RESEARCH OVERVIEW

BRIAN BENSON

The Laplace differential operator appears in the equations of many important physical phenomena such as gravitational potentials, heat, and waves across a uniform medium. One of my primary interests is in studying the connections between this operator and geometry.

I first began studying these connections by considering the eigenvalues of the Laplace operator and the geometry of a Riemannian manifold, an object which generalizes the idea of a surface to higher dimensions. These eigenvalues have a physical interpretation, as they model the frequencies which can resonate from an instrument constructed from a uniform medium.

The relationship between eigenvalues and a geometric invariant related to the isoperimetric problem, related to minimizing the ratio of surface area to volume, has been one focal point in my work. This isoperimetric invariant, called the Cheeger constant, has appeared in my work as both a geometric quantity to bound the Laplace eigenvalues and, in hyperbolic surfaces (2D manifolds with constant Gaussian curvature equal to -1), as a quantity which can be directly computed using other geometric information like the area of the surface and lengths of shortest closed curves. This so-called Cheeger constant is defined as

$$h(M) = \inf_{\Sigma, A, B} \frac{\text{Area}(\Sigma)}{\min\{\text{Vol}(A), \text{Vol}(B)\}}$$

where Σ is a smooth (codimension-1) hypersurface which divides M into 2-colorable regions or sides A and B as illustrated in Figure 1. Independent work of Cheeger and Buser show that the first positive eigenvalue of the Laplacian can be bounded from below and above, respectively, by quadratics in the Cheeger constant [Che70, Bus82]. Specifically, for closed Riemannian manifolds, I found an upper bound for each of the eigenvalues in terms of the Cheeger constant [Ben15]. See Section 1 for more details.

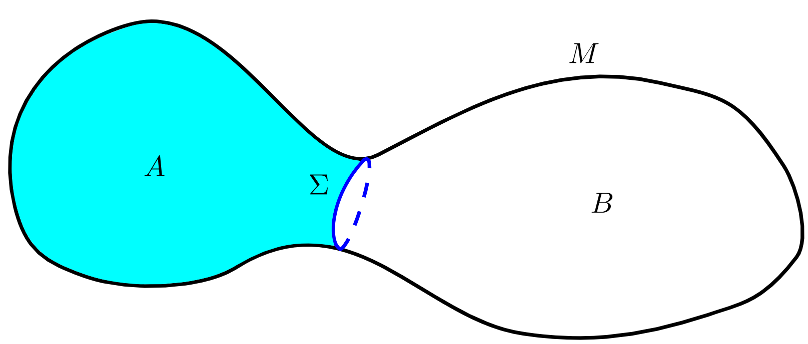


FIGURE 1. A visual illustration of the hypersurface Σ and the regions A and B for the Cheeger constant.

I also found opportunities to study related and analogous phenomena on discrete graphs. One great feature of the Laplace eigenvalues and the Cheeger constant is that analogs of these invariants can also be studied on discrete graphs. Further, versions of the inequalities of Cheeger and Buser have also been shown to hold. For the Cheeger constant, the (codimension-0) volume is given by the counting measure on the vertices, while the area of the splitting is given by a subset of edges. With Prasad Tetali and Peter Ralli, I gave upper bounds on eigenvalues of the graph in terms of both volume growth of the graph and the Cheeger constant. A summary of one of the main theorems of this work is as follows: *There exists an explicit matrix whose entries depend only on pairs of volume growth bounds, one from above and one from below, such that the Laplace eigenvalues of G are bounded above by the eigenvalues of the matrix.* This and other results from this project are summarized in Section 1.

The Cheeger constant and the first positive eigenvalue can also be used to study hyperbolic reflection groups. I worked with Grant Lakeland and Holger Then to explicitly compute these invariants for specific examples of hyperbolic (orbi)surfaces, which was motivated by an experimental approach to questions regarding the enumeration of maximal arithmetic hyperbolic reflection groups. See Section 2 for a summary of our work.

Another way to study the Laplace operator on manifolds is by harmonic functions, i.e. functions with Laplacian identically zero. The mean value property is an important property for harmonic functions in Euclidean space, stating that the integral average of a harmonic function on a metric ball is equal to the value of the function at the center of the ball. While mean value sets for the Laplace operator in Euclidean space are always metric balls [Kur72], the same is not true for Riemannian manifolds! So if mean value sets on Riemannian manifolds are not metric balls, how can we describe their shape and geometry? It turns out that the solution to an obstacle problem, where a Green's function for the Laplacian is the obstacle, provides a way to describe all mean values sets for Riemannian manifolds. Specifically, the singularity of the Green's function corresponds to the analog of the center of the metric ball when constructing these mean value sets on the manifold. See Section 3 for more details on my work with Ivan Blank and Jeremy LeCrone in this direction.

The geometry of solutions to partial differential equations involving the Laplacian is also area of interest. One related question is for convex domains, could the maxima of solutions to semilinear Poisson equations be independent of the non-linearity? Although there is no obvious reason why these points should have this independence, numerical evidence of Cima and Derrick led to the conjecture that they are always the same [CD11]. While working with Laugesen, Minion, and Siudeja, I found a counter-example to this conjecture providing two equations whose solutions have distinct maxima on the right isosceles triangle with hypotenuse having length 2 and on the half-disk of radius 1 [BLMS15]. However, the maxima of these solutions to seemingly very different non-linearities do not differ until the fourth decimal place! See Section 4 for additional details.

1. Eigenvalues and the Cheeger Constant on Riemannian Manifolds and Graphs

Let M be a closed, connected Riemannian n-manifold. The Laplace-Beltrami (Laplacian) operator on M is defined by $\Delta u := -\mathrm{div}\big(\mathrm{grad}(u)\big)$. The eigenvalues of Δ on M are the real numbers λ such that $\Delta u = \lambda u$ where $u \in H^2(M)$ ($u : M \to \mathbf{R}$ in $L^2(M)$ and twice weakly differentiable). The eigenvalues of Δ are discrete and indexed by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots .$$

The set of eigenvalues with multiplicities is called the spectrum of Δ.

Cheeger [Che70] gave the initial motivation for $h(M)$ by first proving

$$\lambda_1 \geq \frac{h(M)^2}{4}.$$

Buser [Bus82] later proved that if $-(n-1)\delta^2$ is a lower bound on the Ricci curvature with $\delta \geq 0$, then

$$\lambda_1 \leq 2\delta(n-1)h(M) + 10h(M)^2.$$

Therefore, we may conclude that the eigenvalue λ_1 is controlled by quadratics in $h(M)$. More recently, in unpublished work, Agol gave a quantitative improvement of Buser's inequality in terms of a parameter implicitly defined by a second order ODE with boundary conditions [Ago].

How much information does the Cheeger constant give about the full Laplace spectrum of M? Buser proved that for any $h > 0$, $k \in \mathbf{N}$, and $\epsilon > 0$, there exists M with $h(M) = h$ and such that $\lambda_k \leq \frac{h^2}{4} + \epsilon$ [Bus78]. This means that there is no analog of Cheeger's inequality for higher eigenvalues! On the other hand, it turns out that there is an analog of Buser's inequality for the higher eigenvalues, which depends on the same manifold invariants, namely h,n, and δ:

Theorem 1. [Ben15] *There exists an explicit Sturm-Liouville problem $\omega = \omega(h, n, \delta)$ such that the eigenvalues μ of ω have the relationship:*

$$\lambda_k \leq \mu_{\lceil \frac{k+1}{2} \rceil}.$$

The Sturm-Liouville problem ω is defined on the interval $(0, T)$ with the following equation and boundary conditions:

$$Lu = \mu u, \qquad u(0) = 0, \qquad u'(T) = 0.$$

The operator L and the endpoint T depend on h, n, δ.

Examples of 2D tori with flat metrics show that the upper bound in Theorem 1 is asymptotically sharp. A qualitative conclusion from this work is that the Cheeger constant provides some information about the higher eigenvalues, but the upper bound is not as asymptotically sharp as when one is able to refer to the volume of the manifold instead of the Cheeger constant. However, work of Cheeger established that the volume cannot provide a lower bound for eigenvalues of manifolds [Che70]. Thus, volume cannot provide upper and lower bounds for eigenvalues in contrast to the Cheeger constant.

Recently, Peter Ralli, Prasad Tetali, and I found a number of analogues of Buser's inequality on graphs. These inequalities use approaches related to my work on manifolds and use the relationship between vertex growth of the graph and several different notions of graph curvature. In addition, since previous versions of Buser's inequality for graphs were proved using heat kernel methods, our work provided a new proof of the classical Buser's inequality for graphs. Instead, our proofs focus on the volume growth of the graph.

Let $G = (V, E)$ be a finite graph, where V is the vertex set and E is the edge set. Given a vertex subset $\Sigma \subseteq V$ and a natural number k, how many vertices of the graph are within k edges of the vertices in Σ. This growth in the number of vertices as k increases is sometimes referred to as the volume growth of the graph.

More explicitly, assume that the vertex subset $\Sigma \subseteq V$ has the property that deleting vertices in Σ from V and, thereby, deleting incident edges from E, results in two subgraphs G^+ and G^-. Define $\mu(i)$ and $\nu(i)$ for $i \in \mathbf{Z}$ such that

$$\mu(i) \leq |\mathrm{dist}_\Sigma^{-1}(i)| \leq \nu(i).$$

Here, we have denoted by $\mathrm{dist}_\Sigma(v)$ the signed distance function induced by counting the least edges among paths from a vertex in Σ to a vertex $v \in V$. Further, to give the distance function a sign, we count edges into G^+ as positive and the edges into G^- as negative.

To state the theorem summarized in the introduction with more precision, define by T^+ to be the maximum integer such that $|\mathrm{dist}_\Sigma^{-1}(T^+)| > 0$ and T^- be the least integer such that $|\mathrm{dist}_\Sigma^{-1}(T^-)| > 0$.

Theorem 2. [BRTar] *For a graph G and any $k, l \in \mathbf{N}$ with $1 \leq k, l \leq \min\{|T^-|, T^+\} =: T$, then there exist explicit symmetric, tridiagonal matrices A^+, indexed by $\{1, \dots, T^+\}$, and A^-, indexed by $\{T^-, T^- + 1, \dots, -2, -1\}$, whose entries depend only on μ and ν, such that*

$$\lambda_{k+l}(G) \leq \frac{1}{2} \max \left\{\rho_k^+, \rho_l^-\right\}$$

where ρ_k^+ and ρ_l^- are the k-th and l-th non-trivial eigenvalues of the respective equations $A^+ g^+ = \rho^+ g^+$ with $g^+ \in W^+$ and $A^- g^- = \rho^- g^-$ with $g^- \in W^-$.

Combining the theorem above with volume growth bounds related to the Cheeger constant, one can bound eigenvalues of the graph from above by the Cheeger constant or higher dimensional analogues thereof. A representative example of such a Buser-type inequality is as follows.

Theorem 3. [BRTar] *Assume that $\nu(i) = 1$ for all $i \in [T^-, T^+]$. If $n \geq 2$ and $h_{out}(n) < 1$, then we have*

$$\lambda_k(G) \leq k^2 h_{out}(n)^2 \left(\frac{27\pi^2}{16} + o(1)\right).$$

Here $h_{out}(n)$ is an analogue of the Cheeger constant for graphs. Specifically, we define

$$h_{out}(n) = \max\left\{\tfrac{|\partial_o V_1|}{|V_1|}, \tfrac{|\partial_o V_2|}{|V_2|}, \dots, \tfrac{|\partial_o V_n|}{|V_n|}\right\}$$

where the V_i are pairwise disjoint subsets of V with $V = \cup_i V_i$ and $\partial_o V_i$ is the set of vertices v with $v \notin V_i$ having an edge between v and a vertex in V_i.

2. Computing the Cheeger Constant of Hyperbolic Surfaces

The Cheeger constant generates additional problems of a purely geometric nature. For example, what can we say about sets whose isoperimetric ratio is equal to the Cheeger constant and can we find and use such sets to compute the Cheeger constant? Independent work of Hass-Morgan and Adams-Morgan provided a pathway for me to develop a theoretical method to directly compute the Cheeger constant of surfaces having a hyperbolic (constant curvature -1) metric and finite area [HM96, AM99]. My first initial motivation in pursuing this direction were questions of Grant Lakeland, who was applying the Cheeger constant and first eigenvalue of the Laplace operator to the enumeration of hyperbolic groups; specifically, counting the number of maximal arithmetic reflection groups in dimension 2. In short, there were a number of hyperbolic orbisurfaces (surfaces having cone point singularities in the metric) for which it would be useful to directly compute their Cheeger constant or their first

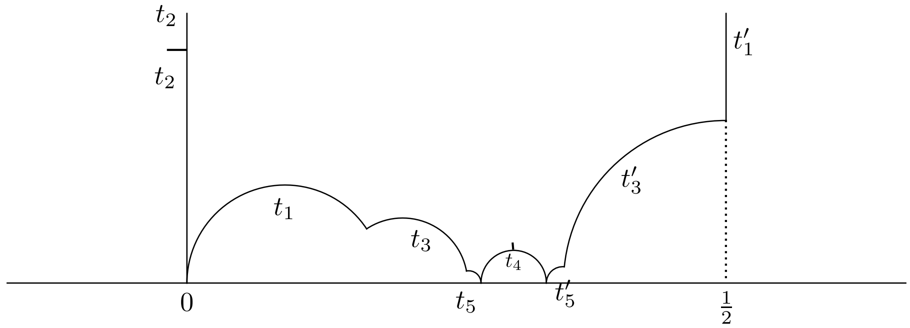


FIGURE 2. A domain for one of the non-congruence surfaces given by Lakeland. In this picture, each arc t_i is glued to t_i' for i even and are glued to itself for i odd.

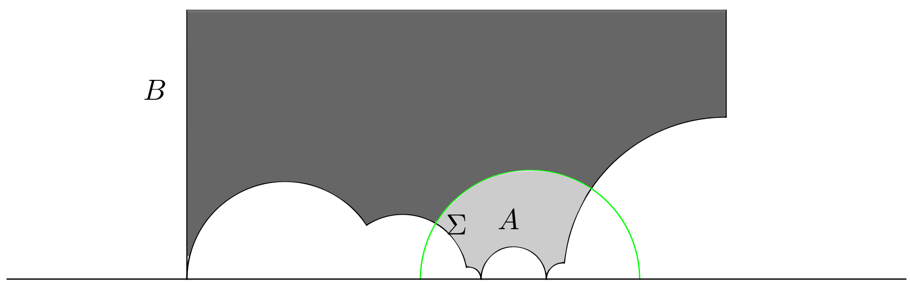


FIGURE 3. The splitting of the surface by Σ whose ratio $\frac{|\Sigma|}{|A|}$ is equal to the Cheeger constant.

positive eigenvalue. In joint work with Lakeland and Holger Then, we were able to produce such computations for both the Cheeger constants and the first positive eigenvalues for each of the non-compact examples related to Lakeland's work[BLT19]. Figure 2 gives an example of a presentation of the orbisurface (discovered by Lakeland [Lak12]) whose Cheeger constant and first eigenvalue of the Laplacian is relevant to the study of maximal arithmetic reflection groups in dimension 2. Figure 3, the curve (hypersurface) Σ and correpsonding regions A and B which realize the Cheeger constant of the orbisurface are given explicitly.

Future Directions. In addition, the ability to directly compute the Cheeger constant of hyperbolic surfaces has the potential to add to our knowledge of the geometry of these surfaces. Selberg conjectured that the first eigenvalues of principal congruence arithmetic surfaces are bounded below by $\frac{1}{4}$, a question related to geometry and number theory which has remained open for about fifty years [Sel65, Sar95]. It is a consequence of Buser's inequality that Selberg's conjecture implies a lower bound on the Cheeger constants. In addition, work of Brook-Zuk give an asymptotic upper bound for the Cheeger constant of these surfaces [BZ02]. In contrast to the Brooks-Zuk result, Jeffrey Meyer and I are currently computing the Cheeger constants of examples of principal congruence arithmetic surfaces. We are using these examples to develop ideas and techniques to make the process of computing the Cheeger constants of hyperbolic surfaces as efficient as possible. An educational goal of our work is to refine the algorithm to use as a tool to teach hyperbolic geometry and distance to advanced undergraduate or graduate students.

3. Mean Value Theorems for Riemannian Manifolds

Another interesting question related to the geometry of the Laplace operator in the setting of Riemannian manifolds is on the geometry of mean value sets of the operator. In Euclidean space, it is well-known that the integral average of a harmonic function f on a metric ball $B_r(x_0)$ of radius $r > 0$ centered at x_0 is equal to the value of the function at the center; that is

$$f(x_0) = \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0)} f(x)\, d\mu(x),$$

where μ is the Lebesgue measure.

In joint work with Ivan Blank and Jeremy LeCrone, we gave a characterization of mean value sets for the Laplace operator on Riemannian manifolds [BBL19]. The original idea for the approach is due to Cafarelli [Caf98], who recognized that one could characterize mean value sets for general divergence form elliptic operators in Euclidean space using a free boundary problem called the obstacle problem.

For each closed subset $S \subseteq M$ and $r > 0$, we consider the following obstacle problem on M :

$$\left\{ \begin{array}{lll} \Delta u & = -\chi_{\{u<G\}} r^{-n} & \text{in } S, \\ u & = G(\cdot, x_0) & \text{on } \partial S, \end{array} \right. \tag{1}$$

where G is the Green's function for Δ on S.

Theorem 4. [BBL19] *Let M be a Riemannian manifold which is non-compact or has non-empty boundary. If u^* solves the obstacle problem (1), then define*

$$D(r, x_0) = \{x \in S : u^*(x) < G(x, x_0)\},$$

called the non-contact set of u^*. Then, if $\Delta f = 0$ and $D(r, x_0) \subset\subset S$, then

$$f(x_0) = \frac{1}{\mathrm{Vol}(D(r, x_0))} \int_{D(r,x_0)} f\, dV. \tag{2}$$

The above theorem also holds more generally for subharmonic ($\Delta f \geq 0$) and superharmonic ($\Delta f \leq 0$) assumptions, with equality in Equation (2) replaced with $\leq$ and $\geq$, respectively.

Future Directions. In addition, a classical theorem of Kuran in Euclidean space implies that if f is a harmonic function and D a domain in Euclidean space containing a point x_0, then if $f(x_0)$ is equal to the integral average of f over D, then D is a metric ball with center x_0 [Kur72]. For Riemannian manifolds which are non-compact or have boundary, geodesic (metric) balls do not necessarily satisfy the mean value property; manifolds with the property that, for every point, a geodesic ball of positive radius satisfies the mean value property are sometimes referred to as harmonic manifolds.

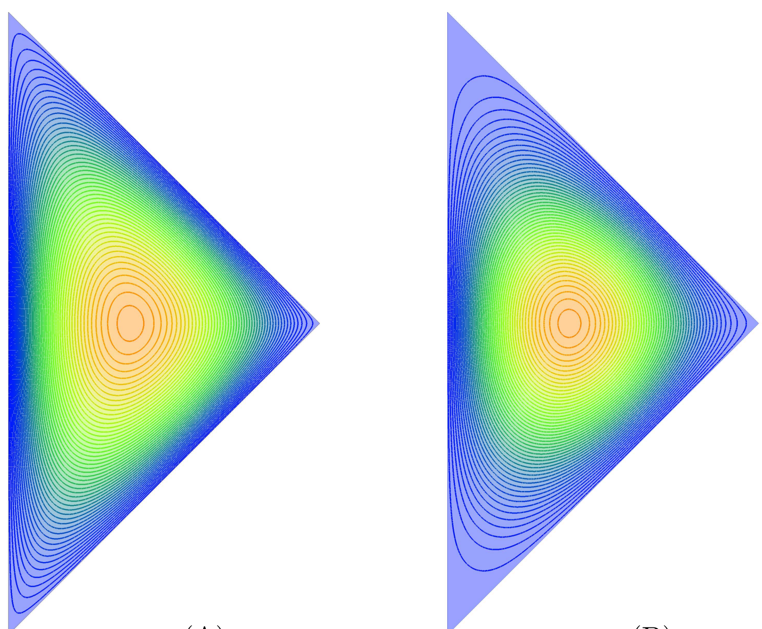
While it is known that every harmonic manifold is Einstein (its Ricci tensor is a constant multiple of its metric), a full characterization of all harmonic manifolds remains unknown [Kre10]. One possible application of the mean value theorem for Riemannian manifolds is to study harmonic manifolds. Recent work of Armstrong-Blank [AB19] makes progress towards showing that every mean value set for the Laplace operator on a manifold must be of the type characterized in Theorem 4.

4. Maxima of Semilinear Poisson Equations

The geometry of solutions to partial differential equations is another area of interest. One related question is for convex domains Ω in Euclidean space: Could the maxima of solutions to semilinear Poisson equations of the form

$$\begin{cases} -\Delta u &= f(u) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{cases}$$

be independent of the non-linearity $f(x)$? Although there is no obvious reason why these points should have this independence, numerical evidence of Cima and Derrick led to the conjecture that they are always the same [CD11]. I worked with Laugesen, Minion, and Siudeja to answer this question. We provided a counter-example to the conjecture, showing that two equations whose solutions have distinct maxima on a half-disk of radius one and on a right isosceles triangle with a hypotenuse of length two [BLMS15]. However, these maxima do not differ until the fourth decimal place, with a difference of approximately 1.5×10^{-4}. The two equations which provide the counter-example correspond to the choices $f(z) = 1$ (the torsion function) and $f(z) = \lambda z$ (the ground state), where $\lambda > 0$ is the first Dirichlet eigenvalue of the Laplacian on Ω. Figure 4 provides a plot of the level sets of the solutions to these equations. Note how close the location of the maxima of the solutions u appear, but how different these functions appear as they approach the boundary the triangle!



(A) (B)

Figure 4. On the right isosceles triangle, the level sets of the torsion function (A) and the level sets of the first eigenfunction (B).

References

[AB19] N. Armstrong and I. Blank, *A Uniqueness Theorem for Mean Value Sets for Elliptic Divergence Form Operators*, arXiv:1907.12523 , 13 (2019).

[Ago] I. Agol, An Improvement to Buser's Inequality.

[AM99] C. Adams and F. Morgan, *Isoperimetric curves on hyperbolic surfaces*, Proc. Amer. Math. Soc. **127**(5), 1347–1356 (1999).

[BBL19] B. Benson, I. Blank and J. LeCrone, *Mean value theorems for Riemannian manifolds via the obstacle problem*, J. Geom. Anal. **29**(3), 2752–2775 (2019).

[Ben15] B. Benson, *Sturm-Liouville Estimates for the Laplacian and Cheeger Constant*, Int. Math. Res. Notices **16**, 7510–7551 (2015).

[BLMS15] B. A. Benson, R. S. Laugesen, M. Minion and B. A. Siudeja, *Torsion and ground state maxima: close but not the same*, ArXiv e-prints (July 2015), 1507.01565.

[BLT19] B. Benson, G. Lakeland and H. Then, *Cheeger constants of hyperbolic reflection groups and Maass cusp forms of small eigenvalues*, arXiv:1908.00199 (2019).

[BRTar] B. Benson, P. Ralli and P. Tetali, *Volume growth, curvature, and Buser-type inequalities in graphs*, Int. Math. Res. Not., arXiv:1802.01952 , 41 (to appear).

[Bus78] P. Buser, *Über eine Ungleichung von Cheeger*, Math. Z. **158**(3), 245–252 (1978).

[Bus82] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. (4) **15**(2), 213–230 (1982).

[BZ02] R. Brooks and A. Zuk, *On the asymptotic isoperimetric constants for Riemann surfaces and graphs*, J. Differential Geom. **62**(1), 49–78 (2002).

[Caf98] L. A. Caffarelli, *The obstacle problem*, Lezioni Fermiane. [Fermi Lectures], Accademia Nazionale dei Lincei, Rome; Scuola Normale Superiore, Pisa, 1998.

[CD11] J. A. Cima and W. Derrick, *A solution of* $\Delta u + f(u) = 0$ *on a triangle*, Irish Math. Soc. Bull. (68), 55–63 (2012) (2011).

[Che70] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, in *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199, Princeton Univ. Press, Princeton, N. J., 1970.

[HM96] J. Hass and F. Morgan, *Geodesics and soap bubbles in surfaces*, Math. Z. **223**(2), 185–196 (1996).

[Kre10] P. Kreyssig, *An Introduction to Harmonic Manifolds and the Lichnerowicz Conjectures*, arXiv:1007.0477 (5), 43 (2010).

[Kur72] U. Kuran, *On the mean-value property of harmonic functions*, Bull. London Math. Soc. **4**, 311–312 (1972).

[Lak12] G. S. Lakeland, *Arithmetic reflection groups and congruence subgroups*, PhD thesis, University of Texas at Austin, 2012.

[Sar95] P. Sarnak, *Selberg's eigenvalue conjecture*, Notices Amer. Math. Soc. **42**(11), 1272–1277 (1995).

[Sel65] A. Selberg, On the estimation of Fourier coefficients of modular forms, in *Proc. Sympos. Pure Math., Vol. VIII*, pages 1–15, Amer. Math. Soc., Providence, R.I., 1965.

Department of Mathematics, University of California, Riverside, CA 92521
E-mail address: brian.benson.math@gmail.com