' $:\!\phi^p(x)\!:$ of the conformally invariant free scalar field on $\widetilde{\mathbf{M}}$ is a continuous sesquilinear form on $\mathbf{K}_\infty$ given by

$$:\!\phi^p(x)\!:(u, u') = 2^{-p/2} \sum_{i=1}^{p} \binom{n}{p} \langle a(x)^p\, u, a(x)^{n-p}\, u' \rangle\ . \tag{2}$$

**Theorem.** *Let $n \geq 3$ be an odd integer and $p$ be an integer greater than 2. Suppose further that $p(n-1) \equiv 0 \bmod 4$. Then for all $u \in \mathbf{K}_\infty$,*

$$\int_C :\!\phi^p\!:(u, v) = 0\ .$$

Proof - Let $\zeta \in \widetilde{\mathbf{G}}$ denote the generator of the infinite cyclic subgroup of $\widetilde{\mathbf{G}}$ that acts on $\widetilde{\mathbf{M}}$ by $\zeta(\tau, u) = (\tau + \pi, -u)$ [7]. For $n \geq 3$ odd the following periodicity result closely related to Huyghen's principle holds [8]:

$$U(\zeta) = (-1)^{(n-1)/2}\ .$$

It follows [2] that

$$:\!\phi^p(\rho, \rho + \pi, \omega)\!: \;=\; (-1)^{p(n-1)/2}\, :\!\phi^p(\rho - \pi, \rho, \omega)\!:\ ,$$

and as $p(n-1)/2$ is even,

$$\int_C :\!\phi^p\!: \;=\; \frac{1}{2} \int_{S^{n-1}} \int_0^{2\pi} :\!\phi^p(\rho - \pi, \rho, \omega)\!:\, \sin^{n-1}\!\rho\, d\rho d\omega\ . \tag{3}$$

By equation (2), we have

$$:\!\phi^p(\rho - \pi, \rho, \omega)\!:(u, v) = 2^{-p/2} \langle a(\rho - \pi, \rho, \omega)^p u, v \rangle\ .$$

Each function $\psi_{jk}$ is the restriction to $S^n$ of a degree-j harmonic homogeneous polynomial on $\mathbb{R}^{n+1}$, so for any $\omega \in S^{n-1}$, $\psi_{jk}(\rho, \omega)$ is a polynomial of total degree $\leq j$ in $\cos\rho$ and $\sin\rho$. It follows from equation (1) that $\langle a(\rho-\pi, \rho, \omega)^p u, v\rangle$ is for each $\omega \in S^{n-1}$ a polynomial in $e^{-i\rho}$ with nonvanishing terms only of degree greater than or equal to $pE_j - pj = p(n-1)/2$. Thus the integral in equation (3) vanishes if $p(n-1)/2 > n-1$, as the hypotheses guarantee. $\square$

# 3 Conclusions

This result for the neutral scalar field generalizes in an obvious manner to multi-component scalar fields. In four dimensions one could also consider the trilinear spinor/one-form interaction of conformal QED, or the Yang-Mills interaction, using the results of Ref. 7. It is unclear, though, whether the annihilation of the free vacuum by the naive lightcone interaction Hamiltonian is of real physical significance. If the form $\int_C :\varphi^p:$ corresponded to a densely defined operator on $\mathbf{K}$, one could hope to show that its sum with the free Hamiltonian is essentially self-adjoint. However, it is not known whether any vectors in $\mathbf{K}_\infty$ other than multiples of the vacuum vector are mapped by $\int_C :\varphi^p:$ into vectors in $\mathbf{K}$.

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