# CONSERVED QUANTITIES FOR THE YANG-MILLS EQUATIONS

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**Abstract**

There are infinitely many gauge-invariant conserved quantities for sufficiently regular solutions of the Yang-Mills equations on 4-dimensional Minkowski space, for example those that extend to $C^2$ classical solutions on the universal cover of the conformal compactification, which may be identified with the Einstein universe, $\mathbf{R} \times S^3$. These conserved quantities are expressed in terms of the 'in' or 'out' fields constructed via the formulation of scattering as a global Goursat problem on the Einstein universe.

## 1. Introduction

A great deal of work has been done on completely integrable nonlinear wave equations in two-dimensional space-time. Less is known about the physically very interesting four-dimensional case, except for the self-dual Yang-Mills equations [1, 2]. Recently, however, a number of Poincaré-invariant nonlinear wave equations in four-dimensional Minkowski space have been shown to be completely integrable, using a method that can be summarized as 'linearization via the wave operator'. At a formal level the idea is simple: the wave operators intertwine the action of the Poincaré group on the solutions of the nonlinear equation with its action on the solutions of an equation that is linear and easily seen to be completely integrable. The difficulty lies in constructing the wave operators as smooth maps on a physically natural space of solutions, for example those having finite energy.

In the massive $\phi^4$ theory, there are smooth wave operators in a neighborhood of the origin of the finite-energy space [3]. In the massless $\phi^4$ theory, infrared problems make it difficult to construct wave operators on the space of finite-energy solutions, but the conformal invariance of the theory permits the construction of wave operators on the space of 'finite-Einstein-energy' solutions [4, 5]. These are solutions extending to finite-energy solutions of a corresponding equation on $\widetilde{\mathbf{M}}$, the universal cover of the conformal compactification of Minkowski space, which can be identified with the 'Einstein universe', $\mathbf{R} \times S^3$. The boundary of Minkowski space as embedded in $\widetilde{\mathbf{M}}$ consists of lightcones $C_\pm$ at 'past and future infinity', and the surface $\{x_0 = 0\}$ in Minkowski space corresponds to the Cauchy surface $\{0\} \times S^3 \subset \widetilde{\mathbf{M}}$ with a single point removed. A finite-Einstein-energy solution may be described by its Cauchy datum $(\phi, \dot{\phi})|_{\{0\}\times S^3}$ or by either of its Goursat data $\phi|_{C_\pm}$. (In the Goursat problem, the solution of a wave equation is determined by its restriction to a characteristic cone [6].) The Goursat data $\phi|_{C_\pm}$ correspond to asymptotics of the solution on Minkowski space as $x_0 \to \pm\infty$, respectively, and may be regarded as 'out' and 'in' fields in the sense of scattering theory. Since the action of the Poincaré group on $\widetilde{\mathbf{M}}$ preserves the cones $C_\pm$, the Poincaré group acts linearly on the space of finite-Einstein-energy Goursat data, and the wave operators linearize the action of the Poincaré group on the space of finite-Einstein-energy solutions.

The physically relevant 'phase space' for the Yang-Mills equations is the space of solutions modulo gauge transformations, which is not a vector space, so one cannot expect a true 'linearization' of the action of the Poincaré group in this case. In four dimensions, however, the Yang-Mills equations are conformally invariant and wave operators can be constructed in terms of a Goursat problem on $\widetilde{\mathbf{M}}$ [7]. Here we show that this permits the construction of infinitely many gauge-invariant conserved quantities for solutions of the Yang-Mills equations on Minkowski space extending to sufficiently regular solutions on $\widetilde{\mathbf{M}}$. This raises the possibility that the Yang-Mills equations in four dimensions are completely integrable on a suitable space of solutions modulo gauge transformations.

Conserved quantities arising from the conformal invariance of the Yang-Mills equations were constructed by Glassey and Strauss [8], and have been further studied by Paneitz [9]. Formal indications in favor of complete integrability of the Yang-Mills equations have been noted by Flato and Simon [10, 11].

## 2. Results

Our notation relating to conformal geometry follows that of [4, 12]. We let $\mathbf{M}_0$ denote $\mathbf{R}^4$ with its Minkowski metric and let $\widetilde{\mathbf{M}}$ denote the universal cover of the conformal compactification of $\mathbf{M}_0$. $\widetilde{\mathbf{M}}$ may be identified with the 'Einstein universe', $\mathbf{R} \times S^3$ with the metric $d\tau^2 - ds^2$, where $d\tau^2$ and $ds^2$ are the usual Riemann metrics on $\mathbf{R}$ and $S^3$, respectively. The conformal embedding of $\mathbf{M}_0$ in $\widetilde{\mathbf{M}}$ is denoted by $\iota$. The group $\widetilde{SO}(2,4)$ acts as conformal diffeomorphisms of $\widetilde{\mathbf{M}}$, and the universal cover $\widetilde{\mathbf{P}}$ of the Poincaré group extended by scale transformations is thus identified with the subgroup of $\widetilde{SO}(2,4)$ preserving $\iota\mathbf{M}_0$. Of particular importance is the 'Minkowski time evolution' one-parameter subgroup of $\widetilde{SO}(2,4)$, denoted by $\exp(t\mathbf{T}_0)$, which satisfies

$$\exp(t\mathbf{T}_0)\iota(x_0, \vec{x}) = \iota(t + x_0, \vec{x}) \ .$$

Let $S^n$ be embedded as the unit sphere in $\mathbf{R}^{n+1}$. Given $u \in S^3$, write $u$ as $(\cos\rho, \omega\sin\rho)$ where $\rho \in [0,\pi]$ and $\omega \in S^2$. Define the subsets $C_\pm \subset \widetilde{\mathbf{M}}$ by:

$$C_\pm = \{(\tau, u) \in \mathbf{R} \times S^3 : \rho = \pm(\pi - \tau)\,,\ 0 \le \rho \le \pi\}\,.$$

In what follows we arbitrarily choose to work only with $C_+$, and let $C^* = C_+ - \{\rho = 0, \pi\}$. $C^*$ is a smooth submanifold of $\widetilde{\mathbf{M}}$, diffeomorphic to $\mathbf{R} \times S^2$, and we give it the coordinates $(s, \omega)$, where $s = -2\cot\rho$ and $\omega$ is defined as above. We give $C^*$ the volume form $ds \wedge d\omega$, where by abuse of notation $d\omega$ denotes the volume form on $S^2$. $C^*$ is preserved by the action of $\widetilde{\mathbf{P}}$. The advantage of the coordinates $(s,\omega)$ is that Minkowski time translation on $C^*$ corresponds to translation in the $s$ variable:

$$\exp(t\mathbf{T}_0)(s,\omega) = (s + t, \omega)\,.$$

Let $G$ be a compact Lie group and $\mathbf{g}$ its Lie algebra. We identify the space $\mathbf{A}$ of $C^2$ $\mathbf{g}$-valued one-forms on $\widetilde{\mathbf{M}}$ with the space of $C^2$ connections on the trivial $G$-bundle over $\widetilde{\mathbf{M}}$ by means of the map taking the $\mathbf{g}$-valued one-form $A$ to the connection $d + A$. The vector space $\mathbf{A}$ is a Frechét space under the $C^2_{loc}$ topology. We let $\mathbf{G}$ denote the group of $C^3$ gauge transformations, that is, $C^3(\widetilde{\mathbf{M}}, G)$ with pointwise multiplication. The group $\mathbf{G}$ acts on $\mathbf{A}$ as continuous affine transformations by:

$$gA = \mathrm{Ad}(g)A - g^{-1}dg\,.$$

The group $\widetilde{SO}(2,4)$ acts as continuous linear transformations of $\mathbf{A}$ by:

$$hA = (h^{-1})^* A\,.$$

Let $\mathbf{F}$ denote the space of $C^1$ $\mathbf{g}$-valued two-forms on $\widetilde{\mathbf{M}}$, a Frechét space with the $C^1_{loc}$ topology. Given $A \in \mathbf{A}$, let $F = dA + \frac{1}{2}[A, A]$. The map $A \mapsto F$ is smooth from $\mathbf{A}$ to $\mathbf{F}$, where here and in what follows 'smooth' means infinitely Frechét-differentiable.

**Theorem.** *Let $f \in C^\infty(\mathbf{g})$ be an Ad-invariant function satisfying $|f(y)| \le \|y\|$ for sufficiently small $y \in \mathbf{g}$, for some norm on $\mathbf{g}$. Let $\phi \in C_0(C^*)$ and let $u, v$ be continuous vector* *fields on $C^*$ invariant under the action of* $\mathbf{R}$ *as Minkowski time translation. Define the function* $H\colon \mathbf{A} \to \mathbf{R}$ *by:*

$$H(A) = \int_{\mathrm{R}} \int_{C^*} \phi(s,\omega)\, f(F(u,v))(t,\omega)\, \phi(s-t,\omega)\, ds \wedge d\omega\, dt\,,$$

*or, more concisely,*

$$H(A) = \int_{C^*} \phi(s,\omega)\, (\phi * f(F(u,v)))(s,\omega)\,.$$

*Then $H$ is smooth. For all $t \in \mathbf{R}$, $\exp(t\mathbf{T}_0)^* H = H$, and for all $g \in \mathbf{G}$, $g^* H = H$.*

Proof - A calculation shows that

$$\|F(u,v)\| \leq c(A)(1+|s|)^{-2}\,,$$

where $c(A)$ is a continuous function of the suprema of the components of $A$ and its first derivatives over a compact set in $\widetilde{\mathbf{M}}$. It follows that $\phi * f(F(u,v)) \in L^1(C^*)$, and the function $H\colon \mathbf{A} \to \mathbf{R}$ is well-defined and smooth.

Suppose that $A \in \mathbf{A}$ and $A' = \exp(t\mathbf{T}_0)^* A$ for some $t \in \mathbf{R}$. Let $F' = dA' + \frac{1}{2}[A', A']$. Then $F' = \exp(t\mathbf{T}_0)^* F$, so

$$f(F'(u,v))(s,\omega) = f(F(u,v))(s+t,\omega)\,,$$

and $H(A') = H(A)$. It follows that the function $H$ is preserved by the action of Minkowski time translation on $\mathbf{A}$.

Suppose that $A \in \mathbf{A}$ and $A' = gA$ for some $g \in \mathbf{G}$. Let $F' = dA' + \frac{1}{2}[A', A']$. As is well-known, $F' = \mathrm{Ad}(g)F$, so $H(A) = H(A')$ by the Ad-invariance of the function $f$. Thus the function $H$ is preserved by the action of $\mathbf{G}$ on $\mathbf{A}$. $\square$

In particular, let $\mathbf{S}$ be the set of $A \in \mathbf{A}$ satisfying the Yang-Mills equations:

$$F = dA + \frac{1}{2}[A, A] \ \ , \ \ d \star F + [A, \star F] = 0\,.$$

By the invariance of these equations under gauge and conformal transformations, $\mathbf{S}$ is preserved by the actions of $\widetilde{SO}(2,4)$ and $\mathbf{G}$, and the functions $H\colon \mathbf{A} \to \mathbf{R}$ restrict to functions on $\mathbf{S}$ that are invariant under Minkowski time translations and gauge transformations. If $A \in \mathbf{S}$ then $\iota^* A$ is a $C^2$ classical solution of the Yang-Mills equations on $\mathbf{M}_0$. Thus the functions $H$ define gauge-invariant conserved quantities for solutions of the Yang-Mills equations on $\mathbf{M}_0$ which extend to $C^2$ classical solutions on $\widetilde{\mathbf{M}}$.

## 3. Conclusions

We have deliberately described a weak version of our result in order to present the basic ideas as simply as possible. Further work will require the use of spaces of solutions in a fixed gauge having initial data in Sobolev spaces. For example, solutions in temporal gauge may be described by Cauchy data in the space $\mathbf{X}$ consisting of pairs $(a, e)$ of $\mathbf{g}$-valued one-forms on $S^3$ such that the components of $a$ lie in the Sobolev space $L^{2,3}(S^3)$, the components of $e$ lie in $L^{2,2}(S^3)$, and the constraint equation $d \star e + [a, \star e] = 0$ holds [7, 10, 13, 14].

To prove that time evolution on a suitable space of solutions of the Yang-Mills equations in four-dimensional Minkowski space are completely integrable, one might attempt to carry out the following program:

1) Remove certain singular points from $\mathbf{X}$ and quotient out the action of the remaining gauge transformations to obtain a symplectic manifold $\mathbf{Y}$, the nonsingular part of the physical phase space.

2) Show that the group $\widetilde{SO}(2,4)$ acts as symplectomorphisms of $\mathbf{Y}$.

3) Show the functions $H\colon \mathbf{X} \to \mathbf{R}$ given in the corollary factor to give smooth functions $J\colon \mathbf{Y} \to \mathrm{R}$.

4) Show that some subset $S$ of the functions $J$ satisfies: a) for all $J, J' \in S$ the Poisson bracket $\{J, J'\}$ is well-defined and vanishes; b) the subspace

$$L = \{v \in T_x\mathbf{Y} \colon \; \forall J \in S \;\; dJ(v) = 0\}$$

is isotropic for all $x$ in a set of second category in $\mathbf{Y}$.

Step 1) has already largely been carried out by Moncrief [15]; see also [16]. Step 4) seems the most difficult, and is likely to require more insight into the global Goursat problem for the Yang-Mills equations. For a) to hold one must find functions $J$ that generate vector fields, which is nontrivial because the symplectic form on $\mathbf{Y}$ will be only weakly nondegenerate. Moreover, there may not be enough conserved quantities with vanishing Poisson brackets for b) to hold, in which case the Yang-Mills equations would be only 'partially integrable'.

The possibility that the Yang-Mills equations are completely integrable on a suitable space of solutions may have been neglected due to the result that time evolution on the finite-dimensional manifold of 'space-independent' solutions (in temporal gauge) is not completely integrable [17, 18]. There need be no contradiction, since for a solution to extend to a solution on $\widetilde{\mathbf{M}}$ it must satisfy certain decay conditions in the spatial directions [14]. In fact, these conditions require the solution to lie in the 'zero charge sector' [19].

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