Categorical Linear Algebra — A Setting for Questions from Physics and Low-Dimensional Topology

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Abstract The 2-category of all small categories equivalent to a (finite) cartesian product of the category of finite dimensional vector spaces over a fixed field, left-exact functors, and natural transformations has structures closely mimicking those found in ordinary linear algebra. We examine these structures, the relation of this category to the 2-categories and weak 2-categories (*née bicategories*) studied by Kapranov and Voevodsky [KV]. In a final section, we give a notion of "matrix theory" applicable to the 2-categories considered, and extending to give a computationally convenient n-categorical setting for studying questions related to QFT's (most especially *topological* QFT's).

This work was inspired by early drafts of Kapranov and Voevodsky's manuscript, which has subsequently been entitled *Braided Monoidal 2-Categories, 2-Vector Spaces and Zamolodchikov Tetrahedral Equations* [KV]. Analogues of some of the results in it are now found in section V.3 of the most recent version of [KV] available to the author. The most important results however are new, specifically, those dealing with dual categories and hom-categories and the reduction of theory of $\mathcal{V}$-module functors and $\mathcal{V}$-modular transformations to exact functors and arbitrary natural transformations, as is the cleaner proof of the coherence properties of the braided monoidal 2-category of $\mathcal{V}$-modules which is afforded by the use of a universal property satisfied by the $\mathcal{V}$-tensor product.

As with Kapranov and Voevodsky [KV] and Lawrence [Law], our purpose is to provide an algebraic footing for the extension to higher dimensions of the successful interaction between 3-manifold topology, quantum field theory and monoidal category theory (cf. [A, C, FY, MS, RT1, RT2, TV, T, Y3]). Specifically, the algebraic structures describe herein provide the algebraic prerequisites for the study of quantum invariants of 2-knots and the formulation of factorization at a corner for (3+1)-dimensional topological quantum field theories.

In a final section, we present a "matrix theory" for categorical linear algebra, and note how it can be extended to n-categories, n-tuple categories, and the n-algebras of Lawrence [Law].

Section 1: One-Categorical Definitions and Results

Fix a field K, and let $\mathcal{V}$ denote the category of finite-dimensional vector spaces over K. Observe that $\mathcal{V}$ is a K-linear abelian category in which all exact sequences split, and is moreover equipped with a symmetric monoidal structure $(\otimes_K, K, \alpha, \sigma, \rho, \lambda)$ (cf. Mac Lane [M]) for which all of the functors $(-) \otimes_K A$ and $A \otimes_K (-)$ are K-linear and exact.

Definition 1 (cf. Yetter [Y1]) *If $\mathcal{M}$ is a monoidal category, left (resp. right) $\mathcal{M}$-module is a category $\mathcal{C}$ equipped with a functor $\odot : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ (resp. $\odot : \mathcal{C} \times \mathcal{M} \to \mathcal{C}$ and natural transformations $a_{M,N,C} : (M \otimes N) \odot C \to M \odot (N \odot C)$, and $l_C : I \odot C \to C$ (resp. $\mathbf{a}_{C,M,N} : (C \odot M) \odot N \to C \odot (M \otimes N)$ and $r_C : C \odot I \to C$) satisfying the obvious pentagon and triangle conditions. A $\mathcal{N}, \mathcal{M}$-bimodule is a category $\mathcal{C}$ equipped with a left $\mathcal{N}$-module structure $\odot_{\mathcal{N}}$, and a right $\mathcal{M}$-module* *structure ${}_{\mathcal{M}}\odot$, together with a natural isomorphism $\mathrm{a}_{N,C,M} : (N \odot_{\mathcal{N}} C)_{\mathcal{M}} \odot M \to N \odot_{\mathcal{N}} (C_{\mathcal{M}} \odot M)$ satisfying two additional (obvious) pentagon relations.*

In the case (which holds throughout this entire paper) where the categories involved are K-linear and abelian, we require that the functor $\odot$ be K-linear and exact in each variable separately.

Definition 2 (cf. [Y1]) *A left $\mathcal{M}$-module functor F from a $\mathcal{M}$-module $\mathcal{C}$ to another $\mathcal{D}$ is a pair $(F, \tilde{F})$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\tilde{F}$ is a natural isomophism $\tilde{F}_{M,C} : F(M \odot C) \to M \odot F(C)$, such that all diagrams of the form in Figure 1 commute*

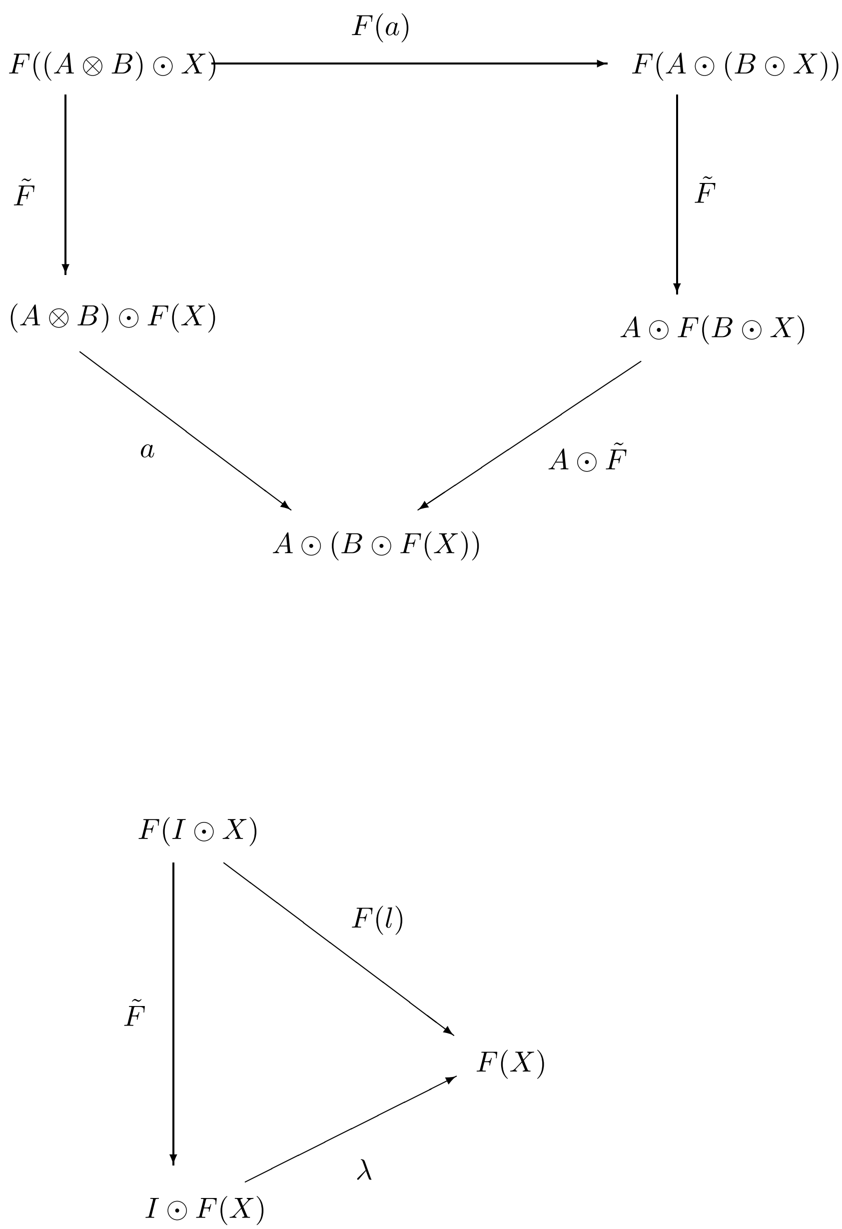

Figure 1: Coherence Conditions for Left-Module Functors

Right module functors are defined analogously.

A $\mathcal{N}, \mathcal{M}$-bimodule functor from $\mathcal{C}$ to $\mathcal{D}$ (both bimodules) is a triple $(F, \tilde{F}, \check{F})$ so that $(F, \tilde{F})$ is a left $\mathcal{N}$-module functor, $(F, \check{F})$ is a right $\mathcal{M}$-module functor and all diagrams of the form in Figure

2 commute (by abuse of notation we denote both actions by $\odot$).

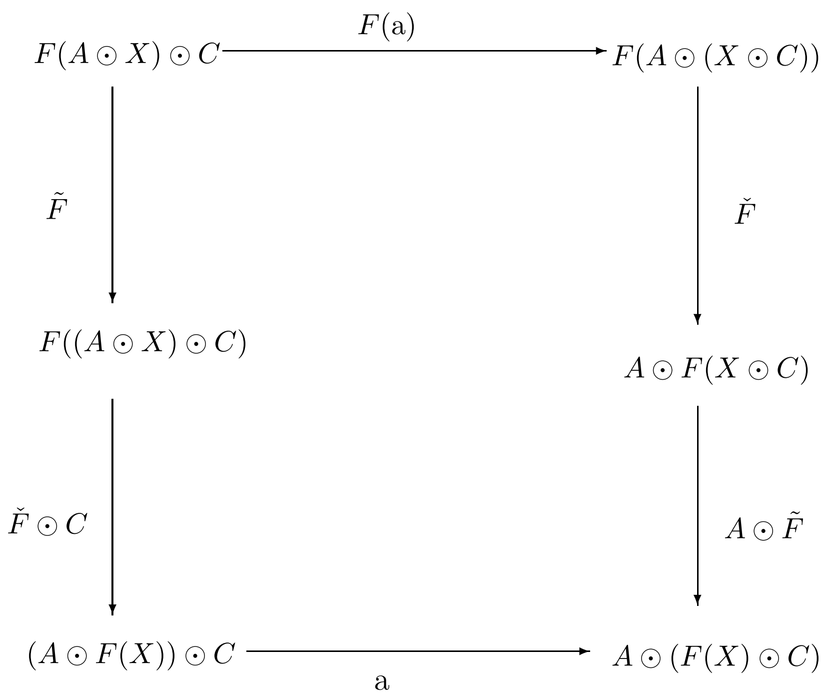

Figure 2: Coherence Condition for Bimodule Functors

When F and G are left $\mathcal{M}$-module functors, a natural transformation ϕ from F to G is a left $\mathcal{M}$-modular transformation if all diagrams of the form in Figure 3 commute. Right $\mathcal{M}$-modular transformations are defined similarly. A $\mathcal{M}$,$\mathcal{N}$-bimodular transformation is a natural transformation between bimodule functors which is both a left $\mathcal{M}$-modular transformation and a right $\mathcal{N}$-modular transformation.

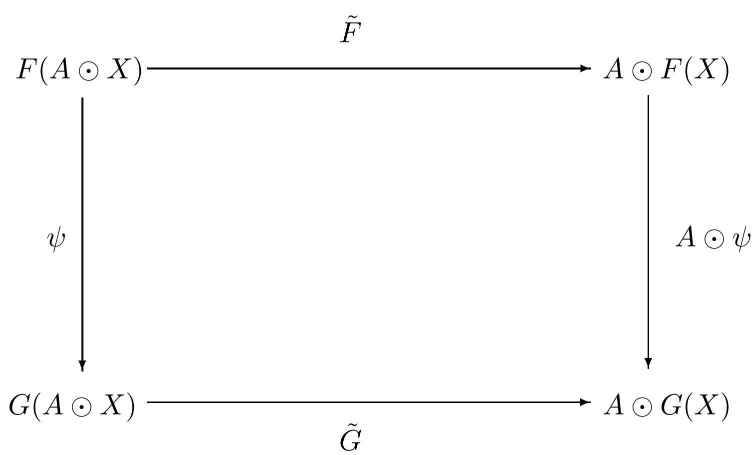

Figure 3: Coherence Condition for Modular Transformations

It is important to observe at this point that we are able to avoid discussion of ring-categories, and coherence for natural distributivity (cf. Kapranov and Voevodsky [KV] and Laplaza [Lap]) by our requirements of abelianness of categories and exactness of action functors in both variables.

(Once one has exactness in each variable, given a choice of direct sum for pairs of objects, all of the additonal coherence conditions for ring-categories and actions of ring-categories follow by use of universal properties.)

We will apply the notions of left and right $\mathcal{M}$-module functors and modular tranformations first to the symmetric monoidal category $\mathcal{V}$. In this case the notions simplify. More generally, we prove

Proposition 3 *Let $\mathcal{M}$ be a braided monoidal category (cf. [JS1, JS2]) with braiding σ. If $\mathcal{C}$ is a left $\mathcal{M}$-module with structure functor $\odot$ and structural transformations $a_{M,N,C}, l_C$ then tw$\odot$ with structural transformations $a^{-1}_{N,M,C}\sigma_{N,M}$, and l_C is a right $\mathcal{M}$-module.*

proof: We must verify that the relevant pentagon and triangle commute. We leave this to the reader (hint: the pentagon is filled by a pentagon, a braiding hexagon and two naturality squares).

$\mathcal{V}$-modules will be our analogues of vector-spaces. Our analogue of finite dimensionality is given by:

Definition 4 *A $\mathcal{V}$-module $\mathcal{X}$ is semi-simple (resp. finitely (or artinian) semi-simple (f.s.s.)) if there exists a set (resp. finite set) $S \subset Ob(\mathcal{X})$ such that*

1. *each object in S is simple (i.e. $Hom(S, S)$ is 1-dimensional)*
2. *every object in $\mathcal{X}$ is isomorphic to a finite limit of objects in S (hence by simplicity) a finite product of objects in S.*

Lemma 5 "The splitting lemma" *If $\mathcal{X}$ is a semi-simple $\mathcal{V}$-module, then all short exact sequences in $\mathcal{X}$ split.*

proof: Consider the s.e.s.

$$0 \to A \to B \to C \to 0.$$

Now since $\mathcal{X}$ is f.s.s. we have a diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{Z\in S} X^{\epsilon_X(A)} & \longrightarrow & \bigoplus_{Z\in S} X^{\epsilon_X(B)} & \longrightarrow & \bigoplus_{Z\in S} X^{\epsilon_X(C)} & \longrightarrow & 0
\end{array}$$

where the bottom maps are named by matrices (of multiple of projections onto elements of S followed by inclusions). (Note only finitely many elements of S occur here.) Standard linear algebra can be used to construct splittings of the bottom s.e.s., and thus the top (isomorphic) s.e.s. also splits. $\square$

Corollary 6 *If F is a functor from an semi-simple $\mathcal{V}$-module to any abelian category then the following are equivalent:*

1. *F is exact*
2. *F is left-exact*
3. *F is right-exact*
4. *F preserves all finite biproducts*

We are now in a position to prove a K-linear analogue of Freyd's representability theorem for set-valued functors (cf. [M]).

Theorem 7 "The Finite Linear Representability Theorem" *If $\mathcal{X}$ is an f.s.s. $\mathcal{V}$-module, then any functor $F : \mathcal{X} \to \mathcal{V}$ is representable if and only if it is exact.*

This will follow from the corollary above and the following theorem (which we will not need in the sequel, but will state and prove forthwith):

Theorem 8 *If $\mathcal{X}$ is any K-linear category equipped with a finite set of object S such that every object is isomorphic to a finite limit of objects in S, then a functor $F : \mathcal{X} \to \mathcal{V}$ is representable if and only if it preserves all finite limits and all intersections of sets of subobjects.*

proof:
"Only if" is immediate since all representable functors preserve all limits which exist.
"If": We imitate the proof of the Special Adjoint Functor Theorem given in Mac Lane [M].

It suffices to show that the underlying functor

$$F_0 : \mathcal{X} \xrightarrow{F} \mathcal{V} \xrightarrow{U} \mathbf{SETS}$$

is representable (where U is the obvious underlying set functor).

To do this, in turn, it suffices to show that the "comma" category $* \downarrow F_0$ has an initial object. Recall that $* \downarrow F_0$ is the category whose objects are pairs $(X,\ p : \{*\} \to F_0(X))$, where X is an object of $\mathcal{X}$, and in which a map from (X, p) to (Y, q) is a map $f : X \to Y$ in $\mathcal{X}$ such that $F_0(f)(p) = q$.

Now, choose a basis $B(J)$ for each $F(J)J \in S$. And consider the object

$$\left(\prod_{J\in S} J^{B(J)},\ p_B : \{*\} \longrightarrow \prod_{J\in S} J^{B(J)}\right)$$

in $* \downarrow F_0$, where p_B maps the point to the element whose coordinate in the factor labelled by $x \in B(J)$ is x.

Now, by the same lemmas used in Mac Lane [M] (pages 117 and 125), $* \downarrow F_0$ has all finite limits, all interections of subobjects and is well-powered.

We claim that the intersection of all subobjects of $(\prod_{J\in S} J^{B(J)},\ p_B : \{*\} \longrightarrow \prod_{J\in S} J^{B(J)})$ is initial. As in the proof of the Special Initial Object Theorem in Mac Lane [M] it is clear that this object admits at most one map to any given object, since otherwise the equalizer of any pair would be a proper subobject, which is impossible.

Thus it remains only to show the existence of a map to any $(A, p) \in * \downarrow F_0$.

Now, there is an isomorphism $A \tilde{\to} \prod_{i \in I} J_i$ for I a finite index set, and J_i not-necessarily-distinct elements of S. We then have a map from $\prod_{J \in S} J^{B(J)}$ to $\prod_{i \in I} J_i$ whose i coordinate map is $\sum_{b_j \in B(J)} a_j p_j$ where $S_i = J$ and p_j is the projection from the product onto the coordinate indexed by $j \in B(J)$ and the i coordinate of the point p is the element of given in terms of the basis $B(J)$ by $\sum a_j b_j$. Observe that this map names a map from $(\prod J^{B(J)}, p_B)$ to (A, p) in $* \downarrow F_0$.

Composing this map with the inclusion of the minimal subobject then gives the required map. □

Section 2: Two-categorical Definitions and Results

Now, if we let $[\mathcal{X}, \mathcal{Y}]$ denote the full sub-category of the functor category $\mathcal{Y}^{\mathcal{X}}$ consisting of K-linear exact functors, we have as a corollary to the last result of the previous section

Corollary 9 *If $\mathcal{X}$ is an f.s.s. $\mathcal{V}$-module, then $[\mathcal{X}, \mathcal{V}]$ is equivalent to $\mathcal{X}^{op}$.*

Thus far we largely have been ignoring the fact that our categories are $\mathcal{V}$-modules, and have been concentration on the consequences of (finite) semi-simplicity. The following corollary of the requirement that a $\mathcal{V}$-action be exact in both variables shows that if we are concerned only with properties "up to equivalence of categories and functors", there is no loss in ignoring the $\mathcal{V}$-action:

Proposition 10 *Given any $\mathcal{V}$-module $\mathcal{X}$, the restriction of the action to the full-subcategory $\mathcal{V}_!$ consisting of the coordinatized vector-spaces K, K^2, K^3, . . . is naturally isomorphic to the "standard action" given by*

$$K^n \odot X = X^n$$

In fact, a much stronger result holds:

Theorem 11 *If $\mathcal{X}$ and $\mathcal{Y}$ are semi-simple $\mathcal{V}$-modules, then every exact functor from $F : \mathcal{X} \to \mathcal{Y}$ admits a unique $\mathcal{V}$-module functor structure. Moreover, every natural transformation between such exact functors is a $\mathcal{V}$-modular transformation.*

It is convenient to first prove

Lemma 12 *If $\Theta, \Xi : \mathcal{X} \to \mathcal{C}$ are exact functors from any semi-simple $\mathcal{V}$-module to any abelian category $\mathcal{C}$, then a natural transformation $\xi : \Theta \Longrightarrow \Xi$ is completely determined by its values on a representative set of simple objects, $S \in \mathcal{X}$. In particular ξ_X for any object X is simultaneously the canonical map induced by the universal property of $\Theta(X)$ as a coproduct of $\Theta(s)$'s for $s \in S$, and the canonical map induced by the universal property of $\Xi(X)$ as a product of $\Xi(s)$'s.*

Proof: By naturality, the component ξ_X must make all squares in a diagram of the form in Figure 4 commute, where the ι_{s_i}'s (resp. π_{s_i}'s) are the inclusions (resp. projections) for the expression of X as a biproduct of object in S. Such a map is uniquely determined by either of the universal properties in the statement of the lemma. Moreover, the lack of non-zero maps between the distinct simple objects in S implies that *any* choice of maps $\{\xi_s : \Theta(s) \to \Xi(s) | s \in S\}$ determines such a natural transformation. □

Proof of Theorem 11: By the previous lemma (applied in each variable separately), observe that the structural natural transformation will be completely determined by its components of the form $\tilde{F}_{K,s}$ for K the ground field and $s \in S$.

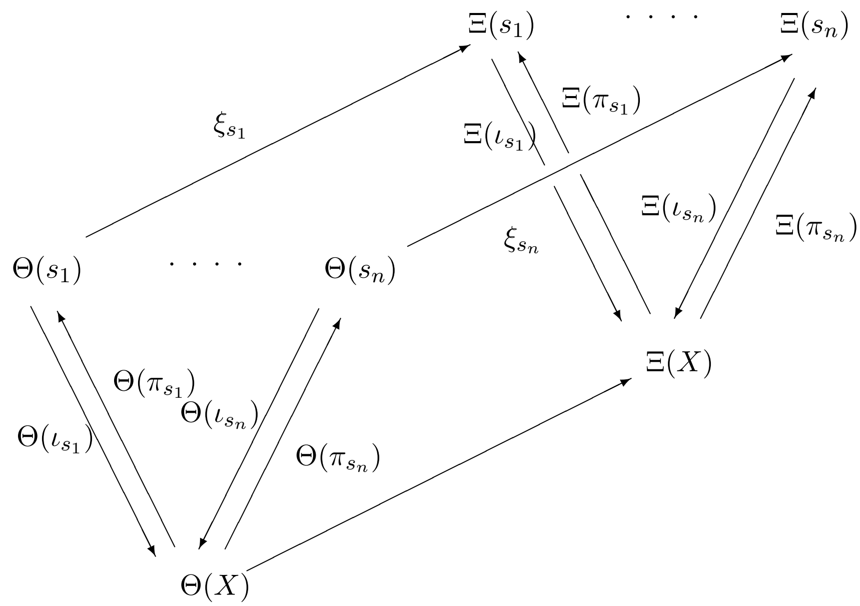

Figure 4: Simple objects determine natural transformations

The triangle condition in the definition of monoidal functoriality then forces these components to be $F(l)l^{-1}$. Thus there is at most one $\mathcal{V}$-module functor structure on any exact functor F. It remains only to show that this choice of structural natural transformations satisfies the pentagon condition.

Note, however, that by the previous lemma it suffices to check the equality of the two ways around the pentagon (each a natural transformation between functors exact in each variable) at triples of objects of the form (K, K, s) for $s \in S$. We leave it to the reader to fill the pentagon with two squares which commute by the naturality of $\tilde{F}$, two triangles which commute by the triangle condition on the $\mathcal{V}$-action, and a triangle of the form $K \odot$ (a triangle expressing $\tilde{F}_{K,s} = F(l)l^{-1}$).

Similarly, to verify that any natural transformation between exact functors between semi-simple $\mathcal{V}$-modules is a $\mathcal{V}$-modular transformation, it suffices to check the commutativity of the required square for pairs of objects of the form (K, s) for $s \in S$. We leave it to the reader to fill the square with two triangles defining $\tilde{F}_{K,s}$ and $\tilde{G}_{K,s}$ and two squares, one commuting by the naturality of the natural transformation in question, the other commuting by the naturality of l^{-1}. □

Now, observe that $\mathcal{V}$ is equivalent to its opposite category (with the functor in each direction being dual-space).

We now wish to show that any f.s.s. $\mathcal{V}$-module is equivalent to a category of the form $\mathcal{V}^n$. In particular, we show

Proposition 13 *If $\mathcal{X}$ is a semi-simple $\mathcal{V}$-module, and S is any choice of representatives of isomorphism classes of simple objects in $\mathcal{X}$, then there exists a canonical equivalence of categories between $\mathcal{X}$, and $\mathcal{X}_S$, the full sub-category of objects of the form*

$$\oplus_{J \in S} V_J \odot J$$

where the V_J's are objects in $\mathcal{V}$. Moreover, the functors and natural transformations in the equivalence are $\mathcal{V}$-module functors and $\mathcal{V}$-modular natural transformations.

proof: Given an object $X \in \mathcal{X}$, let

$$F(X) = \oplus_{J_S} Hom_{\mathcal{X}}(J, X) \odot J.$$

We claim that the inclusion of $\mathcal{X}_S$ and F provide an equivalence of categories between $\mathcal{X}_S$ and $\mathcal{X}$. First, note that F is plainly functorial. Second, observe that $F(X)$ is naturally isomorphic to X, since it suffices to test isomorphism and naturality on hom spaces targetted at the simple objects. Next, the triangle identities for the adjunction conditions can, likewise, be trivially tested on hom spaces. Finally, observe that the exactness properties of $\oplus$ and internal hom immediately show that F is exact. Thus the final statement follows immediately from Theorem 11. $\Box$

We can now show

Proposition 14 *If $\mathcal{X}$ and $\mathcal{Y}$ are f.s.s. $\mathcal{V}$-modules, so is $[\mathcal{X}, \mathcal{Y}]$.*

proof: First observe that since $\mathcal{Y}$ is an f.s.s. $\mathcal{V}$-module, there is an equivalence between $\mathcal{Y}$ and $\mathcal{V}^n$ for some n, and thus an induced equivalence between $[\mathcal{X}, \mathcal{Y}]$ and $[\mathcal{X}, \mathcal{V}]^n$. By Corollary 9, this is equivalent to $(\mathcal{X}^{op})^n$, and thus to $\mathcal{X}^n$. It remains only to show that the $\mathcal{V}$-action on $\mathcal{Y}$ induces an action on $[\mathcal{X}, \mathcal{Y}]$. Thus we need to describe the induced functor

$$\odot_* : \mathcal{V} \times [\mathcal{X}, \mathcal{Y}] \to [\mathcal{X}, \mathcal{Y}]$$

and verify that it, together with the induced natural transformations, satisfy the required coherence conditions.

So let $(V \odot_* F)(X) = V \odot F(X)$ whenever V is a vector-space, and F is a left-exact functor from $\mathcal{X}$ to $\mathcal{Y}$. The structural natural transformations are then induced by those of $\odot$, the $\mathcal{V}$ action on $\mathcal{Y}$, and thus trivially satisfy the coherence conditions. $\Box$

We have thus shown that the 2-category of $\mathcal{V}$-modules is equipped with a kind of internal hom. To define a corresponding tensor product, we need:

Definition 15 *If $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ are $\mathcal{V}$-modules, a $\mathcal{V}$-bilinear functor $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a functor such that for each object $X \in \mathcal{X}$ and each $Y \in \mathcal{Y}$, $F(X, -)$ and $F(-, Y)$ are exact functors.*

We can now make

Definition 16 *A $\mathcal{V}$-module, $\mathcal{P}$, equipped with a $\mathcal{V}$-bilinear functor $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{P}$, and universal among such, in the sense that given any $\mathcal{V}$- bilinear functor $B : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ there exists an exact functor $\lceil B \rceil : \mathcal{P} \to \mathcal{Z}$ such that $\Phi \lceil B \rceil = B$ which is unique up to canonical natural equivalence by natural tranformations inducing the identity transformation on the composite with Φ, is called a $\mathcal{V}$-tensor product of $\mathcal{X}$ and $\mathcal{Y}$. If such a $\mathcal{V}$-module exists for a given $\mathcal{X}$ and $\mathcal{Y}$, we denote it $\mathcal{X} \underline{\otimes} \mathcal{Y}$.*

Before proceeding to construct the $\mathcal{V}$-tensor product for f.s.s. $\mathcal{V}$- modules, let us note that if it exists, it will have the expected relationship with the internal hom:

Proposition 17 *For any $\mathcal{V}$-modules $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$*

$$\mathcal{V} - bilin(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \mathcal{V} - mod(\mathcal{X}, [\mathcal{Y}, \mathcal{Z}])$$

Consequently if $\mathcal{X} \underline{\otimes} \mathcal{Y}$ exists,

$$\mathcal{V} - mod(\mathcal{X} \underline{\otimes} \mathcal{Y}, \mathcal{Z}) \simeq \mathcal{V} - mod(\mathcal{X}, [\mathcal{Y}, \mathcal{Z}])$$

Here $\cong$ denotes isomorphism of categories, and $\simeq$ denotes equivalence of categories.

proof: The second statement follows immediately from the first by the universal property of $\underline{\otimes}$.

On objects, the isomorphisms in each direction in the first statement are given by

$$F : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z} \longmapsto (X \mapsto F(X, -) : \mathcal{Y} \to \mathcal{Z})$$

and

$$\Phi : \mathcal{X} \to [\mathcal{Y}, \mathcal{Z}] \longmapsto ((X, Y) \mapsto \Phi(X)(Y)).$$

Plainly these are inverse maps on the sets of objects. For maps, note that a map in $[\mathcal{Y}, \mathcal{Z}]$ is a natural transformation, and thus, a map in $\mathcal{V} - mod(\mathcal{X}, [\mathcal{Y}, \mathcal{Z}])$ an assignment of maps natural in both $\mathcal{X}$ and $\mathcal{Y}$ variables, the same thing as a map in $\mathcal{V} - bilin(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$. □

We now show

Theorem 18 *If $\mathcal{C}$ and $\mathcal{D}$ are f.s.s. $\mathcal{V}$-modules, then there exists a $\mathcal{V}$-tensor product $\mathcal{C}\underline{\otimes}\mathcal{D}$.*

proof: The objects of $\mathcal{C}\underline{\otimes}\mathcal{D}$ are formal expressions of the form

$$V_1 \cdot (C_1, D_1) \oplus V_2 \cdot (C_2, D_2), ..., V_n \cdot (C_n, D_n)$$

where the V_i's are vector spaces, the C_i's (resp. D_i's) are objects of $\mathcal{C}$ (resp. $\mathcal{D}$). Abbreviating the object above as

$$\bigoplus_{i=1}^{n} V_i \cdot (C_i, D_i)$$

we then have hom-spaces given by

$$Hom_{\mathcal{C}\underline{\otimes}\mathcal{D}}(\bigoplus_{i=1}^{n} V_i \cdot (C_i, D_i), \bigoplus_{i=1}^{m} W_j \cdot (E_j, F_j) = \bigoplus_{i=1,j=1}^{n,m} Hom(V_i, W_j) \otimes Hom_{\mathcal{C}}(C_i, E_j) \otimes Hom_{\mathcal{D}}(D_i, F_j).$$

Composition is given matrix-multiplication-wise on the indices i, j and component-wise on the tensorands of each summand. Explicitly, if

$$\begin{bmatrix} f_{11} \otimes c_{11} \otimes d_{11} & \cdots & f_{1m} \otimes c_{1m} \otimes d_{1m} \\ \vdots & & \vdots \\ f_{n1} \otimes c_{n1} \otimes d_{n1} & \cdots & f_{nm} \otimes c_{nm} \otimes d_{nm} \end{bmatrix}$$

and

$$\begin{bmatrix} g_{11} \otimes a_{11} \otimes b_{11} & \cdots & g_{1p} \otimes a_{1p} \otimes b_{1p} \\ \vdots & & \vdots \\ g_{m1} \otimes a_{m1} \otimes b_{m1} & \cdots & g_{mp} \otimes a_{mp} \otimes b_{mp} \end{bmatrix}$$

are a composable pair of maps, their composite is a matrix of triple tensor products of maps whose i, k entry is

$$\sum_{j=1}^{m} (f_{ij} g_{jk} \otimes c_{ij} a_{jk} \otimes d_{ij} b_{jk}).$$

Observe that this category is K-linear, and has a $\mathcal{V}$-action given by

$$V \odot \bigoplus_{i=1}^{n} V_i \cdot (C_i, D_i) = \bigoplus_{i=1}^{n} (V \otimes V_i) \cdot (C_i, D_i).$$

It is plain that the coherence conditions on the action hold, since they follow from the coherence of the monoidal category structure on $\mathcal{V}$. To see that the action is exact in the second variable we must wait until we have shown that the putative $\mathcal{C}\underline{\otimes}\mathcal{D}$ is abelian.

Likewise, there is an evident functor $\Phi : \mathcal{C} \times \mathcal{D} \to \mathcal{C}\underline{\otimes}\mathcal{D}$ given by

$$\Phi(C, D) = K \cdot (C, D)$$

on objects, and by

$$\Phi(f : C \to C', g : D \to D') = (1 \otimes f \otimes g) \in Hom(K, K) \otimes Hom_{\mathcal{C}}(C, C') \otimes Hom_{\mathcal{D}}(D, D')$$

on maps. Again the exactness properties must wait until we have verified the abelianness of the putative $\mathcal{C}\underline{\otimes}\mathcal{D}$.

To show that the category is abelian, is suffices to show that it is equivalent to a category of the form $\mathcal{V}^n$.

Now, by the construction of the hom-spaces and the composition, it is easy to show

1. that the formal $\oplus$ in the definition of the objects of the category is in fact a biproduct,

and

2. the functor $\Phi' : \mathcal{V} \times \mathcal{C} \times \mathcal{D} \to \mathcal{C}\underline{\otimes}\mathcal{D}$ given by

$$\Phi'(V, C, D) = V \cdot (C, D)$$

on objects, and by

$$\Phi'(v : V \to W, f : C \to C', g : D \to D') = (v \otimes f \otimes g) \in Hom(V, W) \otimes Hom_{\mathcal{C}}(C, C') \otimes Hom_{\mathcal{D}}(D, D')$$

on maps preserves biproducts in each variable (up to canonoical isomorphism).

Now, $\mathcal{C}$ and $\mathcal{D}$ are f.s.s. Let S (resp. T) a finite set of simple objects in $\mathcal{C}$ (resp. $\mathcal{D}$) so that every object in $\mathcal{C}$ (resp. $\mathcal{D}$) is isomorphic to a finite (bi)product of objects in S (resp. T).

It then follows from the observation above that every object of the putative $\mathcal{C}\underline{\otimes}\mathcal{D}$ is isomorphic to a finite (bi)product of objects in

$$S \star T = \{\ K \cdot (s, t) \mid s \in S, t \in T\ \}.$$

Choosing an isomorphism from every object to a biproduct of objects in $S \star T$ (with all occurences of each $K \cdot (s, t)$ grouped together in an ordering), then induces an equivalence of categories between the putative $\mathcal{C}\underline{\otimes}\mathcal{D}$ and $\mathcal{V}^{|S||T|}$. Thus the category is abelian. Moreover, since all exact sequences split in categories equivalent to $\mathcal{V}^n$, it suffices to show that biproducts are preserved to show that a functor targetted there is exact. Thus the functor Φ is a $\mathcal{V}$-bilinear functor, and (by the same condition in the first variable of Φ' and the exactness of $\otimes$ in the second variable in $\mathcal{V}$) the $\mathcal{V}$-action is exact in the second variable.

Finally, we must verify the universal property: given a $\mathcal{V}$-bilinear functor $B : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, we have an exact functor $\lceil B \rceil' : \mathcal{C}\underline{\otimes}\mathcal{D} \to \mathcal{E}$ given on objects by

$$\bigoplus_{i=1}^{n} V_i \cdot (C_i, D_i) \longmapsto \bigoplus_{i=1}^{n} V_i \odot B(C_i, D_i).$$

Similarly, maps are sent to matrices of linear maps tensored with images of maps under B. As it stands, this is not the desired $\lceil B \rceil$. $\lceil B \rceil$ is obtained by modifying $\lceil B \rceil'$ by replacing all direct summands in the image of the form $K \odot B(C_i, D_i)$ (here K is the ground field in its role as the *chosen* monoidal identity object for $\mathcal{V}$) with an isomorphic summand of the form $B(C_i, D_i)$, and suitably composing components of maps by $l^{\pm 1}_{B(C_i, D_i)}$ as needed to make sources and targets match correctly.

Once this is done, it is clear that $\Phi \lceil B \rceil = B$. As for uniqueness, note that the restriction of $\lceil B \rceil$ is fixed by the condition just verified. Since, as we have already seen, every object in the putative $\mathcal{C} \underline{\otimes} \mathcal{D}$ is a finite direct sum of objects in the image of Φ, the existence of a canonical natural equivalence between $\lceil B \rceil$ and any other functor satisfying the requirements follows immediately from the requirement that both functors be exact. $\square$.

With a little modification the proof of the previous theorem can be made to carry the stronger result

Theorem 19 *If $\mathcal{C}$ and $\mathcal{D}$ are semi-simple $\mathcal{V}$-modules, then there exists a $\mathcal{V}$-tensor product $\mathcal{C} \underline{\otimes} \mathcal{D}$.*

Alternatively the tensor product of arbitrary semi-simple $\mathcal{V}$-modules $\mathcal{C}$ and $\mathcal{D}$ can be constructed as follows: choose sets S, T of simple objects in each category as required by the definition of semi-simplicity. Now for each finite subset $A \subset S$ (resp. $B \subset T$) let $\mathcal{C}_A$ (resp. $\mathcal{D}_B$) be the full subcategory of objects isomorphic to a finite biproduct of objects in A (resp. B). $\mathcal{C} \underline{\otimes} \mathcal{D}$ is then the union over all A and B of $\mathcal{C}_A \underline{\otimes} \mathcal{D}_B$.

The following uniqueness result is a consequence of the universal property defining $\mathcal{V}$-tensor products:

Proposition 20 *If $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{P}$ and $\Phi' : \mathcal{X} \times \mathcal{Y} \to \mathcal{P}'$ are two $\mathcal{V}$-tensor products of $\mathcal{X}$ and $\mathcal{Y}$, then they are equivalent by a natural equivalence both of whose functors are exact.*

As with bilinearity, multilinearity has an analogue in $\mathcal{V}$-modules:

Definition 21 *A functor $M : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Z}$ is $n - \mathcal{V}$-multilinear if each functor*

$$M(X_1, ...X_{i-1}, -, X_{i+1}, ..., X_n) : \mathcal{X}_i \to \mathcal{Z}$$

is exact. An n-fold $\mathcal{V}$-tensor product is the universal object for $n - \mathcal{V}$-multilinear functors, with the same sort of uniqueness and canonicity properties as the binary $\mathcal{V}$-tensor product.

A similar proof to that given for the existence of binary $\mathcal{V}$-tensor products shows that semi-simple $\mathcal{V}$-modules have n-fold $\mathcal{V}$-tensor products. It is then trivial to show that

Proposition 22 *Every parenthesization of $\mathcal{X}_1 \underline{\otimes} \mathcal{X}_2 \underline{\otimes} \cdots \underline{\otimes} \mathcal{X}_n$ into an iterate of binary $\mathcal{V}$-tensor products is an n-fold $\mathcal{V}$-tensor product.*

We also have

Proposition 23

$$\mathcal{V}\underline{\otimes}\mathcal{X} \simeq \mathcal{X} \simeq \mathcal{X}\underline{\otimes}\mathcal{V}$$

Moreover, the functors can be chosen so as to identify $\mathcal{X}$ with a full-subcategory of the other categories, so that the inclusion followed by the (bi)reflection is the identity functor on $\mathcal{X}$ (exactly, not up to canonical isomorphism).

sketch of proof: Construct the functors in the equivalences by mapping (V, X) (resp. (X, V)) to $V \odot X$, and X to (K, X) (resp. (X, K)), then modify them to give the "moreover" condition as in the construction of the functor induced by the universal condition in the proof of Theorem 18.

Having shown all this, we now have

Theorem 24 *The 2-category of semi-simple $\mathcal{V}$-modules, exact functors and natural transformations, $\mathcal{V} - mod_{ss}$, equipped with the operation of $\mathcal{V}$-tensor product is a weak monoidal 2-category in the sense of Kapranov and Voevodsky [KV] (i.e. a tricategory with a single object, cf. [GPS]).*

proof: Every one of the 2-diagrams required to commute in the definition of tricategory has as its two sides compositions of canonical natural equivalences between universal functors between a pair of n-fold $\mathcal{V}$-tensor products of the same list of $\mathcal{V}$-modules. It follows from the uniqueness of the canonical natural transformations that they all commute. □

In fact, it follows from the equivalence of 1. and 4. in Corollary 6, Theorem 11 and an examination of the constructions of the tensor products here and in Kapranov/Voevodsky [KV] that

Proposition 25 *The 2-category of f.s.s. $\mathcal{V}$-modules, exact functors, and natural transformations, $\mathcal{V}-mod_{f.s.s}$, equipped with $\mathcal{V}$-tensor product is isomorphic as a weak monoidal 2-category to $2-Vect$ as defined in Kapranov/Voevodsky [KV].*

Similarly, canonicity requirements can by used to verify the necessary 2-commutativities to show

Theorem 26 *$\mathcal{V}$-mod as weak monoidal 2-category is equipped with a 2-braiding in the sense of Kapranov and Voevodsky [KV] induced by the univeral property of the $\mathcal{V}$-tensor product and the fact that*

$$\mathcal{X} \times \mathcal{Y} \stackrel{tw}{\to} \mathcal{Y} \times \mathcal{X} \stackrel{\Phi}{\to} \mathcal{Y}\underline{\otimes}\mathcal{X}$$

makes $\mathcal{Y}\underline{\otimes}\mathcal{X}$ into a $\mathcal{V}$-tensor product of $\mathcal{X}$ and $\mathcal{Y}$ (note the order).

Moreover we can restate Proposition 17 as

Theorem 27 *$\mathcal{V}$-mod is a closed weak monoidal 2-category in the sense that every 2-functor $-\underline{\otimes}\mathcal{X}$ admits a weak right 2-adjoint $[\mathcal{X}, -]$.*

If we restrict our attention to f.s.s. $\mathcal{V}$-modules, the closed structure in fact satisfies a 2-categorical analogue of compact-closed-ness (cf. Kelly/Lapalza [KL]):

Proposition 28 *For any f.s.s. $\mathcal{V}$-modules $\mathcal{X}$ and $\mathcal{Y}$,*

$$[\mathcal{X},\mathcal{Y}] \simeq [\mathcal{X},\mathcal{V}]\underline{\otimes}\mathcal{Y} \simeq \mathcal{X}^{op}\underline{\otimes}\mathcal{Y}$$

by equivalences of categories in which both functors are exact.

proof: We have already observed the equivalence of categories (by exact functors) between $[\mathcal{X},\mathcal{V}]$ and $\mathcal{X}^{op}$. For the first equivalence of categories, we have functors given on objects by

$$F \in [\mathcal{X},\mathcal{Y}] \longmapsto \bigoplus_{s\in S} K \cdot ((X \mapsto Hom_{\mathcal{Y}}(S, F(X))), S)$$

and

$$\bigoplus_i V_i \cdot ((X \mapsto W_i(X)), Y_i) \longmapsto (X \mapsto \bigoplus_i V_i \cdot (W_i(X) \cdot Y_i)).$$

It is easy to verify (using exactness properties of the $\mathcal{V}$-actions and Proposition 13) that these functors induce an equivalence of categories.□

This last observation is important to the use of $\mathcal{V}-mod$ as the foundation for quantum invariants of 2-knots: the 2-braidings of Kapranov/Voevodsky [KV] will not suffice to give 2-knot invariants– as in the case of classical knot one must have suitable notions of dual objects to define a categorical trace. Much work remains to be done here: we do not have a general statement of what coherence conditions are satisfied by weak monoidal 2-categories which are compact closed in this sense. The precise statement would be welcomed, since work of Fischer [F] (cf. also Carter/Saito [CS]) shows that the 2-category of 2-tangles, which plays a role in the theory of knotted surfaces in $\mathbf{S^4}$ analogous to that played by the category of tangles in classical knot theory, has similar duality properties.

It is now possible to reproduce in the categorical setting many of the structures which rely on linear algebra and tensor-products for their formulation: corresponding to algebras, coalgebras, bialgebras, ambialgebras (cf. Quinn [Q]) one has algebra categories, coalgebra categories, bialgebra categories (cf. Crane/Frenkel [CF]), and ambialgebra categories (which in the f.s.s. case are the modular tensor categories of Moore/Seiberg [MS], when a non-degeneracy condition is imposed).

The point of this generalization is this: if a suitable notion of Hopf category (including antipode), and natural examples are found, their 2-category of representations should have the structure required to give solutions to the Zamolodchikov tetrahedral equation with the right kind of duality to construct invariants of 2-knots and 4-manifolds. These applications are, however, beyond the scope of the present paper. In the very special case of the standard truncation of $Rep(U_q(sl_2)$ for q a root of unity, regarded as a weak monoidal 2-category with a single object, invariants of topological 4-manifolds have been constructed by Crane and Yetter [CY]. Crane has conjectured that suitable weak monoidal 2-categories with several objects will lead to non-trivial invariant of PL (and hence smooth) 4-manifolds.

Section 3. "Matrix Theory"

If the structures suggested at the end of the previous section are to be of any use to topology or physics, it must be possible to carry out calculations. Kapranov and Voevodsky [KV] provide a "fully coordinatized" version of $\mathcal{V} - mod$, in which the 2-arrows are matrices of matrices.

In this section, we propose an alternative picture of fully coordinatized 2-arrows which readily generalizes to contruct n-categories, n-tuple categories, and n-algebras (in the sense of Lawrence [Law]) of "matrices".

As a starting point, observe that we can identify an n by m matrix with entries in K with a weighting of the edges of the complete bipartite graph $K(n, m)$ with elements of K, and an ordering on each of the two sets of vertices. More generally, we may identify K-weighted bipartite graphs (K-bigraphs for brevity) in general with n by m matrices over K by taking the i, j entry to be the sum of the weights on edges from i to j (an empty sum being 0). Thus, matrices can be identified with equivalence classes of K-weighted bipartite graphs under the equivelence relation induced by the operations of

1. delete 0-weighted edges,

and

2. replace a pair of edges with common endpoints with a single edge weighted with the sum of the weights.

Thus, the columns (resp. rows) of a matrix correspond to the vertices in the first (resp. second) set of vertices in the bipartite structure. Addition of matrices of the same size corresponds to identitfying corresponding vertices of two K-bigraphs with the same number of vertices in each part. Multiplication of matrices corresponds to the following operation: identify the vertices of the second set of one K-bigraph with the vertices of the second set of another (using the ordering), then form the K-bigraph whose two sets of vertices are the first set of the first K-bigraph and the second set of the second K-bigraph, and whose edges are two-edge paths in the tripartite graph formed by the identification above weighted with the product of the weights on their two edges. Transpose consists in turning the K-bigraph "backwards" (i.e. reversing the roles of the first and second sets of vertices).

Direct sum of matrices corresponds to disjoint union of the K-bigraphs, tensor product to the operation of taking the product of the underlying bipartite graphs (i.e. if $G_i = (V_i, W_i, E_i)$ for $i = 1, 2$ the product is $(V_1 \times V_2, W_1 \times W_2, E_1 \times E_2)$ with the obvious incidence relations), and weighting the edge (e, f) with the product of the edge-weights of e and f.

We illustrate these in Figure 5.

Square matrices correspond to K-bigraphs with the same number of vertices in each set. The identity matrix corresponds to the K-bigraph with only horizontal edges weighted 1. The trace of a K-bigraph is the sum of the weights of the horizontal edges. The determinant of a K-bigraph is the sum over complete matchings of the product of the weights of the edges in the matching times $(-1)^{\#\text{ofcrossings}}$ where the number of crossings of a complete matching is the number of crossings between edges of the complete matching when the bigraph is drawn in the standard way (vertices of each set arranged in column with the second set to the right of the first and all edges drawn as straight lines). (Note: these definitions are independent of which equivalent K-bigraph is chosen to represent a class.)

The point here is not to replace familiar matrices with unfamiliar K-bigraphs, but rather to partially geometrize matrix theory so that its analogues in higher categorical dimensions can be more easily described. The final touch to this geometrization is provided by the observation that a bipartite graph is a graph fibered over the graph with two vertices joined by a single edge.

We can now define the analogues of matrices in higher categorical dimensions:

Definition 29 *An n-dimensional bigonal (resp. simplicial, cubical) multimatrix over K is a (finite) n-dimensional bigonal (resp. simplicial, cubical) complex fibered over a single n-bigon (resp. n-simplex, n-cube) in the sense that each i-cell is mapped to an i-cell in such a way that the boundary relations are preserved in all dimensions, together with a weighting of the n-cells of the complex with elements of K and an ordering on the set of vertices lying above each vertex of the n-bigon (resp. n-simplex, n-cube) over which it is fibered.*

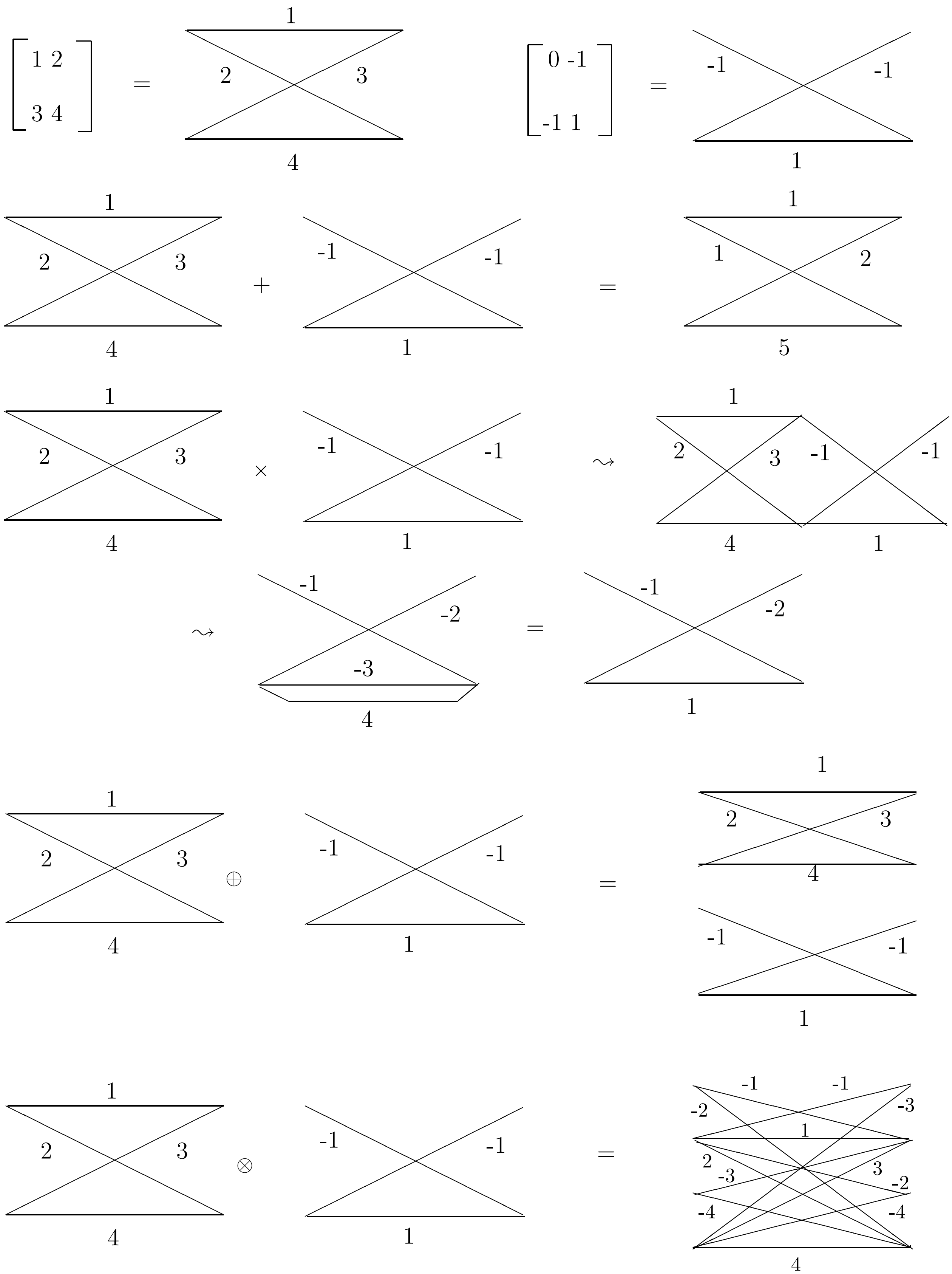

Figure 5: Matrix operations as operations on K-bigraphs

Two multimatrices are equivalent if they are related by the equivalence relation generated by

1. Delete an open n-cell with weight 0 *(note: the boundary is not deleted even if it is not shared by any other n-cell).*

and

2. Replace two n-cells with the same boundary by a single n-cell with that boundary and weighted by the sum of the weights on the deleted n-cells.

We can then use the symmetry and "pasting" properties of bigons and cubes to describe a very rich algebraic structure on each type of multimatrix.

In particular

Theorem 30 *n-dimensional cubical multimatrices over K are the arrows of an n-tuple category in the sense of Ehresmann (cf. [M]) $n-CMM$ as follow:*

The i-source $s_i(A)$ (resp. i-target $t_i(A)$) of a cubical multimatrix A is the $n-1$-complex lying above the 0 *(resp.* 1*) section of the i^{th} projection of the base cube onto the unit interval.*

If A and B are multimatrices with $t_i(A) = s_i(B)$ their i-composite $A \circ_i B$ is the multimatrix obtained as follows: identify $t_i(A)$ with $s_i(B)$ to obtain a space X with a map to the larger n-cube obtained by identifying the 1*-section of the i^{th} projection of the base cube for A with the* 0*-section of the i^{th} projection of the base cube for B. The $A \circ_i B$ has as underlying cubical complex the complex whose n-cubes are global sections of X over the larger n-cube, and whose j-cubes are sections of X over the j-faces of the larger n-cube. Observe that any n-cube in $A \circ_i B$ is the union of an n-cube of A with an n-cube of B; the weight of an n-cube of $A \circ_i B$ is then the product of the weight of the cube from A and the cube from B.*

An identity map for the i^{th} composition is a cubical multimatrix all of whose n-cells have weight 1*, and whose underlying n-complex is the product of the $n-1$ complex lying above the* 0*-section of the i^{th} projection with an interval.*

proof: It is easy to verify that the identity maps are identities, that the composition described above is associative, and that for all $i \neq j$ the middle-four interchange law holds for $\circ_i$ and $\circ_j$. $\square$

Observe that the n-tuple category just described has a tremendous amount of symmetry: to every symmetry of the n-cube there is a functor interchanging the compositions and reversing some of them, and in particular each composition has a notion of duality (or transpose) coming from the relection in the $n-1$ plane perpendicular to the i^{th} axis.

Likewise there is are notions of global direct sums (disjoint union) and tensor product (over each j-face of the base cube take as the set of j-faces the product of the sets of j-faces from two multimatrices, and weight the n-cubes with the product of the weights). Thus, in some sense which remains to be made precise, $n-CMM$ is a "ring n-tuple category".

As a convenience, we use the structure of $n-CMM$ to simplify the description of the structure of $n-BMM$. Observe that an n-dimensional bigon is a quotient of an n-dimensional cube. Explicitly the 1-cube and 1-bigon are identical; if we have expressed the k-bigon β_k as a quotient of the k-cube $[0,1]^k$ by $q_k : [0,1]^k \to \beta_k$, we have

$$q_{k+1} = [0,1]^{k+1} \xrightarrow{id \times q_k} [0,1] \times \beta_k \xrightarrow{\phi} \beta_{k+1}$$

where ϕ is the quotient which collapses each of the 0 and 1 sections of the first projection to a single point.

Pulling back along the quotient map q_n then associates to any bigonal n-multimatrix a cubical n-multimatix.

Theorem 31 *n-dimensional bigonal multimatrices over K are the n-arrows of an n-category $n-BMM$ whose compositions are induced by the composition of the associated cubical n-multimatrics.*

proof: Given the structure on $n-CMM$ it remains only to check that the composite (in any of the n senses) of two pullbacks of bigonal multimatrices is itself the pullback of a bigonal multimatrix, and that the identity for the i^{th} composition is an identity for the j^{th} composition.

For the latter, it suffices to observe that if the pullback of a bigonal complex over q_n is a product of the 0-section of the i^{th} projection onto $[0,1]$ with $[0,1]$, it must be the product of the $(0,...,0)$-section of the projection on the last $n-i+1$ factors with $[0,1]^{n-i+1}$. For the former, it suffices to observe that the property just stated characterizes those cubical complexes fibered over an n-cube which are pullbacks of bigonal complexes fibered over an n-bigon. □

While it is no doubt possible, and perhaps quite fruitful, to formulate a "many-objects" version of Lawrence's d-algebras [Law], we do not do so, but confine ourselves to providing "matrix d-algebras" analogous to the classical matrix algebra (and reducing to them for $d = 2$.) Because of the difference between Lawrence's indexing by the "arity" of the operation valued in the algebra itself, and the natural indexing of multimatrices by dimension of top cell, we adopt the convention $d = n + 1$ throughout the following discussion.

To readily define the matrix d-algebra, we first describe a particular simplicial n-complex fibered over the standard n-simplex Δ_n. Let $K(r, ..., r)$ ($n + 1$ copies of r) be the simplicial n-complex fibered over Δ_n such that the fiber over each vertex of Δ_n is an ordered set of r vertices, and such that $K(r, ..., r)$ satisfies the following strong lifting property: if σ is an i-face of Δ_n for any $i > 0$, any lifting $\phi : \partial\sigma \to K(r, ..., r)$ extends *uniquely* to a lifting of σ.

We then have

Theorem 32 *The set of simplicial n-multimatrices with underlying n-complex $K(r, ..., r)$ admits the structure of a d-algebra (recall $d = n + 1$). We denote the d-algebra by $n - M(r)$.*

proof: First observe that $K(r, ..., r)$ admits a unique action of the alternating group A_d by simplical maps satisfying

1. The fibration is an equivariant map with respect to this action and the action on Δ_n induced by even permutations of the vertices.

and

2. The restriction of the action of any group element γ to the vertices over a vertex v of Δ_n is a monotone map onto the vertices over $\gamma(v)$.

This action then induces the required action on $M = n - M(r)$.

To describe the required operations $m_j : M^{\otimes(d+1-j)} \to M^{\otimes j}$, observe first that M has a basis consisting of "elementary" multimatrices with all but one weight equal to 0, and the remaining weight equal to 1. Next, observe that if D is any n-cell with simplicial boundary, there is a cell complex $D(r)$ fibered over D with r vertices over each vertex of D and the same lifting properties for its cells as $K(r, ..., r)$. Let $V_r(D)$ be the K-vector space of weightings of the n-cells of D. (Note, $V_r(D)$ admits a basis of weights with a single non-zero weight of 1.)

Now, any decomposition κ of D into k n-simplexes (not decomposing the boundary simplexes) gives rise to a map $f_\kappa : M^{\otimes k} \to V_r(D)$ given by forming the complex fibered simplicially over D with r vertices over each vertex of D, and the strong lifting property, and weighting the simplexes with the weights from the copy of M lying over each simplex, then finding an element of $V_r(D)$ by weighting each n-cell by summing the product of the weights of the n-simplexes forming each section over all global sections of the fibration with boundary equal to that of the n-cell.

Now, if κ introduces no vertices, we claim that f_κ is surjective, and admits exactly one splitting mapping the basis elements to tensors of elementary multimatrices. In particular, if there are no vertices introduced in the decomposition of D, the strong lifting property implies that the choice of a vertex lying above each vertex determines uniquely an n-cell of $D(r)$, and thus a basis element of $V_r(D)$ *and* also determines uniquely an n-simplex over each n-simplex of the decomposition. Thus we may split f_κ by carrying the basis element determined by a choice of vertex lifts to the tensor product of the elementary multimatrices with 1-weighted n-simplexes determined by the same choice of vertex lifts. Call this splitting g_κ. The operations on M as a d-algebra are then given as follows: let λ be the subdivision of the n-simplex with a single internal vertex, and let κ_i (resp. κ^i) be the decomposition of the n-cell formed by $d+1-i$ n-simplexes of the boundary of a d-simplex (resp. the decomposition of the same n-cell into the mirror image of the other i n-simplexes of the boundary); then

$$m_1 = f_\lambda$$

$$m_i = f_{\kappa_i} g_{\kappa^i} \text{ for } i = 2, ..., \lfloor \tfrac{d+1}{2} \rfloor$$

All of the required equations can be verified as follows: observe that an evaluation of a product in the sense of Lawrence [Law] is defined independently of a choice of expression in terms of the m_i's and the action by $f_\mu g_\kappa$ where μ is the subdivision defining the product, and κ is the subdivision (without internal vertices) defining the evaluation. It follows that any expression in terms of the m_i's and the action evaluting the same product gives the same result, once we notice that for any two subdivisions ν, π of a cell D, we have $f_\nu g_\pi f_\pi = f_\nu$. □

In the case of $n = 2$ these structures are related to those of the sections 1 and 2 by

Proposition 33 *The 2-category $2-BMM$ is equivalent to the 2-category $2-Vect_{cc}$ of Kapranov and Voevdosky [KV], and it thus (cf. [KV]) the canonical semi-strict rectification of $\mathcal{V}-mod$.*

proof: The translation between bigonal 2-multimatrices and matrices of matrices as used in [KV] is straightforward. The further statement follows from Proposition 25 and results of [KV]. □

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